Revealed Growth: A Method for Bounding the Elasticity of Demand with an Application to Assessing the Antitrust Remedy in the Du Pont Decision

Wallace P. Mullin
George Washington University

Christopher M. Snyder
Dartmouth College
National Bureau of Economic Research

May 2018

Abstract: We propose a method for bounding the elasticity of demand that applies to growing, homogeneous-product markets which requires only minimal data—market price and quantity over a time span as short as two periods. Reminiscent of revealed-preference arguments using choices over time to bound the shape of indifference curves, we use shifts in the equilibrium over time to bound the shape of the demand curve under the assumption that growing demand curves do not cross. We apply the method to assess the effectiveness of the antitrust remedy in the 1952 Du Pont decision, ordering the incumbent manufacturers to license their patents for commercial plastics. Commentators have suggested that the incumbents may have preserved the monopoly outcome by gaming the licensing contracts. The upper bounds on demand elasticities that we compute are close to zero in many post-remedy years. Such extremely inelastic demand is inconsistent with monopoly, suggesting the remedy may have been effective. The bounds are consistent across functional forms (linear, logit) and products.


Keywords: demand estimation · bounds · elasticity · antitrust · plastics · remedy

Contact Information: Mullin: Department of Economics, George Washington University, Monroe Hall 340, 2115 G Street, NW, Washington DC 20052; email: wpmullin@gwu.edu. Snyder: Department of Economics, Dartmouth College, 301 Rockefeller Hall, Hanover NH 03755; email: chris.snyder@dartmouth.edu.

Acknowledgments: The authors are grateful to David Balan, Sharat Ganapati, Paul Novosad, Jose Manuel Paz y Mino, and seminar participants at the 2018 International Industrial Organization Conference (Indianapolis) for valuable discussions, to John Cocklin for insights on locating historical data sources, and to Marvin Lieberman and the Dartmouth FIVE Project for sharing their data on the chemicals industry.
1. Introduction

In antitrust cases, whether structural remedies are effective in encouraging competition has been an ongoing concern for scholars and policymakers. Adams (1951) was an early skeptic. He highlighted a series of earlier cases, including monopolization cases, that were won by the antitrust agency but the imposed remedies failed to restore competition, the so-called pyrrhic victories of antitrust.

To understand why promising remedies might fail to achieve their goals, consider the example of the remedy in the *Du Pont* decision of 1952, compelling the incumbent manufacturers in the commercial plastics market to license their manufacturing patents to all applicants. As Areeda, Kaplow, and Edlin (2013) note, the remedy typically does not set the licensing terms, leaving them subject to the commercial negotiations of the parties. This leaves open the possibility for incumbent manufacturers to preserve something close to the monopoly outcome by specifying exorbitant rates. Another possibility is that the incumbents set rates that a judge or other outside party might deem “reasonable” but do not transfer adequate tacit knowledge (beyond what is written in the patents) that would allow entrants to compete on an even footing with the incumbents.

In this paper, we study the effectiveness of the antitrust remedy ordered in the aforementioned *Du Pont* decision. The case involved two incumbent chemical manufacturers: the U.S. firm Du Pont and the U.K. firm Imperial Chemical Industries (ICI). The two firms signed a Patents and Processes agreement in 1929, granting each company exclusive licenses for the patents and secret processes controlled by the other and dividing the global market into exclusive territories between them. The U.S. government brought suit under Section 1 of the Sherman Act for what they alleged was an illegal market division. District judge Sylvester Ryan ruled in favor of the government, ordering the defendants to cancel their exclusive-territory arrangements requiring them to license the patents behind several of their products, most significantly, two types of plastic widely used for commercial purposes: low-density polyethylene, used to make Tupperware food-storage containers, and high-density polyethylene, used to make Hula Hoops.

On the surface, the remedy appears to have had the desired procompetitive effect (Backman 1964, p. 71). Eleven manufacturers entered by the end of the decade; prices steadily declined and output rose. However, the same price declines and output increases may have arisen in a monopoly
market experiencing substantial cost declines, plausibly realistic for plastics in the 1950s and 60s. The entrants may have merely been producing their share of the monopoly quantity, returning most of the rents to the incumbents.

Our study will provide formal evidence for the effectiveness of the remedy that cuts through these criticisms. Formal study is hindered by a paucity of data for this historical application, just yearly aggregate price and quantity data for polyethylene, and only in years after the remedy was ordered. We offer a new methodology that allows us to draw solid conclusions from these fairly minimal data. The methodology—which we will explain intuitively later in this introduction—provides an upper bound on the elasticity of demand in a given year assuming that demand is growing so that the demand curve in a given year is higher than in previous years and lower than in later years. While bounding methodologies that seem initially promising may turn out to produce such wide bounds as to be uninformative, the bounds we find in the polyethylene market are tightly concentrated around zero in many years, implying that demand cannot be far from perfectly inelastic. Such extremely inelastic demand is inconsistent with monopoly, suggesting the remedy may have been effective. Thus our formal evidence lends support to the use of structural remedies in monopolization cases.

Although the Du Pont decision and implications for antitrust motivate our paper, the bulk of the paper is methodological, focusing on developing the methodology for bounding the elasticity of demand using time-series information on market equilibrium prices and quantities. The methodology applies to markets that, like polyethylene, involve homogeneous products and are growing. Figure 1 can be used to explain how the method works intuitively. In the example in the figure, the researcher has price and quantity data for two years. The equilibrium point in the first year is $e_1$ and in the second is, say, $e'_2$. The researcher wants to bound the steepness of the inverse demand curve in period 1 on which $e_1$ lies. Even if the researcher is willing to assume demand is linear (at least locally, to a first approximation), with so little data, there may be little hope to say anything more than inverse demand lies somewhere between the horizontal dotted line, corresponding to an infinitely elastic demand curve, and the dotted vertical line, corresponding to an infinitely inelastic one. But in fact we can say more. If demand is assumed to be nondecreasing over time, a curve like $D$ is ruled out because that would put the later equilibrium point $e'_2$ on a lower demand curve. The demand curve through $e_1$ must be at least as steep as the line connecting $e_1$ and $e'_2$—the line
labeled $D'$—to preserve nondecreasing demand. The comparison of the two equilibrium points leads to a lower bound on the steepness of inverse demand and an upper bound on the elasticity of demand through $e_1$.

![Figure 1: Intuition for the Use of Time-Series Data to Bound Demand Elasticity](image)

The tightness of the bound is data-driven. Suppose that the observed equilibrium point is $e''_2$ in period 2 rather than $e'_2$. This leads to a tighter bound on the elasticity. The inverse demand through $e_1$ must be at least as steep as $D''$ to keep $e''_2$ from lying on a lower demand curve, violating nondecreasing demand. The funnel in which the demand curve must lie narrows from the entire shaded region to just the dark-shaded part, only leaving room for a relatively inelastic demand. If we find inverse demand this steep—and demand this inelastic—in applications, we may be able to rule out monopoly. Monopoly may be inconsistent with a steep drop from $e_1$ to a point like $e''_2$. A monopolist would never drop price for such a small increase in quantity, even if marginal costs had dropped from something positive to zero. The more likely conclusion is that it was competitive pressure that forced firms out into the inelastic part of the demand curve.

In the example in Figure 1, we have assumed that demand is linear. We develop variants of the methodology that can narrow the bounds on the demand elasticity considerably if the researcher is willing to assume the whole demand curve is linear, from the vertical to the horizontal intercept. The variants incorporate information from the intercepts of other year’s demands to iteratively
narrow the bounds on the elasticity in a given year. We redo the analysis for another widely assumed functional form, logit demand. We also generalize the method to work for any functional form the researcher chooses.

Applying our methodology to the polyethylene market after the Du Pont decision, our results in many years rule out elastic demand, and in some years only allow for extremely inelastic demand. For example, we find that absolute value of the elasticity of demand, $\epsilon_t$, for low-density polyethylene is bounded in $[0, 0.09]$ in 1959 and $[0, 0.08]$ in 1960. These findings are robust, the same whether we assume linear or logit demand or whether we compare those years to all other years or more local comparisons to the two neighboring years. Those are the tightest bounds we find but find that $\epsilon_t$ is bounded well away from unit elastic in all but one year, again holding for a variety of functional forms and variants. The method incorporating intercept information with the linear demand assumption powerfully narrows down the elasticity bounds, but they are so much narrower than the logit bounds that we question whether the use of nonlocal information produces trustworthy results. Perhaps projecting a given functional form too far outside of the range of the observations strains the assumption.

Our work is related to several literatures. One is the literature assessing the effectiveness of structural remedies for antitrust. We already mentioned Adams’ (1951) pioneering study. Studies by the U.S Federal Trade Commission (FTC) staff make important contributions to the literature as well. These studies include U.S. Federal Trade Commission (1999), examining the efficacy of divestiture remedies accompanying mergers approved in 1990 to 1994; Farrell, Pautler, and Vita (2009), describing a similar FTC exercise, but in this case examining ex post outcomes of hospital mergers that had been allowed by the judiciary; and U.S. Federal Trade Commission (2017), studying all 89 FTC merger orders from 2006–12. This last study judged remedies to be a success in 69 percent of cases and a qualified success in 14 percent of cases. Asker (2014) articulates the importance of remedy in merger cases.

We also contribute to the literature on patent licensing. This is a vast literature; most germane are papers on patent pools and also interesting work using historical evidence to shed light on contemporary issues in IP.

Jean Tirole in his Nobel lecture besides surveying his theoretical contributions on intellectual property and patent pools also points out this industry and legal development. “A little known fact
is that, prior to 1945, most high-tech industries of the time were run by patent pools. But the worry about cartelization through joint marketing led to a hostile decision of the US Supreme Court in 1945 and the disappearance of pools until the recent revival of interest.\textsuperscript{1} Of course the chemical industry is a high tech industry of the recent past.

Lerner, Strojwas and Tirole (2007) indicate patent pools are understudied empirically. We agree. Their paper models the licensing or contractual terms that pools would offer if they had complementary patents which solves double marginalization or even worse, or a pool that involves substitute patents, which is much more likely to be anticompetitive. As they point out good pools and bad pools might be indistinguishable to a court or policymaker. The authors model and empirics focuses on whether the pool allows independent licensing of patents or also grantback requirements both of which are more likely to be complementary patents. Their results are consistent with their theoretical predictions. Their sample may include Dupont ICI but their work is complementary to ours since we focus more on remedy than the contractual terms of the pool.

In a recent paper, Watzinger, Fackler, Nagler and Schnitzer (2017) use the 1956 Bell System consent decree to examine induced innovation. Like Du pont, Bell was a monopoly albeit a regulated one. The Truman administration filed a monopolization suit against Bell in 1949, which argued that it had foreclosed entry into various related markets such as telecom equipment. In the consent decree Bell was allowed to remain vertically integrated into telecom but they had to freely license all existing licenses royalty free.

The authors find that Follow-on innovation expands as a consequence of the consent decree. They find no expansion in innovation in the telecom sector, but they do find enhanced innovation in non telecom sectors. This is a very interesting paper with great data work.

Stocking and Watkins (1946) classify the Dupont ICI patents and processes agreement as a patent pool and describe some interesting details.\textsuperscript{2} MOVE TO KAPLOW PRICE DISCRIM THING “...du Pont licenses, or agrees upon request to license, ICI to use all its present and future patents, and secret “know how” for the manufacture and sale within the British Empire, of for example, dyestuffs and explosives, and ICI licenses du Pont similarly for the United States. This


\textsuperscript{2}Stocking and Watkins, 1946, p. 427-428.
is a cartel agreement in fact, whatever it may be its legal status. Each of the parties has the right to use both its own and its licensee’s patents and processes only in its allotted territory.”

Reader devotes several chapters of volume II to a critical evaluation of the U.S. government’s case against ICI and dupont. Although we disagree with Reader’s evaluation, his two volume history of ICI remains a classic scholarly work. He argues that the prosecution arose out of political pressure at the Department of Justice. (p. 417). He criticizes the extra territorial reach of US antitrust law (p.431, p. 440); of Americans “trying to convert the rest of the world to the gospel of anti-trust” (p. 423). He indicates that ICI executives thought any prosecution in the US would stop with Dupont and not reach ICI. (p.419). But of course if the dupont ICI agreement were collusive in intent and effect, bringing a case against ICI does not seem a dramatic stretch of the law.

Hounshell and Smith, criticize the courts verdict for a different reason. “Judge Ryan . . . refused to accept the argument that ICI’s and Du Pont’s agreement resulted in the genuine exchange of scientific and technical information. But the overwhelming historical evidence demonstrates this was indeed the case.”¹³ We find the agreements division of markets geographically to be strong evidence in favor of a collusive intent.

Lampe and Moser (2013) study the first US patent pool the sewing machine pool formed in 1856. They find the creation of the pool is associated with a reduction in innovation by the substitute technology to the pool technology. And welfare consequences as they note depend on the source of this effect. Historical evidence suggests the pool affected innovation through both litigation risks but also differential licence fees for members and non members.

Does intellectual property concerns always trump antitrust, or does antitrust always trumps intellectual property concerns? Louis Kaplow in his (1984) reappraisal of the antitrust patent intersection indicated it depends. Our paper focuses on how the freeing of licenses affected product market competition but we take the patent system as given.

We also contribute to the empirical industrial organization literature developing bounds rather than point estimates on elasticities and other parameters [CITE].

The paper is structured as follows. The first part of the paper covers methodology. Section 2 models the situation to which the methodology will be applied. Section 3 presents a method for bounding demand elasticities incorporating local information from the pairwise comparison

¹³Hounshell and Smith, p. 206.
of equilibria. A series of subsections treats different demand functions, from linear, to logit, to general forms. Section 4 shows how the bounds can be tightened by iteratively incorporating limiting information. Again, a series of subsections treats different demand functions, from linear, to logit, to general forms. The remainder of the paper is devoted to the empirical application to the *Du Pont* decision. Section 5 provides institutional background, Section 6 describes the data, and Section 7 presents our results. Section 8 concludes. Appendix A provides proofs not included in the text, and Appendix B provides supplementary figures.

2. Model

This section lays out a general model of a growing market for a homogeneous product. Each period $t$, the interaction between producers and consumers on the market leads to an equilibrium $e_t \equiv (q_t, p_t)$, where $q_t \geq 0$ is market quantity and $p_t \geq 0$ is market price. The researcher observes the market equilibrium over some time span $t \in \{1, \ldots, T\}$. Let $E \equiv \{e_t | t = 1, \ldots, T\}$ denote the set of time-series observations of equilibrium.

The analysis can be streamlined without much loss of generality if ties between price observations or quantity observations are ruled out, accomplished by assuming $E$ is *distinct* according to the following definition.

**Definition.** $E$ is distinct if and only if, for all $e_t, e_{t'} \in E$ such that $t \neq t'$, we have $q_t \neq q_{t'}$ and $p_t \neq p_{t'}$.

Assuming $E$ is distinct entails little loss of generality if one supposes that two observations of, say, quantity over time are never exactly equal if measurements are carried out to arbitrary precision. We also assume that the market is non-trivial in the sense of involving positive prices and quantities in each period.

**Definition.** $E$ is non-trivial if and only if, for all $e_t \in E$, we have $q_t, p_t > 0$.

We will characterize each side of the market in turn starting with producers. Since the goal of our application will be to determine whether the antitrust remedy was effective in changing producer conduct, it is natural to consider producer conduct as an unknown to be determined. Thus, we will be fairly agnostic about producer behavior in the model. Producers may engage in
perfect competition, in which case it may be possible to characterize their behavior with a supply curve; or they may engage in some form of imperfect competition, perhaps monopoly, whose behavior is characterized by a supply relation derived from a first-order condition.

Next, consider the consumer side of the market. Consumers are price takers whose behavior is captured by the demand curve \( q = D_t(p) \), the functional form of which is posited by the researcher. Assume the functional form obeys the law of demand, i.e., that the demand curve at a given point in time, \( D_t(p) \), is nonincreasing in \( p \).

**Assumption 1 (Law of Demand).** For all \( t \in \{1, \ldots, T\} \), \( D_t(p') \geq D_t(p'') \) for all \( p'' > p' \geq 0 \).

The assumption of a growing market entails growing demand, formally that \( D_t(p) \) is nondecreasing in \( t \).

**Assumption 2 (Growing Demand).** For all \( t', t'' \in \{1, \ldots, T\} \) such that \( t' < t'' \), \( D_{t'}(p) \leq D_{t''}(p) \) for all \( p \geq 0 \).

Not all configurations of equilibria \( E \) are consistent with growing demand. To aid the discussion of which inconsistent configurations are ruled out, it is helpful to introduce notation for subsets determined by compass directions relative to a reference equilibrium point, \( e_t \):

\[
\begin{align*}
NW(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} < q_t, p_{t'} > p_t\} \\
NE(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} > q_t, p_{t'} > p_t\} \\
SE(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} > q_t, p_{t'} < p_t\} \\
SW(e_t) &\equiv \{(q_{t'}, p_{t'}) \in E \mid q_{t'} < q_t, p_{t'} < p_t\}.
\end{align*}
\]

Each equilibrium point divides the nonnegative quadrant into four regions corresponding to four relative compass directions; our notation provides a label for these subregions. Figure 2 depicts these compass sets. As the figure indicates, the axes are taken to be oriented in the usual way when graphing inverse demand, i.e., with quantity on the horizontal axis and price on the vertical axis. The fact that the definitions in (1) involve strict inequalities leaves points on the dotted lines through \( e_t \) in the figure unclassified, but this is without loss of generality for distinct \( E \), which precludes equilibrium points from sharing coordinates.
Building on the definitions in equation (1), we define subsets of the compass sets depending on the time at which the equilibria appear. For example, define

\[
NW^-(e_t) \equiv \{e_t' \in NW(e_t) \mid t' < t\}
\]

\[
NW^+(e_t) \equiv \{e_t' \in NW(e_t) \mid t' > t\}.
\]

In words, \(NW^-(e_t)\) is the subset of equilibrium points in \(NW(e_t)\) that occur before \(e_t\) and \(NW^+(e_t)\) the subset that occur after \(e_t\). Four compass directions and two relative times entail eight possible combinations and thus eight possible subsets. The other six subsets—\(NE^-(e_t)\), \(NE^+(e_t)\), \(SE^-(e_t)\), \(SE^+(e_t)\), \(SW^-(e_t)\), and \(SW^+(e_t)\)—are defined by analogy to equation (2). We will call these time-augmented compass sets.

Consider positioning a new equilibrium point \(e_t'\) that occurs after \(e_t\) in one of the compass sets in the figure. A downward-sloping inverse demand curve through \(e_t\) (such as the solid curve drawn in the figure) slices through regions \(NW(e_t)\) and \(SE(e_t)\). This leaves room to add the new equilibrium point so that it lies on a higher demand curve—thus respecting Assumption 2 that demand is growing—yet still falls in \(NW^+(e_t)\) or \(SE^+(e_t)\). The new equilibrium could also easily be added in \(NE^+(e_t)\) as well, placed on a higher inverse demand in that region. However, there is no way to add the new equilibrium point to \(SW^+(e_t)\) without having it lie on a lower demand curve. Thus, in a market with growing demand, \(SW^+(e_t)\) must be empty for all \(e_t \in E\).
When $E$ has the property that $SW^+(e_t)$ is empty for all $e_t \in E$, we will say that $E$ exhibits \textit{rectangular expansion} because equilibrium points occurring after the reference one $e_t$ lie outside the rectangle determined by $e_t$ and the origin at opposite corners.

\textbf{Definition.} $E$ exhibits \textit{rectangular expansion} if and only if $SW^+(e_t)$ is empty for all $e_t \in E$.

With this definition in hand, we can encapsulate the analysis from the previous paragraph leads with the following proposition.

\textbf{Proposition 1.} Assumption 2 that demand is growing implies that $E$ exhibits rectangular expansion.

Switching perspectives from demand growing as we advance into the future to demand shrinking as we delve into the past, the fact that $SW^+(e_t)$ is empty for all $e_t \in E$ is equivalent to $NE^-(e_t)$ being empty for all $e_t \in E$. Hence an equivalent condition for $E$ exhibiting rectangular expansion is that $NE^-(e_t)$ is empty for all $e_t \in E$. Yet another equivalent condition for $E$ exhibiting rectangular expansion is that for all $e_t',e_{t''} \in E$ such that $t' < t''$, either $q_{t'} \leq q_{t''}$ or $p_{t'} \leq p_{t''}$. That is, $E$ exhibits rectangular expansion if price, quantity, or both grows each period.

\textbf{3. Method Incorporating Local Information}

We begin with our simplest method for bounding demand elasticities. We will refer to this method as \textit{incorporating local information} because it uses the location of other equilibrium points $e_{t'}$ to constrain the position of the demand curve through a given equilibrium point $e_t$. This stands in contrast with a method introduced later that uses limits of demand curves through other equilibrium points to constrain the position of the demand curve through $e_t$. We will refer to this later method as \textit{incorporating limiting information}.

Whether incorporating local or limiting information, all of our methods require the researcher to impose a functional-form assumption on demand. The first subsection analyzes the simple and widely used assumption of linear demand. The next subsection analyzes logit demand. The last subsection extends the methods to general functional forms for demand.
3.1. Linear Demand

Suppose the research imposes the assumption that the sequence of demand curves over time is linear: \( D_t(p) = a_t - b_t p \) for \( t = 1, \ldots, T \). Note that the parameters \( a_t \) and \( b_t \) are subscripted by \( t \) and thus allowed to vary over time, in turn allowing the demand curve to shift over time.

This specification of demand involves two independent parameters, but further requiring the line to pass through the equilibrium point \( e_t \) pins it down to a single-parameter family. Focus for now on the demand slope, \( b_t \), as this key parameter. Letting \( \tilde{D}(p, b_t, e_t) \) be the linear demand with slope \( b_t \) through equilibrium point \( e_t = (p_t, q_t) \), we have

\[
D_t(p) = \tilde{D}(p, b_t, e_t) = q_t + b_t(p_t - p). \tag{3}
\]

For the law of demand, Assumption 1, to hold, \( b_t \geq 0 \).

With knowledge of the slope \( b_t \) and equilibrium point \( e_t \), one can solve for the demand intercept,

\[
a_t = q_t + b_t p_t, \tag{4}
\]

or for the absolute value of the demand elasticity,

\[
\epsilon_t \equiv -\tilde{D}_p(p_t, b_t, e_t) \frac{p_t}{q_t} = \frac{b_t p_t}{q_t}, \tag{5}
\]

where \( \tilde{D}_p \) denotes the partial derivative of the demand function with respect to its first argument. Note that \( b_t \geq 0 \) implies \( \epsilon_t \geq 0 \). For brevity, we will drop the “absolute value” modifier and simply call \( \epsilon_t \) the demand elasticity.

We will derive bounds on demand slope \( b_t \) via pairwise comparisons of that period’s equilibrium point \( e_t \) with the location of the other equilibrium points. Using equation (5), bounds on \( b_t \) can then be translated into the desired bounds on \( \epsilon_t \).

Before proceeding with detailed formal analysis, we provide some intuition on how the method works using Figure 3. Each panel compares a reference equilibrium point \( e_t \) to another equilibrium point \( e_{t'} \). Without further information, all we know is that the linear inverse demand through \( e_t \) must be somewhere between the perfectly horizontal and perfectly vertical ones delineated by the dotted lines. What further information the comparison to \( e_{t'} \) can provide depends on which of
the eight time-augmented compass subsets $e_{t'}$ falls into relative to $e_t$. By Proposition 1, two of these sets are empty when demand is growing, leaving the six non-empty sets into which $e_{t'}$ can fall, corresponding to the panels in the figure: $SE^+(e_t)$, $NW^+(e_t)$, $NE^+(e_t)$, $NW^-(e_t)$, $SE^-(e_t)$, and $SW^-(e_t)$.

In Panel A, $e_{t'} \in SE^+(e_t)$. Drawing a line connecting $e_t$ and $e_{t'}$, we know that the inverse demand through $e_t$ cannot be less steep than that; otherwise $e_{t'}$ would be forced to lie on a lower demand curve, violating Assumption 2 that demand is growing over time since $t' > t$ as $e_{t'} \in SW^+(e_t)$. The line connecting $e_t$ and $e_{t'}$ thus provides a lower bound on the steepness of the inverse demand through $e_t$, allowing us to narrow the location of the inverse demand curve from somewhere between the dotted lines down into the shaded funnel. Recalling that $b_t$ measures the steepness of the direct (i.e., non-inverse) demand, the lower bound on the steepness of the inverse demand translates into an upper bound on $b_{t'}$.

In Panel B, $e_{t'} \in NW^+(e_t)$. With this compass orientation for $e_{t'}$, the line connecting $e_t$ and $e_{t'}$ provides an upper bound on the steepness of the inverse demand through $e_t$; any steeper and $e_{t'}$ would be forced to lie on a lower demand curve, violating Assumption 2. The upper bound on steepness of inverse demand translates into a lower bound on $b_t$. In Panel C, $e_{t'} \in NE^+(e_t)$. In this case, regardless of which inverse demand between the dotted lines we draw through $e_t$, it is always possible to place $e_{t'}$ on a higher demand curve. So in this case the pairwise comparison provides no further information to narrow down the location of the inverse demand through $e_t$.

Panels C–F illustrate cases in which $e_{t'}$ occurs before $e_t$, so $e_{t'}$ falls into the time-augmented compass sets with minus rather than plus superscripts. By examining each panel, the reader can verify that the directions of the bounds are the same as in Panels A–C except for the opposite compass direction. That is, $e_{t'} \in NW^-(e_t)$ generates an upper bound on $b_t$, $e_{t'} \in SE^-(e_t)$ generates a lower bound, and $e_{t'} \in SW^-(e_t)$ provides no information.

To proceed with the formal analysis, first suppose $t' > t$. Then

$$
\tilde{D}(p, b_t, e_t) = D_t(p) \leq D_{t'}(p) = \tilde{D}(p, b_{t'}, e_{t'}),
$$

where the equalities follow from (3) and the inequality from Assumption 2 of growing demand. Since condition (6) holds for all $p \geq 0$, it must hold in particular for $p_{t'}$. Substituting $p_{t'}$ into (6)
Figure 3: Incorporating Local Information from Pairwise Comparisons to Derive Upper and Lower Bounds
yields
\[ q_t + b_t(p_t - p_t') = \bar{D}(p_t', b_t, e_t) \leq \bar{D}(p_t', b_t', e_t') = q_t', \]
(7)
or rearranging,
\[ b_t(p_t - p_t') \leq q_t' - q_t. \]
(8)

The bounds on \( b_t \) that can be derived from equation (8) depend on the compass position of \( e_t' \) relative to \( e_t \). There are three subcases to consider depending on whether \( e_t' \) is in \( NW^+(e_t) \), \( NE^+(e_t) \), or \( SE^+(e_t) \); the subcase in which \( e_t' \in SW^+(e_t) \) can be ignored when demand is growing since \( SW^+(e_t) \) is empty by Proposition 1. To streamline the analysis, let \( B(e_t, e'_t) \) denote the absolute value of the slope of the linear demand through points \( e_t \) and \( e'_t \):
\[ B(e_t, e'_t) = \left| \frac{q_t' - q_t}{p_t' - p_t} \right|. \]
(9)

Suppose \( e_t' \in NW^+(e_t) \). Cross multiplying (8) by \( p_t - p_t' \) yields
\[ b_t \geq \frac{q_t' - q_t}{p_t - p_t'} = B(e_t, e_t'). \]
(10)
The first step follows from \( e_t' \in NW^+(e_t) \), which implies \( p_t < p_t' \) by (1). The second step follows because the numerator and denominator of the middle fraction are both negative for \( e_t' \in NW^+(e_t) \). Condition (10) provides a lower bound on the demand slope.

Next, suppose \( e_t' \in NE^+(e_t) \). Cross multiplying (8) by \( p_t - p_t' \) yields
\[ b_t \geq \frac{q_t' - q_t}{p_t - p_t'} = B(e_t, e_t'). \]
(11)
The right-hand is negative since the numerator is positive and the denominator is negative for \( e_t' \in NE^+(e_t) \). Hence, (11) is a weaker condition than the maintained assumption \( b_t \geq 0 \). Thus this case contributes no useful information to bound \( b_t \).

Next, suppose \( e_t' \in SE^+(e_t) \). Cross multiplying (8) by \( p_t - p_t' \) yields
\[ b_t \leq \frac{q_t' - q_t}{p_t - p_t'} = B(e_t, e_t'). \]
(12)
Cross multiplying did not change the direction of the inequality because \( p_t > p_{t'} \) for \( e_{t'} \in SE^+(e_t) \). The second step follows because both numerator and denominator of the middle fraction are positive for \( e_{t'} \in SE^+(e_t) \). Condition (12) provides an upper bound on the demand slope.

The pairwise comparison of \( e_t \) and \( e_{t'} \) can be repeated for \( t' < t \). Then growing demand implies

\[
q_t + b_t(p_t - p_{t'}) = \hat{D}(p_{t'}, b_t, e_t) \geq \hat{D}(p_{t'}, b_{t'}, e_{t'}) = q_{t'},
\]

or rearranging,

\[
b_t(p_t - p_{t'}) \geq q_{t'} - q_t,
\]

the reverse of (8) There are three subcases to consider, now depending on whether \( e_{t'} \) is in \( NW^-(e_t) \), \( SE^-(e_t) \), or \( SW^-(e_t) \); the subcase in which \( e_{t'} \in NE^-(e_t) \) can be ignored when demand is growing since \( NE^-(e_t) \) is empty by Proposition 1. Sparing the details, using analysis similar to that above, one can show \( b_t \leq S(e_t, e_{t'}) \) when \( e_{t'} \in NW^-(e_t) \) and \( b_t \geq S(e_t, e_{t'}) \) when \( e_{t'} \in SE^-(e_t) \). When \( e_{t'} \in SW^-(e_t) \), (14) implies \( b_t \geq (q_{t'} - q_t)/(p_t - p_{t'}) \). But the right-hand side is negative, so this is a weaker condition than the maintained assumption \( b_t \geq 0 \). Thus the case \( e_{t'} \in SW^-(e_t) \) contributions no useful information to bound \( b_t \).

In sum, pairwise comparison of a reference equilibrium \( e_t \) to the location of another equilibrium point \( e_{t'} \) can be characterized according to which of the eight time-augmented compass subsets that \( e_{t'} \) falls into. The subsets \( NW^+(e_t) \) and \( SE^-(e_t) \) provide lower bounds on \( b_t \); \( b_t \) must be at least as large as the largest of these. The subsets \( SE^+(e_t) \) and \( NW^-(e_t) \) provide upper bounds on \( b_t \); \( b_t \) must be at least as small as the smallest of these. The remaining four subsets are either empty or provide no useful information to bound \( b_t \). We have proved the following proposition.

**Proposition 2.** Suppose the sequence of linear demands \( \hat{D}(p, b_t, e_t) \) defined in (3) satisfies Assumptions 1 and 2. Then \( b_t \in [b_l, b_u] \) for all \( t \in \{1, \ldots, T\} \), where

\[
b_l \equiv 0 \lor \sup_{e_{t'} \in SE^-(e_t) \cup NW^+(e_t)} B(e_t, e_{t'}),
\]

\[
b_u \equiv \inf_{e_{t'} \in NW^-(e_t) \cup SE^+(e_t)} B(e_t, e_{t'}).
\]

A few technical notes about Proposition 2 are in order. The \( \lor \) operator denotes the join; i.e., \( x \lor y \) denotes the maximum of \( x \) and \( y \). The use of this operator in (15) indicates the imposition of a floor of 0 on top of the supremum. We will use this operator repeatedly later as well as \( \land \).
for the meet; i.e., \(x \wedge y\) denotes the minimum of \(x\) and \(y\). We take the supremum in (15) rather than the maximum and take the infimum in (16) rather than the minimum even though the sets involved are discrete to accommodate the possibility that one of these sets is empty. An empty set does not have a maximum or minimum but does have a supremum and infimum; we use the conventional definitions \(\inf \emptyset = \infty\) and \(\sup \emptyset = -\infty\). If the set \(SW^-(e_t) \cup NW^+(e_t)\) over which the supremum is taken in (15) is nonempty, then imposing a 0 floor on the supremum is superfluous because \(B(e_t, e_{t'})\) is defined to be an absolute value. However, if the set over which the supremum is taken happens to be empty, then \(\bar{b}_t = -\infty\) without the imposition of the 0 floor.

Proposition 2 prescribes the following simple algorithm for bounding \(b_t\):\(^4\) To obtain the lower bound \(\bar{b}_t\), pair \(e_t\) with each of the equilibrium points in \(SW^-(e_t)\) and \(NW^+(e_t)\), draw linear demands connecting them, and identify which is steepest (note, we are here referring to the steepness of the direct, not inverse, demand). The steepness of the identified line gives \(b_t\). If both \(SW^-(e_t)\) and \(NW^+(e_t)\) are empty, we set \(b_t\) to respect the non-negativity constraint on \(b_t\); i.e., \(b_t = 0\). To obtain the upper bound \(\bar{b}_t\), pair \(e_t\) with each of the equilibrium points in \(NW^-(e_t)\) and \(SE^+(e_t)\), draw linear demands connecting them, and identify which is the least steep. The steepness of the identified line gives \(\bar{b}_t\). If both \(NW^-(e_t)\) and \(SE^+(e_t)\) are empty, the formula (16) yields \(\bar{b}_t = \inf \emptyset = \infty\), correctly implying we obtain no upper bound in this case. Using equation (5), the bounds on \(b_t\) can be translated into bounds on \(\epsilon_t\); namely, \(\epsilon_t \in [\underline{\epsilon}_t, \bar{\epsilon}_t]\), where \(\underline{\epsilon}_t \equiv b_t p_t/q_t\) and \(\bar{\epsilon}_t \equiv \bar{b}_t p_t/q_t\).

\(^4\) We have coded this and subsequent algorithms in Stata and provide the code to other researchers on request.
Figure 4 provides a simple illustration of how the algorithm works when there are several equilibrium points. Suppose we want to bound the elasticity $\epsilon_1$ of the demand curve through point $e_1$. Panel A shows the bound derived from pairwise comparison of $e_1$ with each of the other equilibrium points $e_2$ and $e_3$. The inverse demand through $e_1$ must lie below $e_2$ and $e_3$ to respect Assumption 2 of growing demand and thus must be steeper than both $\ell_{12}$ and $\ell_{13}$. The steeper of the two, $\ell_{13}$, provides the a lower bound on the steepness of the inverse demand curve through $e_1$, which translates into an upper bound $\bar{b}_1$ on the steepness demand given in Proposition 2 and an upper bound $\bar{\epsilon}_1$ on the demand elasticity. Because $SE^-(e_1)$ and $NW^+(e_1)$ are empty, pairwise comparisons do not yield a lower bound $\underline{\epsilon}_1$ on the elasticity in this example (apart from 0 derived from non-negativity).

### 3.2. Logit Demand

Instead of imposing linear demand, the researcher may choose to impose another popular functional form, logit, microfounded by McFadden (1973), widely used in structural estimation of differentiated-product demand since Berry, Levinsohn, Pakes (1995). In the context of a homogeneous product market under study, logit demand can be specified as

$$D_t(p) = \frac{n_t \exp(-\alpha_t p)}{1 + \exp(-\alpha_t p)} = \frac{n_t}{1 + \exp(\alpha_t p)},$$

where $n_t$ is interpreted as a market-size parameter and $\alpha_t$ as a price-sensitivity parameter. For demand to be nonnegative, $n_t \geq 0$; for Assumption 1 (law of demand) to hold, $\alpha_t \geq 0$. As we did in specifying linear demand, our specification of logit demand here subscripts parameters by $t$ to allow them to vary over time and for demand to shift over time.

This specification of demand involves two independent parameters, but further requiring the curve to pass through the equilibrium point $e_t$ pins it down to a single-parameter family. Focus for now on price sensitivity, $\alpha_t$, as this key parameter. Given $\alpha_t$ and equilibrium point $(q_t, p_t)$, equation (17) can be solved for the market-size parameter:

$$n_t = q_t [1 + \exp(\alpha_t p_t)].$$

(18)
Substituting for $n_t$ from equation (18) into (17) yields an expression for logit demand in terms of the single unknown parameter $\alpha_t$ and known equilibrium point $e_t = (q_t, p_t)$:

$$\tilde{D}(p, \alpha_t, e_t) = \frac{q_t[1 + \exp(\alpha_t p_t)]}{1 + \exp(\alpha_t p_t)}.$$  \hspace{1cm} (19)

The analysis will focus on bounding $\alpha_t$. Bounds on $\alpha_t$ can then be translated into bounds on $\epsilon_t$, the absolute value of the demand elasticity, via the formula

$$\epsilon_t = -\tilde{D}_p(p_t, \alpha_t, e_t) \frac{p_t}{q_t} = \frac{\alpha_t p_t}{1 + \exp(-\alpha_t p_t)}.$$  \hspace{1cm} (20)

Similar to the linear-demand case, in the logit-demand case here, we will derive bounds on $\alpha_t$ from the pairwise comparison of reference equilibrium point $e_t$ to the location of other equilibrium points $e_{t'}$. Suppose $t < t'$. To respect Assumption 2 of growing demand, we must have

$$\tilde{D}(p, \alpha_t, e_t) = D_t(p) \leq D_{t'}(p) = \tilde{D}(p, \alpha_{t'}, e_{t'}),$$  \hspace{1cm} (21)

for all $p \geq 0$. In particular, the inequality must hold for $p = p_{t'}$, implying

$$q_t[1 + \exp(\alpha_t p_t)] = \tilde{D}(p_{t'}, \alpha_t, e_t) \leq \tilde{D}(p_{t'}, \alpha_{t'}, e_{t'}) = q_{t'},$$  \hspace{1cm} (22)

or rearranging,

$$q_t[1 + \exp(\alpha_t p_t)] \leq q_{t'}[1 + \exp(\alpha_{t'} p_{t'})].$$  \hspace{1cm} (23)

Let $A(e_t, e_{t'})$ be the solution to the previous expression treated as an equality, i.e., the value of $\alpha \geq 0$ solving

$$q_t[1 + \exp(\alpha p_t)] = q_{t'}[1 + \exp(\alpha p_{t'})].$$  \hspace{1cm} (24)

One can show—as done in the proof of the next proposition—that if $e_{t'} \in NW^+(e_t)$, then $A(e_t, e_{t'})$ exists, is unique, and provides a lower bound on the $\alpha_t$ satisfying (23). If $e_{t'} \in SE^+(e_t)$, $A(e_t, e_{t'})$ exists, is unique, and provides an upper bound on the $\alpha_t$ satisfying (23). If $e_{t'} \in NE^+(e_t)$, then (24) has no solution over $\alpha \geq 0$ since (23) is satisfied as a strict inequality for all $\alpha_t \geq 0$. In this case, pairwise comparison of $e_t$ and $e_{t'}$ yields no bounding information. The case in which $e_{t'} \in SW^+(e_t)$
can be ignored because the set is empty under Assumption 2 of growing demand.

The pairwise comparison of $e_t$ and $e_{t'}$ can be repeated for $t' < t$. Growing demand then implies the same inequality as (21) with the direction reversed. Similar bounds can be obtained but with compass directions reversed. Details are provided in the proof of the following proposition.

**Proposition 3.** Suppose the sequence of logit demands $\bar{D}(p, \alpha_t, e_t)$ defined in (19) satisfies Assumptions 1 and 2. Then $\alpha_t \in [\bar{\alpha}_t, \tilde{\alpha}_t]$ for all $t \in \{1, \ldots, T\}$, where

\[
\alpha_t \equiv 0 \lor \sup_{e_{t'} \in SE^{-}(e_t) \cup NW^{+}(e_t)} A(e_t, e_{t'}) \tag{25}
\]

\[
\tilde{\alpha}_t \equiv \inf_{e_{t'} \in NW^{-}(e_t) \cup SE^{+}(e_t)} A(e_t, e_{t'}) \tag{26}
\]

Proposition 3 prescribes the following algorithm for bounding $\alpha_t$. To obtain the lower bound $\alpha_t$, pair $e_t$ with each of the equilibrium points in $SW^{-}(e_t)$ and $NW^{+}(e_t)$, compute $A(e_t, e_{t'})$ for each pair by solving the nonlinear equation (24). Equation (24) is well-behaved and can be rapidly solved by a grid search, Newton-Raphson, or other standard methods. The largest of these solutions $A(e_t, e_{t'})$ is taken as $\alpha_t$. If both $SW^{-}(e_t)$ and $NW^{+}(e_t)$ are empty, we set $\alpha_t$ to respect the non-negativity constraint on $\alpha_t$; i.e., $\alpha_t = 0$. To obtain the upper bound $\tilde{\alpha}_t$, pair $e_t$ with each of the equilibrium points in $NW^{-}(e_t)$ and $SE^{+}(e_t)$, compute $\alpha(e_t, e_{t'})$ for each pair by solving the nonlinear equation (24). The smallest of these solutions $A(e_t, e_{t'})$ becomes $\tilde{\alpha}_t$. If both $NW^{-}(e_t)$ and $SE^{+}(e_t)$ are empty, the formula (26) yields $\tilde{\alpha}_t = \inf \emptyset = \infty$, correctly implying we obtain no upper bound in this case. Using equation (20), the bounds on $\alpha_t$ can be translated into bounds on $\epsilon_t$; namely, $\epsilon_t \in [\bar{\epsilon}_t, \tilde{\epsilon}_t]$, where

\[
\epsilon_t \equiv \frac{\alpha_t p_t}{1 + \exp(-\alpha_t p_t)}, \quad \tilde{\epsilon}_t \equiv \frac{\tilde{\alpha}_t p_t}{1 + \exp(-\tilde{\alpha}_t p_t)}. \tag{27}
\]

### 3.3. General Demand

The method for incorporating local information from pairwise equilibrium comparisons is readily generalizable to a broad class of functional forms. Suppose the researcher specifies the form for demand $q = D_t(p)$. This demand curve may start out as a multiple-parameter family; but assume that once it is required to pass through $e_t$, this pins it down to a single-parameter family indexed by $\theta_t$. Thus we have

\[
D_t(p) = \bar{D}(p, \theta_t, e_t). \tag{28}
\]
Assume $\tilde{D}(p, \theta, e_t)$ is continuously differentiable of all orders in all arguments. Assume increases in $\theta \in [0, \infty)$ cause demand to rotate. As an accounting convention, we will assume that the rotation is such that demand becomes steeper (and inverse demand less steep) when $\theta$ increases, consistent with the effect of the price-sensitivity parameter $\alpha$ in the logit case. Thus

$$
\tilde{D}_\theta(p, \theta, e_t) < 0 \text{ if } p > p_t \\
\tilde{D}_\theta(p, \theta, e_t) = 0 \text{ if } p = p_t \\
\tilde{D}_\theta(p, \theta, e_t) > 0 \text{ if } p < p_t,
$$

where $\tilde{D}_\theta$ denotes the partial derivative of demand with respect to its second argument. We also impose the following Inada conditions:

$$
\lim_{\theta \to 0} \tilde{D}(p, \theta, e_t) = q_t \tag{30}
$$

$$
\lim_{\theta \to \infty} \tilde{D}(p, \theta, e_t) = \begin{cases} 
0 & p > p_t \\
\infty & p < p_t.
\end{cases} \tag{31}
$$

Equation (30) implies that $\tilde{D}(p, \theta, e_t)$ becomes infinity inelastic for arbitrarily small $\theta$. Given $\tilde{D}(p, \theta, e_t)$ must pass through $e_t$, as it becomes infinitely inelastic, its inverse approaches a vertical line at quantity $q_t$ for all $p \geq 0$. Equation (31) implies that $\tilde{D}(p, \theta, e_t)$ becomes infinitely elastic as $\theta$ becomes arbitrarily large. Together, (30) and (31) in effect say that the domain of $\theta_t$ is rich enough to allow changes in $\theta$ to trace out all possible demand slopes from infinitely elastic to infinitely inelastic. We verify in the appendix that conditions (29)–(31) hold when $\tilde{D}(p, \theta, e_t)$ is taken to be the logit demand defined in (19) with $\alpha$ playing the role of $\theta$. Hence the general specification nests logit demand as a special case.

As before, we will derive bounds on $\theta_t$ from the pairwise comparison of reference equilibrium point $e_t$ to other equilibrium points $e_{t'}$. Suppose $t < t'$. To respect Assumption 2 of growing demand, we must have

$$
\tilde{D}(p, \theta_t, e_t) = D_t(p) \leq D_{t'}(p) = \tilde{D}(p, \theta_{t'}, e_{t'}), \tag{32}
$$
for all $p \geq 0$. In particular, the inequality must hold for $p = p_t'$, implying

$$\tilde{D}(p_t', \theta_t, e_t) \leq \tilde{D}(p_t', \theta_t', e_t') = q_{t'}.$$ (33)

Let $\Theta(e_t, e_t')$ be the solution to the preceding expression treated as an equality, i.e., the value of $\theta \geq 0$ solving

$$\tilde{D}(p_t', \theta, e_t) = q_{t'}.$$ (34)

Sparing the details filled in by the proof of the next proposition, $\Theta(e_t, e_t')$ is the key component of the bounds on $\theta_t$ for general demands, serving the same role as $A(e_t, e_t')$ for logit demands.

**Proposition 4.** Suppose the sequence of general demands $\tilde{D}(p, \theta_t, e_t)$ defined in (28) satisfies Assumptions 1 and 2. Then $\theta_t \in [\theta_t, \bar{\theta}_t]$ for all $t \in \{1, \ldots, T\}$, where

$$\bar{\theta}_t \equiv 0 \vee \sup_{e_t' \in SE^{-}(e_t) \cup NW^{+}(e_t)} \Theta(e_t, e_t'),$$ (35)

$$\bar{\theta}_t \equiv \inf_{e_t' \in NW^{-}(e_t) \cup SE^{+}(e_t)} \Theta(e_t, e_t').$$ (36)

The proof is provided in the appendix.

Proposition 4 prescribes an algorithm for bounding $\theta_t$ in the case of general demands. This algorithm is analogous to that prescribed by Proposition 3 for bounding $\alpha_t$ in the case of logit demand. The only difference is that instead of solving equation (24) for $A(e_t, e_t')$, the more general equation (34) is solved for $\Theta(e_t, e_t')$. The algorithms are otherwise the same, including that they involve the same time-augmented compass sets for the upper and lower bounds.

Equation (34) is a nonlinear equation in the single unknown variable $\theta$. This equation is well-behaved for $e_t'$ in the relevant subsets contributing to the bounds in Proposition 4: the proof of the proposition shows that for $e_t' \in NW(e_t) \cup SE(e_t)$, the left-hand side of (34) is monotonic in $\theta$. Thus standard methods, including a straightforward grid search, can be used to solve (34) and derive $\Theta(e_t, e_t')$.

Translating the bounds on $\theta_t$ into bounds on the elasticity, defined in the general case as

$$\epsilon_t = -\tilde{D}_p(p_t, \theta_t, e_t)\frac{p_t}{q_t},$$ (37)

requires additional work in the general case because it needs to be shown that condition (29)
implies that $\epsilon_t$ is nondecreasing in $\theta_t$. This is done in the proof of the following proposition, proved in the appendix.

**Proposition 5.** Suppose conditions (29)–(31) hold. Then, defining

$$\epsilon_t \equiv -\tilde{D}_p(p_t, \theta_t, e_t) \frac{p_t}{q_t}, \quad \bar{\epsilon}_t \equiv -\tilde{D}_p(p_t, \bar{\theta}_t, e_t) \frac{p_t}{q_t},$$

we have $\epsilon_t \in [\epsilon_t, \bar{\epsilon}_t]$.

**4. Method Incorporating Limiting Information**

If the researcher is willing to assume that the posited functional form applies throughout the whole domain of the demand curve, the elasticity bounds can be tightened by exploiting information about the relative position of demand curves for limiting values of prices, $p \to 0$ and $p \to \infty$, rather than observed prices. The analysis proceeds as in the previous subsection by analyzing a sequence of functional forms for demand, from linear, to logit, to general.

Before delving into the technical details, we provide some intuition on how limiting information can help tighten the elasticity bounds using Figure 5. The figure revisits the simple example introduced in Figure 4, carrying over the features from that previous figure as the solid lines. Recall the goal of that simple example was to bound the elasticity $\epsilon_1$ of demand through equilibrium point $e_1$. The method of incorporating local information from pairwise comparisons allowed us to conclude that the inverse demand through $e_1$ must be at least as steep as line $\ell_{13}$, providing a lower bound of the steepness of inverse demand, translating into an upper bound $\bar{\epsilon}_1$ on the steepness of demand and an upper bound $\bar{\epsilon}_1$.

The new features of Figure 5, drawn as dotted lines and open circles, show how to tighten the bound by incorporating information on the position of the limits of demand curves through other equilibrium points. The pairwise comparison of points $e_2$ and $e_3$ leads us to conclude that the inverse demand through $e_3$ must be as steep as line $\ell_{23}$ for $e_2$ not to lie on a higher demand curve. But notice that $\ell_{23}$ intersects $\ell_{13}$. Unless the inverse demand curve through $e_1$ is steeper than $\ell_{23}$, parts of the curve will lie above the curve through $e_3$, violating Assumption 2 that demand is growing. For the whole inverse demand through $e_1$ to be lower than the whole inverse demand through $e_3$, the demand curves cannot cross even for prices approaching 0, implying in the case
of linear inverse demands that their horizontal intercepts cannot cross. To ensure their horizontal intercepts do not cross, the inverse demand through $e_1$ must be at least as steep as the dotted line that connects $e_1$ with the horizontal intercept of $\ell_{23}$, drawn as the open circle. This new line through $e_1$ is even steeper than $\ell_{13}$, tightening the lower bound on the steepness of inverse demand, and thus tightening the upper bound on the demand elasticity.

This is just one simple example involving three equilibrium points and linear demands. The general method below will apply to an arbitrary number of equilibrium points and general demand specifications. The general method will preserve the flavor of the example in that the first step is to compute the bounds incorporating local information, and then the second step compares the limiting positions of the bounding demand curves from the first step.

To help clarify the iterative nature of the method, we introduce some new notation. Re-express the bounds incorporating local information presented in the previous section by adding a single star to them to indicate that these bounds constitute the first step of an iterative procedure, thus writing $b_t^* \equiv b_t$, $\alpha_t^* \equiv \alpha_t$, $\bar{\alpha}_t^* \equiv \bar{\alpha}_t$, $\epsilon_t^* \equiv \epsilon_t$, and $\bar{\epsilon}_t^* \equiv \bar{\epsilon}_t$.

### 4.1. Linear Demand

This section provides a formal derivation of the new bounds incorporating limiting information in the case of linear demands. We will analyze new bounds emerging from the limits $p \to 0$ and $p \to \infty$ in turn. Start by considering the limit $p \to 0$ and comparing the equilibrium points $e_t$ and...
\( e_t' \) for \( t' > t \). Assumption 2 of growing demand implies \( \tilde{D}(p, b_t, e_t) \leq \tilde{D}(p, b_{t'}, e_{t'}) \) for all \( p \geq 0 \), implying in particular that \( \tilde{D}(0, b_t, e_t) \leq \tilde{D}(0, b_{t'}, e_{t'}) \). Substituting from for \( \tilde{D} \) from (3) into this inequality, and further substituting \( p = 0 \), yields \( q_t + b_t p_t \leq q_{t'} + b_{t'} p_{t'} \). Notice this is a condition on the relative position of the inverse demand curves’ horizontal intercepts. Rearranging,

\[
b_t \leq \frac{1}{p_t} (q_{t'} - q_t + b_{t'} p_{t'}) \leq \frac{1}{p_t} (q_{t'} - q_t + \bar{b}_{t'}^* p_{t'}),
\]

where the second inequality follows from Proposition 2. Condition (39) provides an additional upper bound on \( b_t \) incorporating information on a different equilibrium point’s upper bound \( \bar{b}_{t'}^* \) from the prior iteration.

Supposing instead that \( t' < t \), we can apply the analysis from the previous paragraph, just reversing the inequalities, to obtain

\[
b_t \geq \frac{1}{p_t} (q_{t'} - q_t + b_{t'} p_{t'}) \geq \frac{1}{p_t} (q_{t'} - q_t + \bar{b}_{t'}^* p_{t'}).
\]

This additional lower bound incorporates information on a different equilibrium point’s lower bound \( \bar{b}_{t'}^* \) from a prior iteration.

Next, consider the limit \( p \to \infty \). Start by supposing \( t' > t \). The fact that \( \tilde{D}(p, b_t, e_t) \leq \tilde{D}(p, b_{t'}, e_{t'}) \) for all \( p \geq 0 \) implies \( \lim_{p \to \infty} \tilde{D}(p, b_t, e_t) \leq \lim_{p \to \infty} \tilde{D}(p, b_{t'}, e_{t'}) \). These limits are just the vertical intercepts of the respective inverse demands. Hence, the preceding inequality implies \( p_t + q_t / b_t \leq p_{t'} + q_{t'}/b_{t'} \), or after rearranging,

\[
b_t \left( p_{t'} - p_t + \frac{q_{t'}}{b_{t'}} \right) \geq q_t.
\]

One can show that the factor in parentheses is always positive when \( t' > t \) and demand is growing.\(^5\)

\(^5\)It is immediate that the factor in parentheses in equation (41) is positive if \( p_{t'} > p_t \) since \( q_{t'} > 0 \) and \( b_{t'} \geq 0 \). Suppose instead that \( p_{t'} < p_t \). Then \( e_{t'} \in SW^*(e_t) \cup SE^*(e_t) \). Proposition 1 rules out \( e_{t'} \in SW^*(e_t) \) under Assumption 2, leaving \( e_{t'} \in SE^*(e_t) \). Then

\[
b_{t'} \leq \bar{b}_{t'} \leq B(e_t, e_{t'}) = \frac{q_{t'} - q_t}{p_t - p_{t'}} < \frac{q_{t'}}{p_t - p_{t'}}.
\]

The first and second steps follow from Proposition 2. The next step follows from the definition of \( B(e_t, e_{t'}) \) from (9) and from \( q_{t'} > q_t \) and \( p_t > p_{t'} \) for \( e_{t'} \in SE^*(e_t) \). The last step follows from \( q_{t'} > 0 \) for non-trivial equilibrium set \( E \). Cross multiplying by \( p_t - p_{t'} \), which is positive, and rearranging proves that the term in parentheses is positive.
Cross multiplying by the positive factor in parentheses yields

$$b_t \geq \frac{q_t}{p_{t'} - p_t + q_{t'}/b_{t'}} \geq \frac{q_t}{p_{t'} - p_t + q_{t'}/b_{t'}^*}.$$  

(42)

Again, we have an additional lower bound incorporating information on a different equilibrium point’s lower bound $b_{t'}^*$ from the prior iteration.

Supposing instead that $t' < t$, we can apply the preceding analysis, just reversing inequalities, to obtain

$$b_t \left( p_{t'} - p_t + \frac{q_{t'}}{b_{t'}} \right) \leq q_t.$$  

(43)

An important difference here is that the factor in parentheses cannot be unambiguously signed when $t' < t$. When the factor in parentheses is non-positive, (43) holds for all $b_t$. In that case, the condition provides no useful bounding information. When the factor in parentheses is positive, cross multiplying by it preserves the direction of the inequality, yielding

$$b_t \leq \frac{q_t}{p_{t'} - p_t + q_{t'}/b_{t'}} \leq \frac{q_t}{p_{t'} - p_t + q_{t'}/b_{t'}^*}.$$  

(44)

To emphasize, condition (44) only provides a potentially useful upper bound if the denominator of the last fraction is positive, i.e.,

$$p_{t'} + \frac{p_{t'}}{b_{t'}^*} > p_t.$$  

(45)

If (45) does not hold, we must ignore (44) lest we conclude that $b_t$ is bounded above by the negative number on the right-hand side of (44), violating the law of demand, Assumption 1, which implies $b_t \geq 0$.

The bounding exercise that generated conditions (39)–(44) can be repeated comparing $e_t$ to all the other equilibrium points $e_{t'}$. The tightest of the resulting bounds incorporating limiting information is tighter than the corresponding bound incorporating local information, we take the former as the new bound, indicated with two stars in the superscript. Otherwise, the new bound is just set to the old bound. We have the following proposition.

**Proposition 6.** Suppose the sequence of linear demands $\tilde{D}(p, b_t, e_t)$ defined in (3) satisfies As-
sumptions 1 and 2. Then $b_t \in [b_t^{**}, \tilde{b}_t^{**}]$ for all $t \in \{1, \ldots, T\}$, where

$$b_t^{**} \equiv b_t^{*} \lor \sup_{t' < t} \left\{ \frac{1}{p_t} (q_{t'} - q_t + b_t^{*} p_{t'}) \right\} \lor \sup_{t' > t} \left\{ \frac{q_t}{p_{t'} - p_t + q_{t'} / b_{t'}^{*}} \right\}$$

(46)

$$\tilde{b}_t^{**} \equiv b_t^{*} \land \inf_{t' < t} \left\{ \frac{q_t}{p_{t'} - p_t + q_{t'} / b_{t'}^{*}} \right\} \land \inf_{t' > t} \left\{ \frac{1}{p_t} (q_{t'} - q_t + b_{t'}^{*} p_{t'}) \right\}.$$  

(47)

Some technical remarks about the proposition are in order. Note that, as do equations (15)–(16), (46)–(47) involve suprema and infima rather than maximums and minimums to accommodate possibly empty sets, as will be the case when computing bounds for the first and last equilibrium points, i.e., $e_1$ or $e_T$. Note also that it is conceivable that continuing the iterations could result in yet tighter bounds, for example computing $\tilde{b}_t^{**}$ by replacing $b_t^{*}$ with $\tilde{b}_t^{**}$ on the right-hand side of (47), and so on. We conjecture that convergence is achieved by the stage shown in (46) and (47), so there is no gain from further iterating.

The bounds $b_t^{**}$ and $\tilde{b}_t^{**}$ incorporating limiting information are at least as tight as $b_t^{*}$ and $\tilde{b}_t^{*}$. This is true by definition because the latter bounds appear on the right-hand side of (46) and (47). If the limiting information does not tighten the bounds, by default they return to the first stage that incorporates just local information. The formulas in (46) and (47) are sufficiently complicated that it is hard to tell whether $b_t^{**}$ and $\tilde{b}_t^{**}$ can ever be strictly tighter than $b_t^{*}$ and $\tilde{b}_t^{*}$. Figure 5, which we now have the language to say derives $\tilde{b}_t^{*}$ in the simple example with those three equilibrium points, suggests that $b_t^{**}$ and $\tilde{b}_t^{**}$ can be strictly tighter than $b_t^{*}$ and $\tilde{b}_t^{*}$ in some cases. We will indeed see that substantial tightening is possible when we turn to the empirical application to the Du Pont decision.

Substituting the new bounds into the elasticity formula (5), we have $\epsilon_t \in [\tilde{\epsilon}_t^{**}, \tilde{\epsilon}_t^{*}]$, where $\tilde{\epsilon}_t^{**} \equiv b_t^{**} p_t / q_t$ and $\tilde{\epsilon}_t^{*} \equiv \tilde{b}_t^{**} p_t / q_t$.

### 4.2. Logit Demand

The method for incorporating limiting information assuming logit demand is similar to that assuming linear demand. Start by considering the limit $p \to 0$ and comparing equilibrium points $e_t$.

---

6Our Stata code that implements the bounds in Proposition 6 allows continued iterations, but in the empirical application to the Du Pont decision and all Monte Carlos we have tried, iterations stop at $b_t^{**}$ and $\tilde{b}_t^{**}$. We have managed to prove in general that iterations stop at $b_t^{**}$ and $\tilde{b}_t^{**}$ for a proliferation of subcases, but a handful of subcases remain open questions.
and \( e_{t'} \) for \( t' > t \). Assumption 2 implies \( \bar{D}(p, \alpha_t, e_t) \leq \bar{D}(p, \alpha_{t'}, e_{t'}) \) for all \( p \geq 0 \). In particular, this inequality must hold for \( p = 0 \): i.e., \( \bar{D}(0, \alpha_t, e_t) \leq \bar{D}(0, \alpha_{t'}, e_{t'}) \). Substituting for \( \bar{D} \) from (3) in the previous inequality yields \( q_t[1 + \exp(\alpha_t p_t)] \leq q_{t'}[1 + \exp(\alpha_{t'} p_{t'})] \), or upon rearranging,

\[
\alpha_t \leq \frac{1}{p_t} \ln \left( \frac{q_{t'}}{q_t} [1 + \exp(\alpha_{t'} p_{t'})] - 1 \right) \leq \frac{1}{p_t} \ln \left( \frac{q_{t'}}{q_t} [1 + \exp(\alpha^*_t p_{t'})] - 1 \right). \tag{48}
\]

For \( t' < t \), we obtain the reverse inequality:

\[
\alpha_t \geq \frac{1}{p_t} \ln \left( \frac{q_{t'}}{q_t} [1 + \exp(\alpha_{t'} p_{t'})] - 1 \right) \geq \frac{1}{p_t} \ln \left( \frac{q_{t'}}{q_t} [1 + \exp(\alpha^*_t p_{t'})] - 1 \right). \tag{49}
\]

To explore the other limit, \( p \to \infty \), suppose \( t' > t \). Expressing its implication in terms of a ratio rather than a difference, Assumption 2 implies

\[
\lim_{p \to \infty} \frac{\bar{D}(p, \alpha_t, e_t)}{\bar{D}(p, \alpha_{t'}, e_{t'})} \leq 1. \tag{50}
\]

Evaluation of this limit is somewhat involved, so the details are deferred to the appendix proof. The result, however, is simple. We show that a necessary condition for (50) is \( \alpha_t \geq \alpha_{t'} \). In words, under logit demand, for demand to grow over time requires the price-sensitivity parameter to shrink over time. Using the result from Proposition 3 that \( \alpha_{t'} \geq \alpha^*_{t'} \), this leads to the lower bound \( \alpha_t \geq \alpha^*_{t'} \) for all \( t' > t \). For \( t' < t \), the reverse inequality is required: \( \alpha_t \leq \alpha_{t'} \leq \bar{\alpha}^*_{t'} \).

The following proposition summarizes the preceding analysis. The appendix proof fills in the omitted detail.

**Proposition 7.** Suppose the sequence of logit demands \( \bar{D}(p, b_t, e_t) \) defined in (19) satisfies Assumptions 1 and 2. Then \( \alpha_t \in [\bar{\alpha}_t^{**}, \alpha_t^{**}] \) for all \( t \in \{1, \ldots, T\} \), where

\[
\alpha_t^{**} \equiv \alpha_t^* \lor \sup_{t' < t} \left\{ \frac{1}{p_t} \ln \left( \frac{q_{t'}}{q_t} [1 + \exp(\alpha_{t'}^* p_{t'})] - 1 \right) \right\} \lor \sup_{t' > t} \alpha_{t'}^*, \tag{51}
\]

\[
\bar{\alpha}_t^{**} \equiv \bar{\alpha}_t^* \land \inf_{t' < t} \bar{\alpha}_{t'}^* \land \inf_{t' > t} \left\{ \frac{1}{p_t} \ln \left( \frac{q_{t'}}{q_t} [1 + \exp(\bar{\alpha}_{t'}^* p_{t'})] - 1 \right) \right\}. \tag{52}
\]

Equations (51)–(52) could be streamlined by consolidating the bounds obtained the first stage, \( \alpha_t^* \) and \( \bar{\alpha}_t^* \), into one of the other sets, but we have kept them separate to emphasize the point that \( \alpha_t^{**} \) and \( \bar{\alpha}_t^{**} \) are weakly tighter than \( \alpha_t^* \) and \( \bar{\alpha}_t^* \).
As done in equation (27), the bounds on $\alpha_t$ can be translated into bounds on $\epsilon_t$; namely, $\epsilon_t \in [\epsilon_t^{**}, \epsilon_t^{* *}]$, where

$$
\epsilon_t^{**} = \frac{\alpha_t^{**} p_t}{1 + \exp(-\alpha_t^{**} p_t)}, \quad \epsilon_t^{* *} = \frac{\bar{\alpha}_t^{**} p_t}{1 + \exp(-\bar{\alpha}_t^{**} p_t)}.
$$

The complex formulas in Proposition 7 for logit demand look very different from those in Proposition 6 for linear demand. We will see in the empirical application to the Du Pont decision, however, that they generate nearly identical bounds on the elasticities. Reassuringly, the bounds incorporating limiting information are more robust to functional-form assumptions than the formulas might suggest.

### 4.3. General Demand

The logic used to derive bounds incorporating limiting information for logit demand carries over to general demand. We have the following proposition, proved in the appendix.

**Proposition 8.** Suppose the sequence of general demands $\bar{D}(p, \theta_t, e_t)$ defined in (28) satisfies Assumptions 1 and 2 as well as conditions (29)–(31). Then $\theta_t \in [\bar{\theta}_t^{**}, \bar{\theta}_t^{* *}]$ for all $t \in \{1, \ldots, T\}$, where

$$
\bar{\theta}_t^{**} = \bar{\theta}_t^{*} \lor \sup_{t' < t} \left\{ \theta \left| \lim_{p \to 0} \frac{\bar{D}(p, \theta, e_t)}{\bar{D}(p, \bar{\theta}_t^*, e_t)} = 1 \right\} \lor \sup_{t' > t} \left\{ \theta \left| \lim_{p \to \infty} \frac{\bar{D}(p, \theta, e_t)}{\bar{D}(p, \bar{\theta}_t^*, e_t)} = 1 \right\} \right\}
$$

$$
\bar{\theta}_t^{* *} = \bar{\theta}_t^{*} \land \inf_{t' < t} \left\{ \theta \left| \lim_{p \to \infty} \frac{\bar{D}(p, \theta, e_t)}{\bar{D}(p, \bar{\theta}_t^*, e_t)} = 1 \right\} \land \inf_{t' > t} \left\{ \theta \left| \lim_{p \to 0} \frac{\bar{D}(p, \theta, e_t)}{\bar{D}(p, \bar{\theta}_t^*, e_t)} = 1 \right\} \right\}.
$$

The algorithm for computing bounds incorporating limiting information prescribed in the proposition is an iterative procedure. For example, the lower bounds $\bar{\theta}_t^*$ incorporating local information are computed in a first iteration; the $\bar{\theta}_t^*$ for all equilibrium points $e_t' \in E$ are then used to compute the new lower bound $\bar{\theta}_t^{**}$ for the single equilibrium point $e_t$ in the second iteration. Lower bounds from the first iteration are used in the computation of lower bounds in the second iteration; upper bounds from the first iteration are used to compute upper bounds in the second iteration.

More specifically, to compute the lower bound $\bar{\theta}_t^{**}$, one solves the equation for $\theta$,

$$
\lim_{p \to 0} \left[ \frac{\bar{D}(p, \theta, e_t)}{\bar{D}(p, \bar{\theta}_t^*, e_t')} \right] = 1
$$

for each $t' < t$ and takes the largest solution. One then solves a similar equation, just involving a
different limit, for \( \theta \),
\[
\lim_{p \to \infty} \left[ \frac{\tilde{D}(p, \theta, e_t)}{\tilde{D}(p, \theta^*_t, e_t)} \right] = 1
\]
(57)
for each \( t' > t \) and takes the largest solution. The new bound \( \theta^{**}_t \) is set to whichever is largest of (a) the bound from the method incorporating local information, \( \theta^*_t \), (b) the largest solution over \( t' < t \) to (56), and (c) the largest solution over \( t' > t \) to (57). The upper bound \( \bar{\theta}^{**}_t \) is computed analogously.

To translate the bounds on \( \theta_t \) into bounds on the elasticity \( \epsilon_t \), the arguments behind Proposition 5, which we will not repeat here, continue to apply. Defining
\[
\epsilon^{**}_t \equiv -\tilde{D}_p(p_t, \theta^{**}_t, e_t) \frac{p_t}{q_t}, \quad \bar{\epsilon}^{**}_t \equiv -\tilde{D}_p(p_t, \bar{\theta}^{**}_t, e_t) \frac{p_t}{q_t},
\]
(58)
we have \( \epsilon_t \in [\epsilon^{**}_t, \bar{\epsilon}^{**}_t] \).

5. Institutional Background

That completes the discussion of general methods. We next turn to the empirical application, beginning with some institutional background on the plastics market and the Du Pont decision.

The use of plastic in consumer goods—in the products themselves and their packaging—is so widespread today that it is hard to envision the world in the 1950s and 1960s when plastic was initially developed and diffused commercially. Polyethylene was among the first commercially developed plastics. We focus on two types, low- and high-density polyethylene. As mentioned in the introduction, low-density polyethylene was famously used to make Tupperware food-storage containers (Clarke 1999, p. 2) and high-density polyethylene to make Hula Hoops (Fenichell 1996, p. 264). Polyethylene remains the highest-volume commercial plastic in terms of global sales.

In the late 1930s, the U.S. Department of Justice sought to prosecute Du Pont, a U.S. manufacturer of polyethylene, and its U.K. co-conspirator Imperial Chemical Industries (ICI), for their Patents and Processes agreement. Signed in 1929, the agreement granted each company exclusive licenses for the patents and secret processes controlled by the other and divided the global market into exclusive territories between them. Views differ on whether the main motive for the agreement was the sharing of complementary technologies or the monopolization achieved by market
division (Hounshell and Smith 1999, p. 190). The Department of Justice viewed the Patents and Processes agreement as an illegal market division, in violation of Section 1 of the Sherman Act. Franklin Roosevelt’s administration lobbied to suspend legal action during the Second World War, especially because of the military significance of the chemical industry (Reader 1975, p. 432). After the war, the case proceeded to trial. The prosecution won a liability verdict in 1951. Judge Sylvester Ryan ordered a remedy in 1952.

Judge Ryan’s order cancelled the exclusive-territory arrangements between Du Pont and ICI and required them to license patents and secret processes involved in the manufacture of several of their products to all applicants at reasonable royalty rates. The incumbents’ most significant patents covered three products: polyethylene, nylon, and neoprene. Judge Ryan did not order the compulsory licensing of neoprene. Of the remaining products, Simon Whitney, later chief economist at the U.S. Federal Trade Commission, contends that only patents for polyethylene ended up garnering “widespread interest” commercially (Whitney 1958, p. 217). Thus, we focus on the effect of Judge Ryan’s structural remedy on the polyethylene market in this paper.

Whether compulsory licensing solves the monopolization problem is the subject of debate in the antitrust literature. In their seminal legal casebook, Areeda, Kaplow, and Edlin (2013) note that the terms of such licensing typically remain subject to commercial negotiations of the parties. This leaves open the possibility that the licensor preserves the monopoly outcome by specifying exorbitant rates. At one point, ICI was asking for a fixed fee of $500,000 and royalties amounting to eight percent of sales revenue (Fortune, 1954). Since this royalty applied to revenue not profit, it would have added to markups, whether as much as the monopoly level may be hard for industry outsiders to know. Expanding the incumbents’ scope to manipulate the licensing outcome in their favor, patents often do not relate all the information necessary for entrants to replicate a production process. While Judge Ryan ordered incumbents to provide accompanying manuals and training, the question remains whether these materials were even adequate to say nothing about allowing the entrants to operate on an even footing with incumbents.

The remedy appeared to have a dramatic effect on the polyethylene market (Backman 1964, 30)

---

7 Judge Ryan himself foresaw the potential difficulty in defining a reasonable royalty rate: “While it is true that as to [royalty rates] question might well be raised as to whether they were arrived at after arm’s length commercial negotiations, they do nevertheless furnish guide posts for future determination of the amount at which such royalties should be set. But, in any event, as to these products and all others, there is also available for judicial finding the sum a prudent licensee would pay under all existing circumstances.” (US vs. ICI, 1952, p. 227–228.)
p. 71). Seven manufacturers entered in 1953–56 and four more in 1956–59. Prices steadily declined and output rose. On the face of these facts, one might be tempted to conclude that Judge Ryan’s remedy achieved its purpose of turning a monopoly into a more competitive market. However, the same price declines and output increases may have arisen in a monopoly market experiencing substantial cost declines, plausibly realistic for plastics in the 1950s and 60s. The fact of entry seems to disprove monopoly unless it is thought that the entrants are merely producing their share of the monopoly quantity, returning most of the rents to the licensor. Thus our study will try to produce evidence for effective remedy that cuts through these criticisms.

To the extent the remedy was successful, a contributing factor may have been that the market was monopolized not by a single firm but by two independent firms, which used the Patents and Processes agreement to enforce exclusive territories, facilitating collusion. With the agreement cancelled by Judge Ryan’s remedy, the independent operators may have been thrown into a competitive fray to license their technologies. This may have disciplined exorbitant licensing fees and thwarted an attempt to preserve the monopoly outcome. That said, the question remains whether incumbents with a history of cooperation needed the explicit legal agreement to maintain cooperation between them.

6. Data

Our dataset consists of annual price and quantity data, aggregated across all firms the U.S. market, for two plastic products, low- and high-density polyethylene, in the period following the imposition of the remedy in the Du Pont (1952) case. Table 1 of Lieberman’s (1984) seminal paper form the foundation of our dataset, which displays the data he obtained from the annual reports, Synthetic Organic Chemicals: United States Production and Sales, issued by the U.S. Tariff Commission.\(^8\) The earliest date that price and quantity are reported for polyethylene is 1958, so Lieberman’s data begin then. Lieberman ends the series in 1972 because in subsequent years the OPEC crisis creates an oil shock which, together with the subsequent recession, disrupted both supply and demand in the plastics market.

Rather than taking Lieberman’s data directly, we returned to the original U.S. Tariff Commis-

\(^8\)Corresponding electronic data are provided by Lieberman (2016).
sion source for two reasons. First, we noticed anomalies in the price series—identical prices to the decimal in adjacent years—suggesting typographical errors, which required correcting. Second, he recorded production for his quantity variable. Since we are interested in estimates of demand, we went back and collected sales for our quantity variable, also reported in the original source. Our final dataset consists of prices (measured in nominal dollars per pound) and sales quantities (measured in million pounds) for low- and high-density polyethylene from 1958–72. The price series is an average wholesale price, computed by dividing total annual industry revenue by industry quantity sold.

Figure 6 displays the data for each of the two products in separate panels. Prices are roughly the same across the products. Quantity sold in the market for low-density polyethylene was about twice that in the market for high-density polyethylene.

Each price-quantity pair can be thought of as an equilibrium point resulting from the interaction of demand and a supply relation. The figure graphs the evolution of these equilibrium points over time. The predominant pattern is for equilibrium to shift to the southeast each year. With one exception, the equilibrium never shifts southwest over time, thus exhibiting rectangular expansion. The lone exception is 1963 for low-density polyethylene. Our analysis excludes that product-year. As a robustness check, we redo the analysis preserving rectangular expansion by dropping 1962 rather than 1963. The results, reported in Figure B2 in Appendix B, are quantitatively quite similar.

A strict interpretation of our model would contend that the southwest shift in the equilibrium point from 1962 to 1963 is a rejection of our assumption that demand is growing in the low-density polyethylene market. The 1963 equilibrium point must lie on a lower demand curve than the 1962 equilibrium. Given our assumption is violated in that year, there may be other years, unknown to us, in which it is violated as well. We adopt a more liberal interpretation that 1963 is a minor exception to a pattern that holds across several products and over several decades. The latter interpretation does have consequences when assessing our results later, increasing our burden to

---

9Production equals the sum of sales, change in inventories, contract production using the customer’s raw materials, and production consumed directly by the manufacturer. While we believe sales best captures quantity demanded, we repeat the empirical analysis using production as our quantity measure. The results, presented in Figure B1 in Appendix B, are qualitatively close to those using sales for quantity.

10Another maintained assumption in the methodology section is that $E$ is distinct. Our data are consistent with this assumption in that no price or quantity is observed twice over time. Prices or quantities that appear similar in fact differ when measured to sufficient precision. The data are also consistent with $E$ being non-trivial, in that prices or quantities are positive in all cases in our sample.
Figure 6: Evolution of Equilibrium in the Plastics Market. Source: U.S. Tariff Commission (various years).

demonstrate broad patterns that hold across many product-years, rather than drawing conclusions from a single elasticity estimate.

7. Empirical Results

This section presents empirical results using our methods for bounding elasticity of demand $\epsilon_t$. The nature of our dataset happens to allow only meaningful upper, not lower, bounds to be obtained. We are not able to obtain meaningful lower bounds because $SE^{-}(\epsilon_t)$ and $NW^{+}(\epsilon_t)$ happen to be
empty for all $e_t \in E$ in our dataset. We obtain a default lower bound of 0 as a consequence of the
law of demand, Assumption 1. Thus $e_t$ will always be bounded in an interval extending from 0 to
the upper bound we derive. We perform all our analysis assuming both linear and logit demands
to gauge robustness to functional form and for two products, low- and high-density polyethylene.

Figure 7 presents elasticity bounds for the low-density polyethylene market. The top panel
presents results using our first method, that incorporating local information from pairwise equi-
librium comparisons, generating an upper bound denoted $\bar{\epsilon}_t^*$. Focus first on the light-shaded bars,
representing elasticity bounds under the assumption of linear demand. The results are remarkable.
Aside from the first year, in which the upper bound is quite high at 2.5, the upper bound on the
elasticity is consistently low, never higher than 0.68, which is still quite inelastic. The upper bound
can be much lower than this: in particular, 0.09 and 0.08 in 1959 and 1960, respectively. These ex-
tremely low elasticities rule out anything but almost perfectly inelastic demand in those years. This
result would be inconsistent with a monopoly outcome, which in theory leads to an equilibrium in
the elastic region of demand. Only in the first year, 1958, does the upper bound rise into the elastic
region. We suspect that the high bound in 1958 results from a lack of information from previous
years to bound the demand curve in that year rather than a symptom of an idiosyncratically high
elasticity in that year; we have no evidence to suggest that demand for low-density polyethylene
was markedly different in that compared to later years. We will see repeatedly that our methodol-
gy works best with pre- and post-year equilibrium information to produce satisfactorily narrow
bounds.

The dark-shaded bars represent elasticity bounds assuming logit rather than linear demand. As
under linear demand, under logit demand, the elasticity continues to be bounded well within the
inelastic region every year but the first. The bound can be as low as 0.08, almost perfectly inelastic,
as we see in 1960. The bound can be tighter or looser than that under linear: logit generates a
tighter bound in the first year and one other isolated year; linear generates a tighter bound in the
other years.

The lower panel presents results from our second method, incorporating limiting information,
generating an upper bound labeled $\bar{\epsilon}_t^{**}$. The elasticity bounds are sharply tighter than in the first
panel. When limiting information is incorporated, the upper bounds on the elasticity shrink down
to around 0.4 in the first year and very close to 0 in all later years. These results strengthen the
Panel A: Method Incorporating Local Information

Panel B: Method Incorporating Limiting Information

Figure 7: Elasticity Bounds in Market for Low-Density Polyethylene

conclusion that demand was almost perfectly inelastic in the post-remedy years, far from the range consistent with monopoly. The bounds are virtually numerically identical whether linear or logit demand is assumed. The crucial assumption behind this method appears to be not the specific functional form assumed but, whatever that form is, that it is consistent across the domain of prices.

Figure 8 presents elasticity bounds for the other product, high-density polyethylene. In this market, there is a longer initial period in which only wide elasticity bounds are attainable. The
Panel B: Method Incorporating Limiting Information

Figure 8: Elasticity Bounds in Market for High-Density Polyethylene

bounds narrow considerably in 1962 and 1963, the upper bound reaching as low as 0.34 for linear demand and 0.35 for logit demand. The bounds remain well within the inelastic region under linear demand, but under logit demand the upper bound rises up above 1 in 1966 and remains around that level for the rest of the sample. The bounds for logit demand do not imply that the elasticity was this high but that this method at least cannot rule such high elasticities out. Overall, the upper bound on the elasticity is higher in the high-density market than low-density market, but even in the high-density market, elastic demand and monopoly is precluded in some years in the
post-remedy period.

The second panel of Figure 8 applies the second method, that incorporating limiting information, to the high-density polyethylene market. As in the top panel, the bounds are not particularly tight in the first four years; but by 1962 the upper bound has shrunk to around 0.5, and continues to shrink steadily, down to around 0.16 by 1972. The results in this panel are quantitatively similar across the two functional forms. Regardless of the functional form assumed, incorporating limiting information provides extremely tight bounds on the elasticity of demand. For over a decade, demand in each market was almost perfectly inelastic according to this method.

8. Conclusion

This paper provided a methodology for bounding the elasticity of demand that works in growing markets for homogeneous products. The method requires minimal information, working with even as few as two time-series observations on aggregate prices and quantities. The method is based on the idea that the demand curve through a given equilibrium price-quantity pair cannot be either so steep or so flat that it passes below earlier equilibria or above later equilibria without violating the assumption that demand is nondecreasing over time. These inequality conditions place bounds on the elasticity of demand in any given year.

A potential drawback of any methodology delivering bounds rather than point estimates is that the resulting bounds may be so wide as to be uninformative. In our empirical application to the polyethylene market in the 1958–72 period after a licensing remedy was ordered, the methodology turned out to quite informative. Using the more powerful method incorporating limiting information, the upper bound on the elasticity was very close to zero for all years but the first. This was true regardless of the functional form (linear, logit) assumed. Even in the first year, the elasticity was bounded below 0.41. The more conservative method incorporating only local information generated looser bounds, but the bounds were still extremely low—between 0.08 and 0.09—in several years regardless of functional form. In other years, all but the first, we do not obtain such low bounds using the more conservative method, but the elasticity is still bounded well below the elastic region of demand. The findings were weaker in the high-density polyethylene market, but even there, the elasticity was bounded below 0.5 for over a decade under either linear or logit demand.
Based on our finding that the elasticity of demand was bounded well away from the elastic region, virtually zero in some cases, we reject the contention that monopoly behavior effected by the Patents and Processes agreement between Du Pont and ICI continued after Judge Ryan’s remedy in the *Du Pont* case.

The *Du Pont* application serves as a proof of concept for our bounding methodology, which could be applied to any growing market involving homogeneous products. We hope other researchers will find value in our methodology and find it easy to apply using the Stata code available on request.

One might suspect that analogous elasticity bounds could be obtained in the opposite case of a declining rather than growing market simply by reversing all the signs and inequalities. In theory, this is true but with several caveats. First, even for markets experiencing secular declines, it may be hard to contend that the demand curve is definitively shifting back each year. Consumers may be losing their taste for a product, but natural population growth may offset this taste change. Second, even if suppliers are not investing much, the spillover of technology from other industries may lower costs in the market, shifting supply out. Equilibrium points may end up shifting to the southwest over time, offering little useful information to bound elasticities.
Appendix A: Proofs

Proof of Proposition 3: The text defines \( A(e_t, e_t') \) as the solution for \( \alpha \) in (24), rearranged here as
\[
g(\alpha, p_t, p_{t'}) = \frac{q_{t'}}{q_t}, \tag{A1}
\]
where
\[
g(\alpha, p_t, p_{t'}) = \frac{1 + \exp(\alpha p_t)}{1 + \exp(\alpha p_{t'})}. \tag{A2}
\]
Before proceeding, we prove several useful facts about \( g \). First,
\[
g(0, p_t, p_{t'}) = 1. \tag{A3}
\]
Next, differentiating (A2),
\[
\frac{\partial g(\alpha, p_t, p_{t'})}{\partial \alpha} = \frac{p_t \exp(\alpha p_t) - p_t \exp(\alpha p_{t'}) + (p_t - p_{t'}) \exp(\alpha p_t) \exp(\alpha p_{t'})}{[1 + \exp(\alpha p_{t'})]^2}. \tag{A4}
\]
Equation (A4) implies \( g \) is monotonic in \( \alpha \), increasing for \( p_t > p_{t'} \) and decreasing for \( p_t < p_{t'} \). To compute its limit as \( \alpha \to \infty \), we first rearrange (A2) as
\[
g(\alpha, p_t, p_{t'}) = \frac{1 + \exp(-\alpha p_t)}{\exp(\alpha (p_t - p_{t'})) + \exp(-\alpha p_t)}. \tag{A5}
\]
Then it is apparent that
\[
\lim_{\alpha \to \infty} g(\alpha, p_t, p_{t'}) = \begin{cases} 
0 & p_t < p_{t'} \\
\infty & p_t > p_{t'}.
\end{cases} \tag{A6}
\]
To apply these facts, first suppose \( p_t > p_{t'} \). Then the facts from the previous paragraph imply that \( g \) increases from 1 to \( \infty \) as \( \alpha \) increases from 0 to \( \infty \). Equation (A1) has a solution if and only if \( q_{t'} > q_t \). But \( p_t > p_{t'} \) and \( q_{t'} > q_t \) imply that \( e_{t'} \in SE(e_t) \). Next, suppose \( p_t > p_{t'} \). Then the facts from the previous paragraph imply that \( g \) decreases from 1 to 0 as \( \alpha \) increases from 0 to \( \infty \). Equation (A1) has a solution if and only if \( q_{t'} < q_t \). But \( p_t > p_{t'} \) and \( q_{t'} < q_t \) imply that \( e_{t'} \in NW(e_t) \). Therefore, (A1) has a solution if and only if \( e_{t'} \in SE(e_t) \cup NW(e_t) \). If (A1) has a solution, the monotonicity of \( g \) implies that this solution is unique.

The text analyzes the case in which \( t < t' \), showing that (23) is a necessary condition for Assumption 2 not to be violated. One can verify that (23) is satisfied for all \( \alpha \geq 0 \) if \( e_{t'} \in NE^+(e_t) \), implying that this subset does not contribute to bounds on \( \alpha_t \). One can verify that (23) is satisfied for all \( \alpha \geq A(e_t, e_{t'}) \) if \( e_{t'} \in NW^+(e_t) \), implying that this subset contributes lower bounds on \( \alpha_t \). One can verify that (23) is satisfied for all \( \alpha \leq A(e_t, e_{t'}) \) if \( e_{t'} \in SE^+(e_t) \), implying that this subset contributes upper bounds on \( \alpha_t \).

We complete the proof by analyzing the case in which \( t > t' \). To respect Assumption 2, we must have
\[
\mathcal{D}(p, \alpha_t, e_t) = D_t(p) \geq D_{t'}(p) = \mathcal{D}(p, \alpha_{t'}, e_{t'}). \tag{A7}
\]
Substituting \( p = p_{t'} \),
\[
\frac{q_t [1 + \exp(\alpha_t p_t)]}{1 + \exp(\alpha_t p_{t'})} = \mathcal{D}(p_{t'}, \alpha_t, e_t) \geq \mathcal{D}(p_{t'}, \alpha_{t'}, e_{t'}) = q_{t'}, \tag{A8}
\]
39
or rearranging,
\[ q_t [1 + \exp(\alpha_t p_t)] \geq q_{t'} [1 + \exp(\alpha_t p_{t'})]. \] (A9)

One can verify that (A9) is satisfied for all \( \alpha \geq 0 \) if \( e_{t'} \in SW^-(e_t) \), implying that this subset does not contribute to bounds on \( \alpha_t \). One can verify that (A9) is satisfied for all \( \alpha \geq A(e_t, e_{t'}) \) if \( e_{t'} \in SE^-(e_t) \), implying that this subset contributes lower bounds on \( \alpha_t \). One can verify that (A9) is satisfied for all \( \alpha \leq A(e_t, e_{t'}) \) if \( e_{t'} \in NW^-(e_t) \), implying that this subset contributes upper bounds on \( \alpha_t \). Q.E.D.

**Verifying Logit Satisfies General Demand Conditions:** Let \( \hat{D}(p, \theta, e_t) \) be the logit demand defined in (19) after substituting \( \theta = \alpha \). We will verify that this \( \hat{D}(p, \theta, e_t) \) satisfies conditions (29)–(31).

We verify (29) by direct differentiation. Differentiating \( \hat{D}(p, \theta, e_t) \) with respect to its first argument and substituting \( p = p_t \),
\[ \hat{D}_p(p_t, \theta, e_t) = \frac{-\theta q_t}{1 + \exp(-\theta p_t)}. \] (A10)

Differentiating (A10) with respect to \( \theta \) and substituting \( \theta = \theta_t \),
\[ \hat{D}_{\theta}(p_t, \theta_t, e_t) = \frac{-q_t [1 + (1 + \theta_t p_t) \exp(-\theta_t p_t)]}{[1 + \exp(-\theta_t p_t)]^2}, \] (A11)

which is negative, verifying (29).

To verify (30), we can use the notation introduced in the proof of Proposition 3 to write \( \hat{D}(p, \theta, e_t) = q_t g(\theta, p_t, p) \). Equation (A3) then implies \( \hat{D}(p, 0, e_t) = q_t \), verifying (30).

To verify (31),
\[ \lim_{\theta \to \infty} \hat{D}(p, \theta, e_t) = q_t \lim_{\theta \to \infty} g(\theta, p_t, p) = \begin{cases} 0 & p > p_t \\ \infty & p < p_t \end{cases}, \] (A12)

where the last equality follows from (A6). Q.E.D.

**Proof of Proposition 4:** The proof closely follows the logic of the proof of Proposition 3, so we omit most of it for brevity. Here, we fill in the detail that under conditions (29)–(31) hold and \( e_{t'} \in \text{NW}(e_t) \cup SE(e_t) \), then (34) has a unique solution.

Suppose \( e_{t'} \in \text{NW}(e_t) \). Condition (30) implies \( \lim_{\theta \to 0} \hat{D}(p_{t'}, \theta, e_t) = q_t < q_{t'} \), where the inequality follows from \( e_{t'} \in \text{NW}(e_t) \). Condition (31) implies \( \lim_{\theta \to \infty} \hat{D}(p_{t'}, \theta, e_t) = \infty \) since \( p_{t'} < p_t \) for \( e_{t'} \in \text{NW}(e_t) \). Condition 29 implies that \( \hat{D}(p_{t'}, \theta, e_t) \) is increasing in \( \theta \) since \( p_{t'} < p_t \). Together, these results imply that \( \hat{D}(p_{t'}, \theta, e_t) \) is below \( q_{t'} \) for low \( \theta \) and monotonically increases in \( \theta \) until it exceeds \( q_{t'} \) for high \( \theta \). Thus, the solution \( \Theta(e_t, e_{t'}) \) of (34) exists and is unique.

Suppose \( e_{t'} \in \text{SE}(e_t) \). Condition (30) implies \( \lim_{\theta \to 0} \hat{D}(p_{t'}, \theta, e_t) = q_t > q_{t'} \), where the inequality follows from \( e_{t'} \in \text{SE}(e_t) \). Condition (31) implies \( \lim_{\theta \to \infty} \hat{D}(p_{t'}, \theta, e_t) = 0 \) since \( p_{t'} > p_t \) for \( e_{t'} \in \text{SE}(e_t) \). Condition 29 implies that \( \hat{D}(p_{t'}, \theta, e_t) \) is decreasing in \( \theta \) since \( p_{t'} > p_t \). Together, these results imply that \( \hat{D}(p_{t'}, \theta, e_t) \) is above \( q_{t'} \) for low \( \theta \) and monotonically decreases in \( \theta \) until it falls below \( q_{t'} \) for high \( \theta \). Thus, again, the solution \( \Theta(e_t, e_{t'}) \) of (34) exists and is unique. Q.E.D.
Proof of Proposition 5: The first step of the proof is to show that (29) implies $\tilde{D}_{p\theta}(p_t, \theta, e_t) \leq 0$. Suppose for the sake of contradiction that $\tilde{D}_{p\theta}(p_t, \theta, e_t) > 0$. Since $\tilde{D}$ is continuously differentiable of all orders, $\tilde{D}_{p\theta}(p_t, \theta, e_t)$ is continuous in its first argument, implying that there exists $\delta > 0$ such that $\tilde{D}_{p\theta}(p, \theta, e_t) > 0$ for all $p \in (p_t - \delta, p_t + \delta)$. Thus

$$0 \leq \int_{p_t-\delta}^{p_t+\delta} \tilde{D}_{p\theta}(p, \theta, e_t) dp = \tilde{D}_\theta(p_t + \delta, \theta, e_t) - \tilde{D}_\theta(p_t - \delta, \theta, e_t).$$

(A13)

But (29) implies $\tilde{D}(p_t - \delta, \theta, e_t) > 0$ and $\tilde{D}(p_t + \delta, \theta, e_t) < 0$, implying their difference is negative, not positive as in (A13), a contradiction.

The fact that $\tilde{D}_{p\theta}(p_t, \theta, e_t) \leq 0$ implies $\frac{\partial}{\partial p}[-\tilde{D}(p_t, \theta, e_t)]$, implying $e_t$ as defined in (37) for general demand is nondecreasing in $\theta$, implying $e_t \in [\bar{e}_t, \tilde{e}_t]$. Q.E.D.

Proof of Proposition 7: For concreteness, suppose $t' > t$. (The proof supposing $t' < t$ is similar and omitted.) It remains to show that $\alpha_t \geq \alpha_{t'}$ is a necessary condition for (50). We have

$$\lim_{p \to \infty} \left\{ \frac{\tilde{D}(p, \alpha_t, e_t)}{\tilde{D}(p, \alpha_{t'}, e_{t'})} \right\} = \lim_{p \to \infty} \left\{ \frac{q_t[1+\exp(\alpha_t p_t)]}{q_{t'}[1+\exp(\alpha_{t'} p_{t'})]} \cdot \frac{1+\exp(\alpha_t p)}{1+\exp(\alpha_{t'} p_{t'})} \right\}$$

(A14)

$$= \frac{q_t[1+\exp(\alpha_t p_t)]}{q_{t'}[1+\exp(\alpha_{t'} p_{t'})]} \lim_{p \to \infty} \exp(p(\alpha_{t'} - \alpha_t))$$

(A15)

$$= \begin{cases} \infty & \alpha_t < \alpha_{t'} \\ \frac{q_t[1+\exp(\alpha_{t'} p_{t'})]}{q_{t'}[1+\exp(\alpha_t p)]} & \alpha_t = \alpha_{t'} \\ 0 & \alpha_t > \alpha_{t'} \end{cases}$$

(A16)

Equation (A14) follows from substituting for $\tilde{D}$ from (19), (A15) follows from dividing numerator and denominator by $\exp(\alpha_{t'} p)$ and taking limits, and (A16) follows from evaluating the remaining limit. We see that (50) is violated if $\alpha_t < \alpha_{t'}$. Q.E.D.

Proof of Proposition 8: Start by analyzing the limit $p \to 0$. For $t' > t$, we have

$$\lim_{p \to 0} \left[ \frac{\tilde{D}(p, \theta_t, e_t)}{\tilde{D}(p, \theta_{t'}, e_{t'})} \right] \geq \lim_{p \to 0} \left[ \frac{\tilde{D}(p, \theta_{t'}, e_{t'})}{\tilde{D}(p, \theta_{t'}, e_{t'})} \right] = 1.$$ \hspace{1cm} (A17)

The first inequality follows from Assumption 2. To see the second inequality, note that in the limit as $p \to 0$, eventually $p \leq p_{t'}$. But (29) then implies $\tilde{D}_\theta(p, \theta, e_t) \geq 0$, in turn implying $\tilde{D}(p, \theta_{t'}, e_{t'}) \leq \tilde{D}(p, \theta_{t'}, e_{t'})$ in the limit as $p \to 0$ since $\theta_{t'} \leq \theta_{t'} \leq \theta^*_{t'}$ by Proposition 4.

Similarly, one can argue that (29) implies that the numerator on the left-hand side of (A17), and thus the entire left-hand side, is increasing in $\theta_t$. Condition (A17) treated as an equality,

$$\lim_{p \to 0} \left[ \frac{\tilde{D}(p, \theta_t, e_t)}{\tilde{D}(p, \theta_{t'}, e_{t'})} \right] = 1,$$ \hspace{1cm} (A18)

thus provides an equation in $\theta_t$ that can be solved to provide a lower bound on $\theta_t$. The new bound $\underline{\theta}_t^*$ in (54) is set to be no lower than the highest of the solutions to (A18) for $t' > t$.

The analyses for $t' < t$ and for the other limit, $p \to \infty$ are similar and omitted for brevity. Q.E.D.
Appendix B: Additional Figures

Figure B1: Elasticity Bounds in the Low-Density Polyethylene Market Using Production Data for Quantity. Notes: The whole set of equilibrium points $E$ exhibits rectangular expansion, so there is no need to drop 1963 as was the case using sales data for quantity.
Figure B2: Elasticity Bounds in the Low-Density Polyethylene Market Dropping Alternative Year. Notes: Rather than dropping 1963 to preserve property of rectangular expansion for equilibrium points, the same goal can be achieved dropping 1962, done in this figure.
References


