DARTMOUTH
GK-12 PROGRAM
A PARTNERSHIP WITH THE
U.S. NATIONAL SCIENCE FOUNDATION

Domino Tilings and Fibonacci Numbers

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Overview

This lesson is an exploration into the mathematical world of tiling. It is an incredible fact that the number of domino tilings of a 2 x n rectangle is precisely the Fibonacci numbers. Students will work in groups to construct the tilings of 2 x n rectangles with dominoes. They will then conjecture how many tilings there are of an arbitrary 2 x n rectangle and work to explain why their conjecture is true.

Objectives

In this lesson, students will:

- *Use geometric intuition and a systematic approach to enumerate the different domino tilings of a 2 x n rectangle.*
- *Learn the meaning of a conjecture and make one of their own.*
- *Generalize observations on tilings and Fibonacci numbers to all sizes/numbers.*
- *Use knowledge of the Fibonacci numbers to construct an explanation (proof) of their conjecture.*

Background

This activity assumes that the students are acquainted with the Fibonacci numbers (though a review is included as part of the lesson).

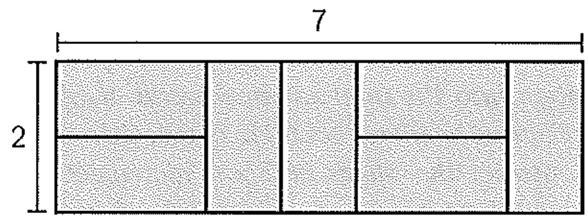
Materials

- *Dominoes (6 dominoes per group - the worksheets are sized for dominoes with dimension approximately $\frac{3}{4}$" x $1\frac{1}{2}$", though any size will work)*
- *Relevant worksheets:* **Tiling with Dominoes, Fibonacci Numbers and Domino Tilings**
- *Accompanying Lesson Plan for reference*

Lesson Plan

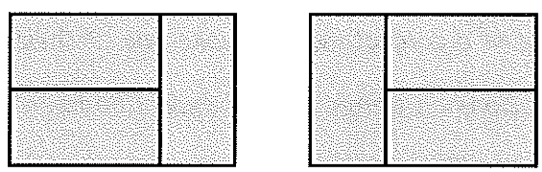
1. Introduce students to the concept of tilings and mention that you will be focusing on a very specific problem: tiling 2 x n rectangles with dominoes that are of size 1 x 2. At this point giving a couple of examples of what you mean by 2 x n and dominoes works well.

Example of a tiling of a 2x7 rectangle:



2. There are a couple rules that need to be followed when tiling the rectangles:
   - There can be no unfilled spaces on the rectangles
   - Symmetry doesn't count - if you rotate a tiling and it looks different, then it's a different tiling.

   For example, these two tilings are different:



3. Give each group of students 6 dominoes along with the worksheet **Tiling with Dominoes**. Give them time to work on the sheet.

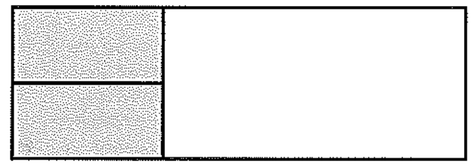
   NOTE: You may notice that on the table provided on the worksheet, the number of tilings of a 2 x 0 rectangle is 1. This corresponds to the "empty tiling" - simply not using any tiles. This is to help students see the Fibonacci sequence which traditionally starts with two 1's.

4. Let groups proceed through the worksheet. As they come up with conjectures about the pattern, they will raise their hands to check with the teacher. If they have conjectured that the pattern is the Fibonacci Numbers, then give them the next worksheet: **Fibonacci Numbers and Domino Tilings**.

5. The worksheet **Fibonacci Numbers and Domino Tilings** will help students explain *why* the tilings following the Fibonacci pattern. Depending on their level, they may need assistance, but your main goal should be to interfere *as little as possible*. Below is an explanation of the phenomenon to help with teaching:

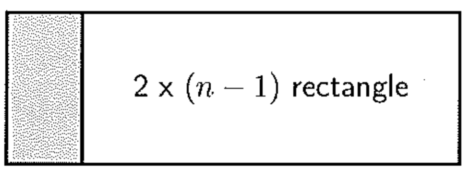
- The Fibonacci numbers are recursive, meaning if we want to calculate a term in the sequence, we can use previous terms to do so. Specifically, if you want to get a number in the Fibonacci sequence, you add the previous two terms. In a similar way, we will show that you can get the number of domino tilings of a particular 2 x n rectangle by adding the number of domino tilings of a 2 x $(n-1)$ and 2 x $(n-2)$ rectangle. This is precisely the pattern of the Fibonacci sequence and is the explanation of why the number of tilings match the Fibonacci sequence.
- In order to get the recursion with the tilings, you have to consider how you can *start* a tiling. Let's say you want start tiling on the left side of your rectangle. There are two ways to completely cover the left edge:

Case 1

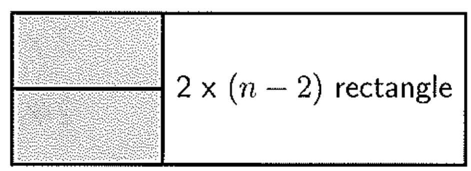


Case 2

- So now, you are either in Case 1 or Case 2, depending on how you decided to start your tiling. At this point, we can count how many tilings their are in each case. The key observation is that we simply need to consider how many tilings there are of the white space in the pictured rectangles.



Case 1



Case 2

- So the number of tilings that look like Case 1 = the number of tilings of a 2 x $(n-1)$ rectangle and the number of tilings that look like Case 2 = the number of tilings of a 2 x $(n-2)$ rectangle. You add those together to get the number of tilings of a 2 x n rectangle.

Name:____________________________________ Date:_______________

Tiling with Dominoes

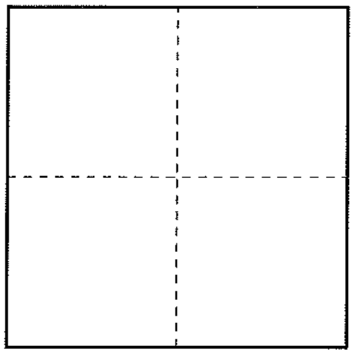
Using the dominoes provided, tile the rectangles on the following page. For each, determine the number of distinct tilings possible. **Record the configuration of all the tilings you find, so you can keep track of them.** When you and your partner(s) believe you have found them all, record the number of tilings in the table below.

Rectangle Dimension	Number of Tilings
2 x 0	1
2 x 1	
2 x 2	
2 x 3	
2 x 4	
2 x 5	
2 x 6	

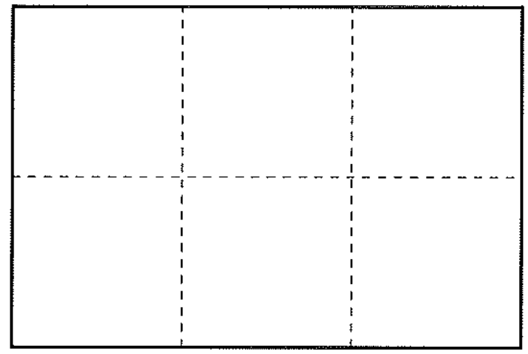
FACT: A statement that has not yet been proven true (though is believed to be true) is what mathematicians call a **conjecture**.

Examine the pattern of these numbers - can you make a conjecture about the pattern? When you have written down your conjecture in complete sentences, raise your hand and explain it to your teacher.

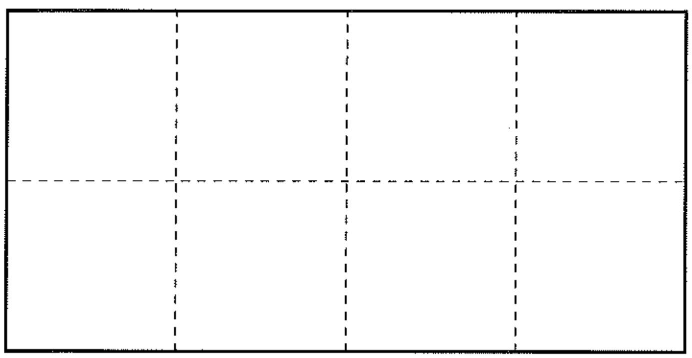
2 x n Rectangles



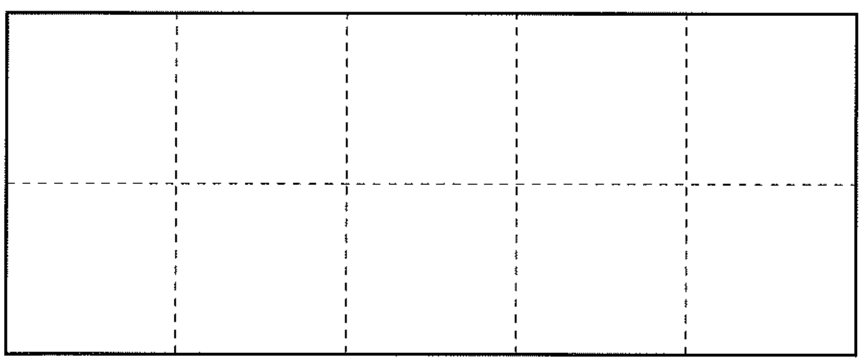
2 x 2



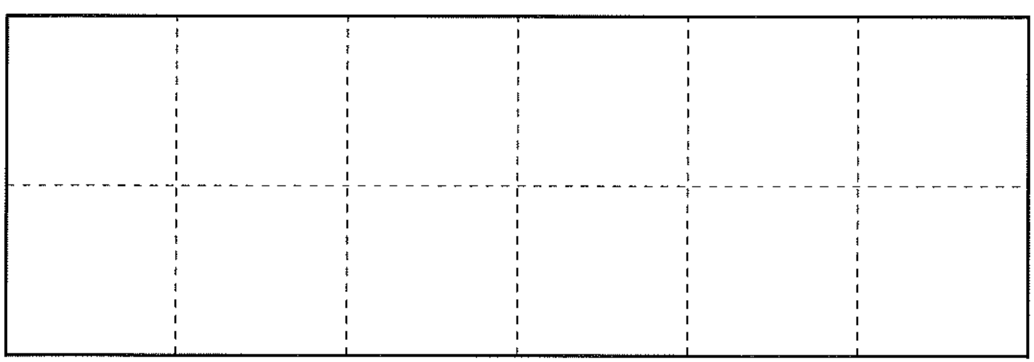
2 x 3



2 x 4



2 x 5



2 x 6

Name:______________________________________ Date:________________

Fibonacci Numbers and Domino Tilings

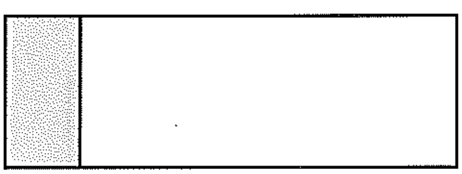
1. Let's recall a little bit about the Fibonacci Numbers.

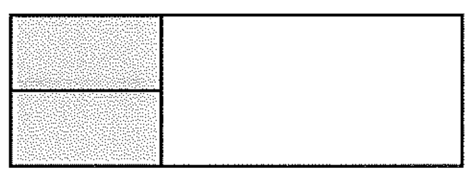
   (a) The beginning of the Fibonacci sequence is: 1,1,2,3,5,8,. . .
   What number comes after 8 in the sequence? How did you figure that out?

   (b) If you have the 84th and the 85th Fibonacci numbers, how can I find the 86th Fibonacci number? State your answer with a complete sentence.

   (c) Complete the following sentence:
   If you have two consecutive Fibonacci numbers, you can get the next Fibonacci number by ______________________________________.

2. Consider the process of tiling a rectangle with dominoes. Let's say you start tiling on the left side of the rectangle. There are two ways you can fit your dominoes the totally cover the left side (the gray rectangles represent your dominoes):





Case 1 Case 2

   (a) Now, look at all your tilings of the 2 x 6 rectangle. Put your tilings into two different groups - Group 1 is made up of tilings that look like Case

1 and Group 2 is made up of those tilings that look like Case 2. Count up the number of tilings in each group.

(b) Compare the sizes of Group 1 and Group 2 with the number of tilings of 2 x 5 and 2 x 4 rectangles. In complete sentences, record how these numbers relate.

(c) Is this a coincidence or a pattern? Look at the tilings of the 2 x 5 rectangle and break them into two groups (as in 2(a)). Is there a similar phenomenon?

3. Now consider any 2 x n rectangle (the n can be any positive whole number you want). If we were to look at all the tilings of the 2 x n rectangle and split them into groups like in 1(a), how big do you think the groups will be? (Hint: Your answer will require you to consider the number of tilings of 2 x $(n-1)$ and 2 x $n-2$ rectangles)

4. Consider Fibonacci numbers and the phenomenon you just witnessed with tilings. How do they relate? Write this relationship in complete sentences.