Overview of Laplace Solutions, Transfer Functions, Impedance, and Frequency Response

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Note: All of this works only on LTI systems!

Two observations on Laplace Solution:

\[ V_C(s) = \frac{v_c(0)}{s^2 + \zeta \omega_n s + \omega_n^2} + \frac{\cdots}{s^2 + \zeta \omega_n s + \omega_n^2} \]

With constant, step or zero input, Responses are all:

\[ C + A_1 e^{\xi_1 t} + A_2 e^{\xi_2 t} + \cdots \]

Characterize by roots of denominator, \( \{p_1, p_2, p_3, \ldots\} \)

Real roots \( \Rightarrow \) exponentials
Complex roots \( \Rightarrow \) complex exponentials (damped oscillations)
Describe with \( \zeta, \omega_n \)

Poles of \( H(s) \) \( \Leftrightarrow \) response with step or constant input

Obtaining \( H(s) \) (for zero init condts)

\[ V_{\text{out}}(s) = \frac{\cdots}{s^2 + \zeta \omega_n s + \omega_n^2} V_m(s) \]

This is total response for zero init condts.

Note that it can be written:

\[ V_{\text{out}}(s) = \frac{\cdots}{s^2 + \zeta \omega_n s + \omega_n^2} V_m(s) \]

Thus, for zero init condts.,

\[ V_{\text{out}}(s) = H(s) \cdot V_m(s) \]

and \( V_{\text{out}}(t) = \mathcal{L}^{-1}\{H(s) \cdot V_m(s)\} \)

Note that this does not mean

\[ v_{\text{out}}(t) = \{h(t) \cdot v_m(t)\} \]
Response using $H(s)$

For zero init. condits., $V_{out}(s) = H(s) \cdot V_{in}(s)$

$V_{out}(t) = e^{-t} \{H(s) \cdot V_{in}(s)\}$

Example: $H(s)$ from last page, plus $v_{in}(t) = u(t) \cos(\omega t)$ (cosine that turns on at time zero)

Calculating this complete response gets nasty, so nasty, that actually....

I used a numerical solution:

% Plot response of LRC circuit
% (described in lrc.m) % to sinusoidal input.% crs[t,y]=ode45('lrc',[0,1e-3],[0;0]);
subplot(2,1,2)
plot(t*1e6,y(:,2))bigylabel('Capacitor voltage [v]')xlabel('time [\mu s]')subplot(2,1,1)
plot(t*1e6,0.8*cos(2*pi*5e3*t))big
ylabel('Input Voltage')

function xd = lrc(t,x)
% Derivative function for ode45
% LRC circuit
% C Sullivan
% Set Parameters
R = 8; % ohms C = 0.06e-6; % farads L = 575e-6; % Henries
v0 = 0.8; % volts
w = 2*pi*5e3;
A = [-R/L -1/L ; 1/C 0];
B = [1/L ; 0];
xd = A*x + B*v0*cos(w*t);

What good is $H(s)$ with sinusoidal input?

Ignore this

Look at this: “Sinusoidal Steady-State” need:
$G$ (one in this case)
$\phi$ (zero in this case)

Amazingly simple result:
$G = |H(j\omega)|$
$\phi = \angle (H(j\omega))$

Evaluating $H(j\omega)$

- Use MATLAB as a complex-number calculator.
- Get qualitative idea
- Evaluate $|H(j\omega)|$ and $\angle (H(j\omega))$ algebraically
- Express $H(s)$ as

$$H(s) = \frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)(s-p_3)}$$

$z$ is a zero, a place on the $s$-plane where $H \rightarrow 0$ (plotted o)
$p$ is a pole, where $H \rightarrow \infty$ (plotted x)

- Now plug in $s = j\omega$ $H(j\omega) = \frac{(j\omega-z_1)(j\omega-z_2)}{(j\omega-p_1)(j\omega-p_2)(j\omega-p_3)}$

Each term is a distance (a complex vector distance on the $s$-plane) between $p$ or $z$ and $j\omega$, a point on the imaginary axis at a height corresponding to the input frequency we want to study.

- If we want to know $|H(j\omega)|$, then all we need is

$$|H(j\omega)| = \frac{|z_1-j\omega| |z_2-j\omega|}{|p_1-j\omega| |p_2-j\omega| |p_3-j\omega|}$$

Each term in this is just the length of a distance vector.