Physics 82, PROBLEM SET 8, due Wednesday, March 9

1. Show how to factor 65 by finding the order of $x$ mod 65, including the following cases: 
   (a) Show that $x = 4$ fails, and explain why. (b) Show how $x = 14$ works.

2. In this problem we consider rotating the plane of linear polarization of photons. We shall do this for one or both of the photons in a Bell state. For convenience, you may denote such rotate states by a subscript, $|\psi\rangle_\chi \equiv \rho(\chi) |\psi\rangle$.

   (a) As background, show that rotating the plane of linear polarization of a single photon through an angle $\chi$ in the H-V plane is equivalent to rotating through $2\chi$ about the y-direction on the Bloch sphere, i.e., $\rho(\chi) = R_y(2\chi)$. This reviews a connection found in an earlier problem.

   (b) Suppose that two photons are in the Bell state $|\beta_{11}\rangle$, one going toward Alice and the other toward Bob. Suppose that Bob rotates his photon as in part (a), which he can do with an optically active medium like sugar. Define the correlation function

   $$C(\chi) = \langle \Psi | Z_1 Z_2 | \Psi \rangle,$$

   and calculate it for the Bell state as transformed by Bob. Show that $C(\chi) = -\cos 2\chi$, in agreement with the result found in class. This definition of $C(\chi)$ is indeed equivalent to one given in class, but it is based on operators and may be more intuitive. Its more general form, valid for any polarization states including elliptical polarizations, is $C(\Psi) = \langle \Psi | \vec{\sigma}_1 \cdot \vec{\sigma}_2 | \Psi \rangle$.

   (c) Start with the same Bell state, $|\beta_{11}\rangle$, but now let Alice and Bob each rotate their photon’s plane of polarization by $\chi$. Show that the state is unchanged (i.e., $|\beta_{11}\rangle_r = |\beta_{11}\rangle$). (This invariance does not hold for other Bell states, as is easily demonstrated by trying it on the state $|\beta_{01}\rangle$, which differs from $|\beta_{11}\rangle$ by only a sign change!)

3. In NMR applications, a useful model Hamiltonian for two interacting nuclear spins in a constant external magnetic field ($\vec{B} = B\hat{z}$) is

   $$H = -\frac{1}{2}\mu_1 B Z_1 - \frac{1}{2}\mu_2 B Z_2 + \frac{1}{4}\hbar J Z_1 Z_2,$$

   where $\mu_1$ and $\mu_2$ differ because the gyromagnetic ratios of the two nuclei differ. The interaction term is called the “Ising” interaction; this arises from the Heisenberg interaction ($\sim \vec{\sigma}_1 \cdot \vec{\sigma}_2$) when a large external magnetic field causes precession of the nuclear spins about the z-axis, making the $X_1X_2$ and $Y_1Y_2$ terms oscillate rapidly and time-average to zero.
Note that the precession frequencies of the two spins differ, $\omega_{p1} = \mu_1 B/\hbar \neq \omega_{p2}$. This is actually advantageous because it means that their Rabi flopping frequencies differ, thus allowing us to address the spins individually by tuning applied rf pulse frequencies to the appropriate Rabi frequency. The choice of one qubit gate is made by selecting the rf field direction (e.g., along x or y), and the rf pulse length.

In this problem, we shall be concerned with the evolution $U(t)$ induced by applied rf pulses (which are not included in $H$ above), and the Ising interaction term. We shall not be concerned with the evolution induced by the constant-B terms in $H$, since their only effective role is to determine the Rabi frequencies of the individual qubits and to convert the Heisenberg interaction into the Ising interaction.

(a) Find the combination of rf pulse length $T_{rf}$ and field strength $B_{rf}$ required to rotate a spin by $90^\circ$ about the x or y axis. What rf frequency $\omega_{rf}$ is required to apply this gate to the second qubit?

(b) Derive the usual (cos-sin) identity for the Ising evolution operator $U_I(t)$ using the fact that $(Z_1Z_2)^2 = I$, and find the time $T$ that makes $U_I(T) = (I - iZ_1Z_2)/\sqrt{2}$.

(c) Show that the circuit below is the CNOT gate modulo phases.* (One possible approach is to deduce the effective operator acting on the second qubit. One can write this operator as $(I + Z_1)$ times something that does not flip the second spin, plus $(I - Z_1)$ times something that does flip it.)

(d) What one thing would you change to reverse the CNOT gate (interchange control and target roles)?

* Interpreting this sequence of gates geometrically can make an intuitively convincing argument that they indeed amount to a CNOT. It is a sequence of rotations of the second qubit, each through $90^\circ$, about x, z, and y axes, respectively. The sense of the second rotation (right or left-handed) is conditioned on the state of the first qubit.