Solutions to Problem Set 7

1a) \( M = N / 50 \Rightarrow \tan \theta \approx \sqrt{\frac{m}{N-M}} = \frac{1}{4} \approx \theta \)

\[(2V_{opt} + 1) \theta = \frac{\pi}{2} \Rightarrow V_{opt} = \frac{1}{2}(\frac{\pi}{2\theta} - 1) = 5 \text{ nearest whole number} \]

(b) \( \Rightarrow \Theta_{opt} = (10 + 1) \theta = \frac{11}{7} \) (but see below)

(c) Coefficient for success \( \alpha = \sin \frac{\pi}{7} = 1 - 2.0 \times 10^{-7} \)

Probability of failure \( P_F = 1 - |\alpha| = 4.0 \times 10^{-7} \) (see below)

This is so small that we must use \( \theta = \arctan \frac{1}{4} \) instead of \( \frac{11}{7} \) ! So

\[ \Theta_{opt} = 11 \times \arctan \frac{1}{4} = 1.56087 \]

\[ \alpha = \sin \Theta_{opt} = 1 - 4.9 \times 10^{-5} \]

\[ P_F = 1 - |\alpha| = 1.0 \times 10^{-4} \]

Expected "worst case scenario" probability of failure is based on assuming that

\[ \Theta \approx \frac{\pi}{2} - \theta \]

\[ P_F^{(wc\#)} = 1 - \sin^2 \left( \frac{\pi}{2} - \theta \right) = \sin^2 \theta \approx \frac{M}{N-M} \approx 0.02 \]

(Close to \( (wc\#) \) occurs when \( M = N / 59 \) or \( N / 42 \).)
2) Writing $FT = U_1 U_2 \ldots U_n$ as a sequence of unitary gates, we can write its inverse as $(FT)^+ = U_n^+ \ldots U_2^+ U_1^+$, i.e., reverse the order and take the inverse of every gate in the sequence. The swap gate is its own inverse, so that the inverse Fourier transform is

\[
\begin{array}{c}
J_1 \\
\vdots \\
J_n
\end{array}
= \begin{array}{c}
\begin{array}{c}
R_1^+ \\
\vdots \\
R_n^+
\end{array}
\end{array}
\]

where

\[
\begin{array}{c}
R_1^+ \\
\vdots \\
R_n^+
\end{array}
= \begin{array}{c}
\begin{array}{c}
R_1^+ \\
\vdots \\
R_n^+
\end{array}
\end{array}
\]

\[
\text{if } n \text{ even, } \quad \begin{array}{c}
\begin{array}{c}
R_1^+ \\
\vdots \\
R_n^+
\end{array}
\end{array}
\]

\[
\text{or } \quad \begin{array}{c}
\begin{array}{c}
R_1^+ \\
\vdots \\
R_n^+
\end{array}
\end{array}
\]

\[
\text{if } n \text{ odd, } \quad \begin{array}{c}
\begin{array}{c}
R_1^+ \\
\vdots \\
R_n^+
\end{array}
\end{array}
\]

and

\[
R_2^+ = \begin{array}{c}
\begin{array}{c}
R_2^+ \\
H
\end{array}
\end{array}
\]

\[
\begin{array}{c}
R_3^+ \\
\vdots \\
R_n^+
\end{array}
= \begin{array}{c}
\begin{array}{c}
R_3^+ \\
R_2^+ \\
\vdots \\
H
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
R_n^+
\end{array}
= \begin{array}{c}
\begin{array}{c}
R_n^+ \\
R_n^+
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array}
\]
The conditional phase gates involve the inverses of the $R_k$ gates, i.e.,

$$\begin{bmatrix} R_k^+ \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2\pi i/2^k} \end{bmatrix},$$

and the $H$ gate is its own inverse, $H^2 = I$.

3) The state $|\psi_0\rangle$ entering the controlled-$U$ gate is a uniform superposition of all basic states,

$$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_j |j\rangle = \frac{1}{\sqrt{N}} \sum_{j_1=0}^1 \cdots \sum_{j_t=0}^1 |j_1 \cdots j_t\rangle,$$

where the sum is over all $N = 2^t$ binary numbers in reg. 1. Consider the action of the controlled-$U$ gates on each component $|j_1 j_2 \ldots j_t\rangle$ of this state. The first gate ($U$) acts if $j_t = 1$ and multiplies this component by its eigenvalue; $U \rightarrow e^{2\pi i \varphi j_t}$. The second gate ($U^2$) acts if $j_{t-1} = 1$ and multiplies by $U^2 \rightarrow e^{2\pi i \varphi (2j_{t-1})}$. In general,

$$U^{2^k} \rightarrow e^{2\pi i \varphi (2^k j_{t-k})},$$

all the way to $U^{2^{t-1}} \rightarrow e^{2\pi i \varphi (2^{t-1} j_{t-1})}$.
In this way, the sequence of $U^k$ gates multiplies the component $|j_1 j_2 \ldots j_t\rangle$ by the phase factor 
\[ \exp[2\pi i \varphi (2^{t-1} j_1 + 2^{t-2} j_2 + \ldots + 2 j_{t-1} + j_t)] = e^{2\pi i \varphi j}, \]

since the sum in round brackets is just a representation of the binary number $j = (j_1 j_2 \ldots j_t)$. In summary,

\[ [U^k \text{- gates}] |j\rangle |u\rangle = e^{2\pi i \varphi j} |j\rangle |u\rangle \]

\[ \Rightarrow [U^k \text{- gates}] |\psi_0\rangle |u\rangle = \frac{1}{\sqrt{N}} \sum_j e^{2\pi i \varphi j} |j\rangle |u\rangle \equiv |\psi_0\rangle |u\rangle, \] where $|\psi_0\rangle = \text{FT} |\varphi_1 \varphi_2 \ldots \varphi_t\rangle$.

\[ 4a) \] If we input a superposition of input states $|u\rangle$ into the second register, then $U$ will multiply each component by its corresponding eigenvalue,

\[ U |\phi\rangle = U \sum_u C_u |u\rangle = \sum_u C_u e^{2\pi i \varphi_u} |u\rangle. \]

The sequence of controlled-$U^k$ gates will act on each component $|u\rangle$ as in the boxed expression above, producing the state $|\Psi'\rangle$ of 1st & 2nd registers:

\[ |\Psi'\rangle = [U^k \text{- gates}] |\psi_0\rangle |\phi\rangle = \frac{1}{\sqrt{N}} \sum_u C_u \sum_j e^{2\pi i \varphi_j} |j\rangle |u\rangle \]

\[ \frac{1}{\sqrt{N}} \sum_j |j\rangle \rightarrow \sum_u C_u |u\rangle \]
(b) The inverse Fourier transform of this state is, by definition,

\[ (FT)^+ |\Psi'\rangle = \frac{1}{N} \sum_{k} \sum_{u} C_u \sum_{j} e^{2\pi i j k/N} e^{2\pi i j \Phi_u} |k\rangle |u\rangle \]

Now it is useful to write \( N\Phi_u = b_u + \delta_u \), where \( b_u = \Phi_1 \Phi_2 \ldots \Phi_t \) is the closest binary integer less than or equal to \( \Phi_u \), and \( \delta_u = 0. \Phi_{t+1} \Phi_{t+2} \ldots \) is the remainder (and \( 0 \leq \delta_u < 1 \)).

\[ \Rightarrow (FT)^+ |\Psi'\rangle = \frac{1}{N} \sum_{k} \sum_{u} C_u \left( \sum_{j} e^{2\pi i j (b_u + \delta_u - k)/N} \right) |k\rangle |u\rangle \]

\[ = \sum_{k} C_u \sum_{u} a_k^{(u)} |k\rangle |u\rangle \]

where \( a_k^{(u)} = \frac{1}{N} \sum_{j} e^{2\pi i j (b_u + \delta_u - k)/N} \).

(c) If all \( \Phi_u \) are exactly expressible with \( t \) bits, this means that \( \Phi_u = b_u \), with \( \delta_u = 0 \).

This means that \( a_k^{(u)} = \delta_k, b_u \), so that the above expression (upper box) reduces to

\[ (FT)^+ |\Psi'\rangle = \sum_{u} C_u |b_u\rangle |u\rangle \].
As (1) the readout probabilities are nonzero only for $k = b u = (u_1, u_2, \ldots, u_t)$, and the probability of each such outcome is $P_u = |C u|^2$.

(But we are certain to get one of the eigenvalues exactly.)

(2) Reading the first register "projects" the second register onto the eigenstate $|u\rangle$ corresponding to the result $|q_u\rangle$ found on the first register. (see boxed expression at bottom of p. 7.5)

(c) If $\delta_u \neq 0$, then the probability dist. $|\lambda_k^{(u)}|^2$ is peaked around the eigenvalues $(N u_0 = b_u + \delta_u)$ rather than being perfectly sharp.

Assuming that these peaks are nonoverlapping, the probability of getting a particular value $k$ is

$$P_u(k) = |C u|^2 |\lambda_k^{(u)}|^2$$

If we pick $t = N + \log(2 + \frac{2}{\pi^2} \epsilon)$, then we will find $q_u$ to $N$-place accuracy with a probability no less than

$$P_u = |C u|^2 (1 - \epsilon)$$

$N$-place accuracy means we fall inside the tolerance window $N q_u \pm E$, where $E = 2/\pi^2 \epsilon$.

* "Peaks nonoverlapping" means peaks are separated by more than $2E$. 