Solutions to Problem Set 5

1) A magnetic field $\vec{B} = B\hat{z}$ applied for a time $T$ generates the gate

$$ U(T) = e^{-iHT/t}, \quad H = -\frac{i}{2}\mu_0 B\hat{z} $$

$\Rightarrow U(T) = e^{i\mu_0 Bt^2/2t}$

$$ U(T) = \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} = e^{i\xi} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\xi} \end{pmatrix} $$

Where $\xi = \mu_0 Bt/2t$. To make the $R_k$ gate $R_k = -\pi/2k$, so that $B$ must be negative (i.e., the field $\vec{B}$ must be applied in the negative $z$-direction, and for a time

$$ T_k = \frac{2\pi t}{2k\mu_0 |B|} \quad \text{and} \quad X_k = -\xi = \frac{\pi}{2k}. $$

2) We'll say $M(\beta_{00})$ detects the state $|\beta_{00}\rangle$, and only that state.

(A) What does $X \cdot M(\beta_{00})$ detect?

In order for $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |111\rangle)$ to enter the detector, we must have the state $|\Psi\rangle$ incident prior to Bob's application of the NOT gate, i.e. such that $X |\Psi\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |111\rangle)$. 

But $X_2$ is its own inverse, $X_2^2 = 1$, so that

$$|\psi\rangle = X_2 \frac{1}{\sqrt{2}} (|100\rangle + |111\rangle) = \frac{1}{\sqrt{2}} (|101\rangle + |110\rangle) = |\beta_{01}\rangle$$

In previous notation. As we detect $|\beta_{01}\rangle$, i.e.,

$$M(\beta_{01}) = M(\beta_{00})$$

In photon terms,

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}} (|HV\rangle + |VH\rangle).$$

(b) Suppose we rotate by $90^\circ$ in the right-handed sense, then

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow |\psi\rangle' = \begin{pmatrix} \beta \\ -\alpha \end{pmatrix},$$

which corresponds to the gate $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We need a state $|\psi\rangle$ incident on Bob's device such that

$$A |\psi\rangle = |\beta_{00}\rangle,$$

and again, since $A^2 = I$, we have

$$|\psi\rangle = \frac{1}{\sqrt{2}} A (|100\rangle + |111\rangle) = \frac{1}{\sqrt{2}} A (|101\rangle - |110\rangle) = |\beta_{11}\rangle$$

in same notation. And $|\beta_{11}\rangle = \frac{1}{\sqrt{2}} (|HV\rangle - |VH\rangle)$. 
(c) Suppose we perform the NOT gate followed by the 90° rotation, and demand that

\[ A_2 \text{ NOT}_2 |\psi\rangle = \frac{1}{\sqrt{300}} |\psi\rangle \]

\[ \Rightarrow |\psi\rangle = \text{NOT}_2 A_2 |\psi\rangle \]

\[ = (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) (\begin{pmatrix} 0 \\ -1 \end{pmatrix}) |\psi\rangle \]

\[ = (\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}} (|10\rangle - |11\rangle) \]

We detect \[ |\beta_{10}\rangle = \frac{1}{\sqrt{2}} (|100\rangle - |111\rangle) \].

(d) Our NOT (half wave plate) is the usual \[ \text{NOT} \].

Our 90° rotation, \( A \), is \( A = -i Y \).

Our combined \( A_2 \cdot \text{NOT} = (\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix})(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = (\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}) \).

3(a) \[ a \times a' \]

\[ b \times b' \]

\[ \Rightarrow \text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

(6) \[ |\psi\rangle = (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \times \frac{1}{\sqrt{2}} \]
\[
\begin{aligned}
\Rightarrow \quad \ket{14} &= \frac{1}{2} \left( \begin{array}{c} x \\ y \\ \beta x \\ \beta y \\ \beta z \\ \beta \delta \\
\end{array} \right) \\
\end{aligned}
\]

\[
\text{and } \text{SWAP} \ket{\psi} = \frac{1}{2} \left( \begin{array}{c} x \\ y \\ \beta x \\ \beta y \\ \beta z \\ \beta \delta \\
\end{array} \right)
\]

\[
= \frac{1}{2} \left( \ket{x10} + \beta \ket{y11} + \beta \delta \ket{z11} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{\sqrt{2}} \left( \ket{x10} + \beta \ket{y11} + \beta \delta \ket{z11} \right) \right)
\]

\[
= \frac{1}{2} \left( \ket{x10} + \beta \ket{y11} \right) \left( \ket{\alpha 10} + \beta \ket{\alpha 11} \right) = \ket{\psi_2} \ket{\psi_1},
\]

meaning qubit 1 is in \ket{\psi_2}, and qubit 2 in \ket{\psi_1}. This is what SWAP should do.

4(b) To show that \( C(2) = \)

\[
\begin{align*}
\text{First, if } c &= 0, \text{ then } t \rightarrow H^2 t = t. \quad \text{If } c &= 1, \\
\text{then } \ket{t} &= \left( \begin{array}{c} x \\ y \\ \alpha - \beta \\ \alpha + \beta \\ \beta \\ \delta \\
\end{array} \right) \rightarrow \frac{1}{\sqrt{2}} \left( \begin{array}{c} \alpha + \beta \\ \alpha - \beta \\ \beta \\ \delta \\
\end{array} \right) \rightarrow \frac{1}{\sqrt{2}} \left( \begin{array}{c} \alpha - \beta \\ \alpha + \beta \\ \beta \\ \delta \\
\end{array} \right) \\
&= \frac{1}{\sqrt{2}} \left( \alpha - \beta + (\alpha + \beta) \right) = (\frac{\alpha}{\beta}) = 2(\frac{x}{y}) \\
&= \frac{1}{\sqrt{2}} \left( \alpha - \beta - (\alpha - \beta) \right) = (\frac{-\beta}{\beta}) = 2(\frac{x}{y})
\end{align*}
\]

so we have exactly a \( C(2) \) gate!

\[ \text{(b) } \begin{array}{c} H \end{array} \begin{array}{c} Z \end{array} \begin{array}{c} H \end{array} = \begin{array}{c} H^2 \times H^2 \end{array} = \begin{array}{c} \text{became} \end{array} \begin{array}{c} H^2 = 1 \end{array} \text{ in the finite operation fields.} \]
5 - Not an official part of the problem set, but in case you did it -

In every case we construct the truth table, and from this the matrix:

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>a'</th>
<th>b'</th>
<th>a' b'</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

\[ (a) \]

\[
X_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

\[ (b) \]

\[
X_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

and also from the truth table,

\[
X_1 X_2 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

One can easily check that the product of the matrices is the \(X_1 X_2\) matrix, as anticipated by the notation.