Problem 1. (Based on Sakurai, Problem 5.1)
Consider a one-dimensional harmonic oscillator perturbed by a constant force
\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 - Fx \]

(a) Show that the first order perturbation to the energy vanishes. (20pts)

\[ E_n^{(1)} = -F \langle \Psi_n^{(0)} | x | \Psi_n^{(0)} \rangle \]

\[ = -F \langle x \rangle \]

\[ = 0 \] for the harmonic oscillator.

**Detail:** \[ \langle x \rangle = \frac{\hbar}{\sqrt{2m}} \]

\[ \langle n | x | n \rangle \]

\[ = \frac{\hbar}{\sqrt{2m \omega}} \left( \langle n | a^+ | n \rangle + \langle n | a | n \rangle \right) \]

\[ = \frac{\hbar}{\sqrt{2m \omega}} \left( \sqrt{n+1} \delta_{n,n+1} + \sqrt{n} \delta_{n,n-1} \right) \]

\[ \downarrow \]

\[ 0 \]

\[ 0 \]
Problem 1(b). Calculate the eigenenergies $E_n$ to second order in the perturbation. (25pts)

From Eq. [2.14], page 226:

$$E^{(2)}_n = (-F)^2 \sum_{m \neq n} \frac{|| <m^{(0)} | \times | n^{(0)} > ||^2}{E^{(0)}_n - E^{(0)}_m}$$

The matrix elements:

$$<m | \times | n> = \sqrt{\frac{\hbar}{2mw}} \left( \sqrt{n+1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1} \right)$$

and

$$||<m | \times | n>||^2 = \frac{\hbar}{2mw} \left( (n+1) \delta_{m,n+1} + n \delta_{m,n-1} \right)$$

\* Note: Cross terms vanish, $\delta_{m,n+1} \delta_{m,n-1} = 0$.

So the infinite series has only 2 non-zero terms.

$$E^{(2)}_n = F^2 \left( \frac{\hbar}{2mw} \right) \left( \frac{n+1}{E^{(0)}_n - E^{(0)}_{n+1}} + \frac{n}{E^{(0)}_n - E^{(0)}_{n-1}} \right)$$

Recall $E_n = (\frac{1}{2} + n) \hbar \omega$.

$$E^{(2)}_n = -\frac{F^2}{2mw^2}$$

or

$$E_n \approx (n + \frac{1}{2}) \hbar \omega - F^2 / 2mw^2$$
Problem 2. Consider a one-dimensional harmonic oscillator with a perturbation to the Hamiltonian of the form $\beta x^4$. (This can be called an anharmonic oscillator.)

$$H = \frac{p^2}{2m} + \frac{1}{2}m \omega^2 x^2 + \beta x^4$$

What is the first-order correction to the ground state energy? (45 pts)

$$E_0^{(1)} = \beta \langle 0 | \hat{x}^4 | 0 \rangle = \beta \langle 0 | \hat{x}^2 \hat{x}^2 | 0 \rangle$$

$\hat{x}^2$ is Hermitian.

$$= \beta \langle x^2 | 0 | x^2 | 0 \rangle.$$ 

Now

$$x^2 | 0 \rangle = \frac{\hbar}{2m\omega} \left( (a^\dagger)^2 + a a^\dagger + a^\dagger a + a^2 \right) | 0 \rangle$$

but $a | 0 \rangle \equiv 0$.

So

$$x^2 | 0 \rangle = \frac{\hbar}{2m\omega} \left( \sqrt{2} | 2 \rangle + | 1 \rangle \right)$$

and

$$E_0^{(1)} = \beta \frac{\hbar^2}{4m^2 \omega^2} \left( \langle 2 | \sqrt{2} + \langle 1 | 1 \rangle \right) \left( \sqrt{2} | 2 \rangle + | 1 \rangle \right)$$

$$= 3 \beta \left( \frac{\hbar}{2m\omega} \right)^2$$
Problem 3. Please turn to page 222 of Griffiths. It is assumed that the energy eigenkets and the energy spectrum of the unperturbed problem \( H^0 \) only are completely known.

Equations [6.7] and [6.8] give conditions on the corrections to the energies and wavefunctions for the perturbed problem (assuming the series converges). Write down the next equation in this sequence, i.e. collect the terms of third order. (10pts)

\[
\lambda^3 : \quad H^0 \phi_n^{(3)} + H' \phi_n^{(2)} = E_n^{(0)} \phi_n^{(3)} + E_n^{(1)} \phi_n^{(2)} + E_n^{(2)} \phi_n^{(1)} + E_n^{(3)} \phi_n^{(0)}.
\]

OR

\[
\left( H^{(0)} - E_n^{(0)} \right) \phi_n^{(3)} = \left( E_n^{(1)} - H' \right) \phi_n^{(2)} + E_n^{(2)} \phi_n^{(1)} + E_n^{(3)} \phi_n^{(0)}.
\]