Solutions to Homework Set 6

1(a) In the "B-only" region the electron's path is circular, so $|\vec{v}|$ and $|\vec{p}|$ are time-independent, and
\[
\vec{F} = -e \vec{v} \times \vec{B} = \frac{d\vec{p}}{dt}
\]
becomes, in magnitude,
\[
euvB = p \frac{d\theta}{dt} = p \frac{v}{r}
\]
\[
\Rightarrow r = \frac{p}{eB} \quad \text{and} \quad p = mv \quad \text{with relativistic mass} \quad m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

(b) The center of the circle lies directly below the exit collimating slit of the velocity selector, so the impact point lies a horizontal distance $l$ and a vertical distance $(r-y)$ from this center. Therefore,
\[
r^2 = l^2 + (r-y)^2 = l^2 + r^2 - 2ry + y^2
\]
\[
\Rightarrow r = \frac{(l^2 + y^2)}{2y}
\]
and also,
\[
y^2 - 2ry + l^2 = 0
\]
\[
\Rightarrow y = r \pm \sqrt{r^2 - l^2} \quad \text{where only $\Theta$ soln. makes sense because $y < r$ (or no contact)}
Incidentally (not required in your answer) the + solution corresponds to the "return" contact point that you would see if you removed the upper portion of the screen.

\( U = \frac{E}{B} = \frac{V}{Bd} = \frac{1060 \text{ volts}}{0.0177 \text{ T}(2.51 \times 10^{-4} \text{ m})} \)

\( \Rightarrow U = 2.39 \times 10^8 \text{ m/s}, \quad \frac{U}{c} = 0.795, \quad \gamma = \frac{1}{\sqrt{1 - \frac{U^2}{c^2}}} = 1.65 \).

Correct \( r \):

\[ r = \frac{\gamma m_0 V}{e B} = \frac{1.65 \times (9.11 \times 10^{-31} \text{ kg})(2.39 \times 10^8 \text{ m/s})}{(1.60 \times 10^{-19} \text{ kg})(0.0177 \text{ T})} = 0.132 \text{ m} \]

\( \Rightarrow y = r^2 - \sqrt{r^2 - (0.0247 \text{ m})^2} = 0.0023 \text{ m} \)

Incorrect \( r \):

\[ r = \frac{m_0 V}{e B} = \frac{0.132 \text{ m}}{1.65} = 0.080 \text{ m} \]

\( y = r^2 - \sqrt{r^2 - (0.0247 \text{ m})^2} = 0.0039 \text{ m} \)

Neumann's experiment showed \( y = (0.0024 \pm 0.0005) \text{ m} \), so the (correct) relativistic prediction agrees with experiment while the nonrelativistic calculation clearly does not!

**Footnote:** If you fear that you lost theoretical precision in calculating \( y \), note that the approximation \( y \approx \frac{l^2}{2v} \) (valid because \( l \ll v \)) gives you exactly the same results to two significant figures.
2) Helmholtz coils are placed at $z = \pm \frac{1}{2}a$, both centered on the $z$-axis as shown. Using the expression for $B$ along the axis of a single coil, we can write the expression for $B$ along the $z$-axis due to both Helmholtz coils:

$$B = \frac{1}{2} \mu_0 NI a^2 \left\{ \frac{1}{[a^2 + (z + \frac{1}{2}a)^2]^{3/2}} + \frac{1}{[a^2 + (z - \frac{1}{2}a)^2]^{3/2}} \right\}$$

$$= \frac{1}{2} \mu_0 NI a^2 f(z).$$

3) Find $x$ such that $f'(x)$ and $f''(x)$ both vanish:

$$f'(x) = -\frac{3(x + \frac{1}{2}a)}{\lbrack \ldots + \ldots \rbrack^{5/2}} - \frac{3(x - \frac{1}{2}a)}{\lbrack \ldots - \ldots \rbrack^{5/2}}$$

$$f''(x) = -\frac{3}{\lbrack \ldots + \ldots \rbrack^{7/2}} - \frac{3}{\lbrack \ldots - \ldots \rbrack^{7/2}}$$

$$+ \frac{15(x + \frac{1}{2}a)^2}{\lbrack \ldots + \ldots \rbrack^{7/2}} + \frac{15(x - \frac{1}{2}a)^2}{\lbrack \ldots - \ldots \rbrack^{7/2}}$$

Now $f'(x) = 0$ for all $x$, but not so for $f''(x)$;

$$f''(x) = -\frac{6}{\lbrack a^2 + \frac{1}{4}a^2 \rbrack^{5/2}} + \frac{30(\frac{1}{4}a^2)}{\lbrack a^2 + \frac{1}{4}a^2 \rbrack^{7/2}}$$

Put this expression over the common denominator $\lbrack \ldots \rbrack^{7/2}$.
\[ f''(0) = -6 \left[ a^2 + \frac{1}{4} d^2 \right] + 30 \left( \frac{1}{4} d^2 \right) \left[ a^2 + \frac{1}{4} d^2 \right]^{7/2} = 0 \]

\[ \Rightarrow -6a^2 - \frac{3}{2}d^2 + \frac{15}{2}d^2 = 0 \]

\[ \Rightarrow d = a \]

\[ (b) \quad f(0) = \frac{2}{\left[ a^2 + \frac{1}{4} d^2 \right]^{3/2}} \quad \Rightarrow \quad \frac{2}{\left( \frac{5}{4} a^2 \right)^{3/2}} = \frac{16}{\sqrt{125} \ a^3} \]

\[ B = \frac{1}{2} \mu_0 N I a^2 f(0) = \frac{8 \mu_0 N I}{\sqrt{125} \ a} \]

Incidentally, we can compare the magnetic field strength at the center and at the position of either coil:

\[ f\left( \frac{d}{2} \right) = \frac{1}{\left[ a^2 + a^2 \right]^{3/2}} + \frac{1}{a^3} \rightarrow \left( \frac{1}{2\sqrt{2}} + 1 \right) \frac{1}{a^3} \]

\[ f\left( \frac{d}{2} \right) = 1.354/a^3 \quad \text{whereas} \quad f(0) = 1.431/a^3 \]

and the ratio is \[ \frac{f\left( \frac{d}{2} \right)}{f(0)} = 0.946 \]

so B falls slowly!
3(a) \[ P_{E_{\text{max}}} = \frac{kQ_1Q_2}{R_n} = \frac{k(2e)(13e)}{R_n} = \frac{26ke^2}{R_n} \],

\[ R_n = (1.2 \times 10^{-15} \text{ m}) \cdot \frac{A}{3} = 3.6 \times 10^{-15} \text{ m} \] (since \( A = 27 \)).

\[ \Rightarrow P_{E_{\text{max}}} = \frac{26ke^2}{3.6 \times 10^{-15} \text{ m}} = 10.4 \text{ MeV}. \]

(b) Initially, only the \( \alpha \) is moving, with

\[ K_{E_{\text{max}}} = \frac{1}{2} m_\alpha U_{\text{max}}^2 \]

At moment of closest approach, the \( \alpha \) and \( \text{Al} \) move together (at "contact" velocity \( U_c \)), and total energy is conserved:

\[ K_{E_{\text{max}}} = P_{E_{\text{max}}} + \frac{1}{2} (m_\alpha + m_{\text{Al}}) U_c^2. \]

But momentum conservation requires that

\[ m_\alpha U_{\text{max}} = (m_\alpha + m_{\text{Al}}) U_c \]

\[ \Rightarrow U_c = \frac{m_\alpha}{(m_\alpha + m_{\text{Al}})} U_{\text{max}}, \text{ so substitute:} \]

\[ K_{E_{\text{max}}} = P_{E_{\text{max}}} + \frac{m_\alpha^2}{2(m_\alpha + m_{\text{Al}})} U_{\text{max}}^2 = P_{E_{\text{max}}} + \left( \frac{m_\alpha}{(m_\alpha + m_{\text{Al}})} \right) K_{E_{\text{max}}} \]

\[ \Rightarrow K_{E_{\text{max}}} = \frac{m_\alpha + m_{\text{Al}}}{m_{\text{Al}}} P_{E_{\text{max}}} = \frac{31}{27} P_{E_{\text{max}}}, \]

or \[ K_{E_{\text{max}}} = 11.9 \text{ MeV}. \]