1a) middle C: \[ \lambda_c = \frac{v}{f} = \frac{340 \text{ m/s}}{256 \text{ /s}} = 1.3 \text{ m} \quad (4.3 \text{ ft.}) \]

   lowest: \[ \lambda_L = \frac{340 \text{ m/s}}{20 \text{ /s}} = 17 \text{ m} \quad (55 \text{ ft.}) \]

   highest: \[ \lambda_H = \frac{340 \text{ m/s}}{20,000 \text{ /s}} = 1.7 \text{ cm} \quad (.66 \text{ in.}) \]

1b) Observed frequencies and (closed/open) conditions are, in ascending order of \( f \),

<table>
<thead>
<tr>
<th>frequency</th>
<th>left end</th>
<th>right end</th>
</tr>
</thead>
<tbody>
<tr>
<td>128 s(^{-1})</td>
<td>open</td>
<td>open</td>
</tr>
<tr>
<td>192 &quot;</td>
<td>open</td>
<td>closed</td>
</tr>
<tr>
<td>256 &quot;</td>
<td>open</td>
<td>open</td>
</tr>
<tr>
<td>320 &quot;</td>
<td>open</td>
<td>closed</td>
</tr>
</tbody>
</table>

Frequencies correspond to \( f_n = 64n \text{ (s}^{-1}) \) with \( n = 2, 3, 4, 5 \) (the fundamental is absent), so

\[ n = 2 \]

\[ \begin{array}{c}
\text{3} \\
\text{4} \\
\text{5}
\end{array} \]

\[ \sqrt{\text{one wavelength fits in pipe when } n = 4, \text{ so length of pipe is}} \]

\[ L = \frac{\lambda_4}{64 \times 4 \text{ /s}} = 1.3 \text{ m} \]

(It's middle C)
2a) Cartesian:
\[ z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) \]

Polar: \[ z_1z_2 = r_1r_2 e^{i(A + B)} \]

Euler's identity says that \( x_1 = r_1 \cos A \); \( y_1 = r_1 \sin A \), etc. so Cartesian form can be written as
\[ z_1z_2 = r_1r_2 \left[ \cos A \cos B - \sin A \sin B + i (\cos A \sin B + \sin A \cos B) \right] \]

Polar is just \[ z_1z_2 = r_1r_2 \left( \cos(A + B) + i \sin(A + B) \right) \].

Equating last two expressions (real and imaginary parts separately):
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \quad (i) \]
\[ \sin(A + B) = \cos A \sin B + \sin A \cos B \quad (ii) \]

2b) Change \( B \rightarrow -B \) and use \( \cos(-B) = \cos B \) and \( \sin(-B) = -\sin B \):
\[ \cos(A - B) = \cos A \cos B + \sin A \sin B \quad (iii) \]
\[ \sin(A - B) = \sin A \cos B - \cos A \sin B \quad (iv) \]

2c)
\[ (i + iii) \Rightarrow \cos(A + B) + \cos(A - B) = 2 \cos A \cos B \]
\[ (i - iii) \Rightarrow \cos(A + B) - \cos(A - B) = -2 \sin A \sin B \]
\[ (ii + iv) \Rightarrow \sin(A + B) + \sin(A - B) = 2 \sin A \cos B \]
\[ (ii - iv) \Rightarrow \sin(A + B) - \sin(A - B) = 2 \cos A \sin B \]
3a) On handout, \( \Psi = A \sin 2\pi f_1 t + A \sin 2\pi f_2 t \).

Define average, \( \bar{f} = \frac{1}{2} (f_1 + f_2) \), and difference, \( \Delta f = f_2 - f_1 \).

Invert, \( f_1 = \bar{f} - \frac{1}{2} \Delta f \), \( f_2 = \bar{f} + \frac{1}{2} \Delta f \);

substitute, \( \Psi = A \left[ \sin 2\pi (\bar{f} - \frac{1}{2} \Delta f) t + \sin 2\pi (\bar{f} + \frac{1}{2} \Delta f) t \right] \);

and use a trig identity (\( A = 2\pi \bar{f} t \) and \( B = \pi \Delta f t \)):

\( \Psi = 2A \sin 2\pi \bar{f} t \cos \pi \Delta f t \)

interior wave, envelop wave

The frequency of either wave is the coefficient of \( 2\pi t \),

so the interior wave has frequency \( \bar{f} \) and the

envelop wave has frequency \( \frac{1}{2} \Delta f \).

3b) The intensity of the envelop wave is

\( I_{en} = 4A^2 \cos^2(\pi \Delta f t) \)

(The coefficient is unimportant and you could ignore it in this problem. If you time-average the interior wave, the coefficient is then \( 2A^2 \).) In any case, use

the first trig identity and set \( A = B \):

\( I_{en} = 2A^2 (1 + \cos 2\pi \Delta f t) \).

The frequency of this wave is \( \Delta f \).
3c) Including spatial as well as temporal dependence and making the waves left-directed,

\[ \Psi(x,t) = A \left[ \sin 2\pi \left( \frac{x}{\lambda_1} + f_1 t \right) + \sin 2\pi \left( \frac{x}{\lambda_2} + f_2 t \right) \right] \]

\[ = A \left[ \sin (k_1 x + \omega_1 t) + \sin (k_2 x + \omega_2 t) \right] , \]

where \( k_1 = \frac{2\pi}{\lambda_1} \), \( \omega_1 = 2\pi f_1 \), etc. According to part (a), the interior wave depends on the average of the two arguments,

\[ \bar{\Theta} = \bar{k} x + \bar{\omega} t , \text{ where } \bar{k} = \frac{1}{2} (k_1 + k_2) , \text{ etc.} , \]

and the envelope wave depends on the difference times \( \frac{1}{2} \):

\[ \frac{1}{2} \Delta \Theta = \frac{1}{2} (\Delta k \cdot x + \Delta \omega \cdot t) , \text{ where } \Delta k = k_2 - k_1 , \text{ etc.} \]

So

\[ \Psi(x,t) = 2A \sin (\bar{k} x + \bar{\omega} t) \cos \frac{1}{2} (\Delta k \cdot x + \Delta \omega \cdot t) . \]

(i) The speed of the two "original" waves, \( \bar{\omega} \), is given by

\[ \bar{\omega} = \lambda_1 \bar{f}_1 = \lambda_2 \bar{f}_2 = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} \quad (2\pi \text{ factors cancel}) \]

and it is left-directed. A crest of the envelope wave moves according to

\[ (\Delta k \cdot x + \Delta \omega \cdot t) = 0 \quad \Rightarrow \quad x = -\frac{\Delta \omega}{\Delta k} t , \]
which shows that it is left-directed ("-" sign) and its speed \( v_{en} \) is

\[
v_{en} = \frac{\Delta \omega}{\Delta k} = \frac{\omega_2 - \omega_1}{k_2 - k_1}.
\]

We must show that this ratio is equal to \( v \). To show this, use the fact that \( \omega_1 = \nu k_1 \) and \( \omega_2 = \nu k_2 \) so that

\[
v_{en} = \frac{\nu k_2 - \nu k_1}{k_2 - k_1} = \nu.
\]

(ii) The intensity of the envelope wave may be written immediately by comparing with (36):

\[
I_{en} = 2A^2(1 + \cos(\Delta k \cdot x + \Delta \omega \cdot t)).
\]

The wavelength of this wave (call it \( \lambda_1 \)) is

\[
\lambda_1 = \frac{2\pi}{\Delta k} = \frac{2\pi}{(k_2 - k_1)} \quad \varepsilon\left(\frac{2\pi \text{ times inverse}}{\text{coefficient of } x}\right)
\]

Substitute \( k_2 = \frac{2\pi}{\lambda_2} \) and \( k_1 = \frac{2\pi}{\lambda_1} \) and you get

\[
\lambda_1 = \frac{1}{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}} = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}.
\]

This can be much greater than either \( \lambda_1 \) or \( \lambda_2 \).
4) \( \psi(y, t) = A \left[ \cos(kr_1 - \omega t) + \cos(kr_2 - \omega t) \right] \),

where \( r_1 \) and \( r_2 \) are

(using approx. from class)

\[
\begin{align*}
    r_1 &\approx r - y \frac{\lambda}{2r} \\
    r_2 &\approx r + y \frac{\lambda}{2r}
\end{align*}
\]

\[\Rightarrow \psi(y, t) \approx A \left[ \cos(kr - \frac{kyd}{2r} - \omega t) + \cos(kr + \frac{kyd}{2r} - \omega t) \right] \]

\[= 2A \cos(kr - \omega t) \cos \left( \frac{kyd}{2r} \right) \]

same as in class, but different trig. identity.

\[\psi^2(y, t) = 4A^2 \cos^2(kr - \omega t) \cos^2 \left( \frac{kyd}{2r} \right) \text{ and time-ave. of} \]

\[\cos^2(kr - \omega t) \text{ is } \frac{1}{2}, \text{ so time-ave. intensity is} \]

\[s(y) = 2A^2 \cos^2 \left( \frac{kyd}{2r} \right) , \]

which is identical to result derived in class, where we started with \( \sin \) rather than \( \cos \).

Incidentally, note that we can use the same identity as in (36) to write \( s(y) \) as

\[s(y) = A^2 \left( 1 + \cos \left( \frac{kyd}{r} \right) \right) \]

which shows that the period in \( y \) (called \( \Delta y \)) is

\[\Delta y = \frac{2\pi r}{k \lambda} = \frac{\lambda}{\lambda} \frac{r}{\lambda} \text{ spacing between maxima} \]