1. In this problem we shall account for the combined relativistic effects of gravity and velocity in an orbiting satellite. The satellite is in Earth orbit at radius \( R = 4R_E \), where \( R_E = 6000 \) Km is the radius of the Earth.

   (a) How much time \( \Delta t \) is gained (or lost), \( \Delta t \) positive (or negative), by a clock on the satellite, per unit of "Earth" time, \( t \), due to the gravitational effect alone? (You may use the fact that the gravitational acceleration at the surface of the Earth is \( g = GM/R_E^2 = 9.8 \text{m/s}^2 \).)

   (b) Include the velocity effect on \( \Delta t/t \). The velocity (or \( v^2 \)) can be expressed in terms of \( G, M, \) and \( R_E \), or just \( g \), and \( v/c \) is small enough that you can expand the usual square root factor. As a result, the velocity contribution to \( \Delta t/t \) can be expressed a ratio of the gravitational contribution. What is this ratio, and what is its sign? What is the final result for \( \Delta t/t \)?

   (c) The Global Positioning System (GPS) consists of an array of very precise atomic clocks orbiting the Earth on satellites at about \( 4R_E \). The resolution of the GPS system, which in capable of determining distances on the Earth's surface to within about \( 10 \text{m} \), is limited by the stability of the atomic clocks, which is about 1 part in \( 10^{11} \). Are the relativistic effects found above large enough that the GPS system must correct for them?

2. How much mass \( M \) must be placed inside a sphere of radius \( R \) in order to make a photon execute circular motion at this radius? (This is an estimate of the mass required to create a black hole - an object from which even photons cannot escape.)

3. (a) A photon Compton-scatters from an electron through 90°. Find a simple expression for the fractional energy loss by the photon, defined as \( \mathcal{F} \equiv (f - f')/f \), as a function of the incident photon energy in units of the electron rest energy, \( hf/m_o c^2 \).

   (b) Evaluate \( \mathcal{F} \) for the two cases, \( hf/m_o c^2 = 1/4 \), and \( hf/m_o c^2 = 4 \).

   (c) At what angle \( \phi \) does the electron scatter in each case (in degrees)?

Comment: These results demonstrate that the collision is almost elastic when \( hf/m_o c^2 \) is small (the photon is "light"), and very inelastic when \( hf/m_o c^2 \) is large (the photon is "heavy" compared to an electron at rest).

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4. A particle and its antiparticle, both of rest mass $m_o$, annihilate with the emission of two photons. If these particles are at rest, the two photons must come out in opposite directions and with equal frequencies in order to conserve momentum. Now suppose that the two particles are moving together with velocity $\vec{v}$, and the two photons are ejected parallel to this velocity, one forward and one backward.

(a) Use energy and momentum conservation to find the frequencies of the two photons. Show that one is “blueshifted” ($f_b$) and the other “redshifted” ($f_r$) with respect to the photons emitted from particles at rest, ($f$):

$$f_b = f \sqrt{\frac{1 + v/c}{1 - v/c}} \quad \text{and} \quad f_r = f \sqrt{\frac{1 - v/c}{1 + v/c}}.$$

(b) As a check on these results, explain how you could have obtained them by applying the conventional theory of the Doppler effect for light, with formulae as discussed on p. 12 of Beiser, for example.