Week #1: Friday, March 28  
Reading: Bender & Orszag, Chapter 6  
Homework: Due in class, Friday, April 4.

1. Dimensionless nonlinear pendulum.  
The equation of motion for a pendulum is  
\[ mL \frac{d^2 \Theta}{dt^2} = -mg \sin \Theta \]

where \( m \) is the swinging mass and \( L \) the length of the pendulum (assumed massless) and \( \Theta \) the angle with respect to vertical, in radians. Let the initial conditions be \( \Theta(0) = \alpha \) and \( \frac{d\Theta}{dt} = 0 \). Assume \( \alpha << 1 \). Find a dimensionless version of the differential equation, and solve it to first order in the dimensionless small parameter of your choice. Of course, the zeroth order solution will be the simple harmonic oscillator.

2. In lecture we found an expansion for the motion of a projectile shot straight up from the surface of Planet Earth (with no atmosphere). Plot the zeroth order solution and the perturbed solution for various values of the small parameter \( \epsilon \).

3. Find the next term (order \( \epsilon^2 \)) in the expansion for the projectile problem. For what values of \( \epsilon \) is the second order term necessary? What is the corresponding initial velocity?

4. Divergent asymptotic expansion may be better than convergent expansion. 
(a) Derive the asymptotic expansion of \( E_{i b}(x) \) and hence of \( \gamma(a, x) \), for \( x \to \infty \).  
(b) How many terms must be kept in order to calculate \( \gamma(\frac{1}{2}, 10) \) to 1% accuracy?  
(c) How many terms must be kept in the (convergent) Taylor series for the same accuracy?

Note:
\[ \Gamma(a) \equiv \int_0^\infty dt e^{-t} t^{a-1}, \quad a > 0 \text{ for convergence} \]
\[ \gamma(a, x) \equiv \int_0^x dt e^{-t} t^{a-1} = \Gamma(a) = E_{i -a}(x) \]

compare
\[ E_{i b}(x) = \int_x^\infty dt e^{-t} t^{-b}, \quad \text{any } b \]

Convergent expansion:
\[ \gamma(a, x) = \int_0^x dt (1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots) t^{a-1} = \frac{x^a}{a} - \frac{x^{a+1}}{a + 1} + \frac{x^{a+2}}{(a + 2)2!} - \ldots = x^a \sum_{n=0}^\infty \frac{(-1)^n x^n}{(a + n)n!} \]

(Here “=” means convergent.) Divergent expansion:
\[ \gamma(a, x) \sim \Gamma(a) - x^a \sum_{n=1}^\infty \frac{c_n}{x^n} e^{-x} \]

You’ll compute the \( c_n \) and show that the divergent series needs many fewer terms.