Section IV.7: Generalized Induction

We now give a generalized induction principle for proving statements about any well ordered set (see Anderson).

The Generalized Induction Principle
Let $X$ be a set well ordered by a relation $<$. Let $S(x)$ be a statement about an element $x$ of $X$. To prove that $S(x)$ is true for all $x \in X$, it suffices to prove the following.

(i) $S(x_0)$ is true, where $x_0$ is the least element of $X$.

(ii) For any $x \in X$ such that $x_0 < x$, if $S(y)$ is true for all $y < x$, then $S(x)$ is also true.

As with other forms of induction, (i) is called the basis, (ii) is the induction step, and the antecedent of the conditional in (ii) is the induction hypothesis. If $X$ is the set of integers greater than or equal to some $x_0$ and $<$ is the usual order relation on the integers, then the generalized induction principle is identical to the strong induction principle.

We can appeal to the properties of well ordering to prove the validity of the generalized induction principle. As a special case, the following theorem also asserts the validity of the strong induction principle.

Theorem IV.7.1: The Validity of the Generalized Induction Principle
The generalized induction principle is a valid proof technique.

Proof:
Suppose that $X$ is a set well ordered by the relation $<$ and that $S(x)$ is a statement about an element $x$ of $X$ for which we have proven (i) and (ii) in the statement of the generalized induction principle. We must show that it then follows that $S(x)$ is true for all $x \in X$. We do so by contradiction. So suppose that $S(x)$ is not true for all $x \in X$. Then let

$$A = \{ x \in X \mid S(x) \text{ is false} \}$$

Since (by assumption) $S(x)$ is not true for all $x \in A$, $A$ is a nonempty subset of $X$. Since $X$ is well ordered by $<$, $A$ contains a least element $a_0$. By definition, $S(a_0)$ is false, and, furthermore, $a_0$ is the least element of $X$ for which $S(x)$ is false. Thus, $S(y)$ is true for all $y < a_0$ (if there are any such $y$). There are two cases, depending on whether or not $a_0$ is the least element in $X$.

Case 1 ($a_0$ is the least element in $X$): In this case, the falsity of $S(a_0)$ contradicts the fact that $S(a_0)$ was proven true by step (i) in the generalized induction principle.

Case 2 ($a_0$ is not the least element in $X$): In this case, $S(y)$ is true for all $y < a_0$ (since $a_0$ is the least element of $X$ for which $S(x)$ is false). From this and from step (ii) in the generalized induction principle, it follows that $S(a_0)$ is true − contradicting the falsity of $S(a_0)$.

Thus, in either case we have a contradiction. Hence $S(x)$ is true for all $x \in X$ (that is, $A$ is empty, and there is no least element $a_0$ of $A$).
Example IV.7.1:
Consider the following sequence of natural numbers:
\[ s_{0,0} = 0 \]
For any pair \( (m,n) \in \mathbb{N}^2 \) except \( (0,0) \),
\[ s_{m,n} = \begin{cases} 
  s_{m-1,n} + 1 & \text{if } n = 0 \\
  s_{m,n-1} + 1 & \text{otherwise}
\end{cases} \]
We wish to show that, for all pairs \( (m,n) \in \mathbb{N}^2 \), \( s_{m,n} = m + n \). We do so by generalized induction on the set \( \mathbb{N}^2 \), which is well ordered by the simple lexicographical ordering \( \prec \).

**Proof:**

**Basis:** We prove that \( s_{m,n} = m + n \) is true where \( (m,n) \) is the least element of \( \mathbb{N}^2 \). Since \( (0,0) \) is the least element in \( \mathbb{N}^2 \), we need to show that
\[ s_{0,0} = 0 + 0 = 0 \]
But this is true by the definition of \( s_{0,0} \).

**Induction Step:** We prove, for all \( (m,n) \in \mathbb{N}^2 \) such that \( (0,0) \prec (m,n) \), that, if
\[ s_{m',n'} = m' + n' \] is true for all \( (m',n') \prec (m,n) \)
then
\[ s_{m,n} = m + n \]
So assume (as the induction hypothesis) that
\[ s_{m',n'} = m' + n' \] is true for all \( (m',n') \)
It remains, then, to prove that \( s_{m,n} = m + n \). To do so, we distinguish two cases, following the definition of \( s_{m,n} \).

**Case 1** \( (n = 0) \): In this case, we have by definition \( s_{m,n} = s_{m-1,n} + 1 \). But
\( (m-1,n) \prec (m,n) \), so, by the induction hypothesis, \( s_{m-1,n} = (m - 1) + n \). Therefore,
\[ s_{m,n} = s_{m-1,n} + 1 = (m - 1 + n) + 1 = m + n, \]
as required.

**Case 2** \( (n \neq 0) \): In this case, we have by definition \( s_{m,n} = s_{m,n-1} + 1 \). But
\( (m,n-1) \prec (m,n) \), so, by the induction hypothesis, \( s_{m,n-1} = m + (n - 1) \). Therefore,
\[ s_{m,n} = s_{m,n-1} + 1 = (m + n - 1) + 1 = m + n, \]
as required.
Exercises:

(1) Consider the following sequence of positive integers:
\[ s_{1,1} = 5 \]
For any pair \( \langle m, n \rangle \) of positive integers except \( \langle 1, 1 \rangle \),
\[ s_{m,n} = \begin{cases} 
  s_{m-1,n} + 2 & \text{if } n = 1 \\
  s_{m,n-1} + 2 & \text{if } n \neq 1 
\end{cases} \]
Prove by (generalized) induction that, for all pairs of positive integers \( \langle m, n \rangle \),
\[ s_{m,n} = 2 \cdot (m + n) + 1 \]
(Note that the induction here is over pairs of positive integers, with least element \( \langle 1, 1 \rangle \).)

(2) Consider the following sequence of natural numbers:
\[ s_{0,0} = 0 \]
For any pair \( \langle m, n \rangle \in \mathbb{N}^2 \) except \( \langle 0, 0 \rangle \),
\[ s_{m,n} = \begin{cases} 
  s_{m-1,n} + 1 & \text{if } n = 0 \\
  s_{m,n-1} + n & \text{if } n \neq 0 
\end{cases} \]
Prove by (generalized) induction that, for all pairs of natural numbers \( \langle m, n \rangle \),
\[ s_{m,n} = \frac{n(n+1)}{2} + m \]