Topic 1b: Uncertainty

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Based Primarily on Varian, Ch. 12

Uncertainty

- Choice under uncertainty is choice between (risky) prospects or lotteries.
  - Choice between not carrying and carrying an umbrella if you don’t know what the weather will be:
    - either: do not carry umbrella when
      - it is raining
      - it is not raining
    - or: carry umbrella when
      - it is raining
      - it is not raining
  - How do we make such choices?
Uncertainty, cont’d

- There are a (fixed) number of states of the world ("states of nature"), and they are mutually exclusive:
  - e.g. either it is raining or it is not raining
- In each state of nature, there is an outcome for the agent:
  - e.g. get wet or stay dry; inconvenience of carrying an umbrella
- We can make choices that influence the outcome in each state:
  - e.g. carry an umbrella or not carry an umbrella
- Assumption: we can ascribe probabilities to each state of nature happening, and probabilities of exhaustive and mutually exclusive states sum to one.

<table>
<thead>
<tr>
<th>choice: umbrella</th>
<th>state of nature: rain</th>
<th>outcome: stay dry + carry umbrella</th>
</tr>
</thead>
<tbody>
<tr>
<td>no umbrella</td>
<td>no rain</td>
<td>stay dry + carry umbrella</td>
</tr>
<tr>
<td></td>
<td>rain</td>
<td>get wet + don’t carry umbrella</td>
</tr>
<tr>
<td></td>
<td>no rain</td>
<td>stay dry + don’t carry umbrella</td>
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</tbody>
</table>
Uncertainty, cont’d

- Each prospect (or, lottery) is associated with different outcomes (one outcome in each state of nature):
- Choice depends on:
  - how likely each outcome is
  - how “good” / “bad” each outcome is
- We can assign probabilities to outcomes (e.g. the chance of rain is 80%, or: it will rain with probability 0.8)

Lotteries

- Choice under uncertainty is choice between prospects or “lotteries”:
  - if you do not carry an umbrella (lottery A):
    - rain (probability 0.8) ⇒ get wet
    - no rain (probability 0.2) ⇒ stay dry
  - if you carry an umbrella (lottery B):
    - rain (probability 0.8) ⇒ stay dry but carry umbrella
    - no rain (probability 0.2) ⇒ pointlessly carry umbrella
- Lotteries associate outcomes (or, events) with probabilities of those events happening.
Lotteries with Money Prizes

- We are often particularly interested in lotteries that have monetary outcomes. And we will only consider lotteries with at most two outcomes. For instance:
  - share price falls - share price increases
  - accident - no accident
  - loss - no loss

Lotteries: Notation

- **Notation**: Outcome 2 happens with probability \( p \). Therefore, outcome 1 happens with probability \( 1 - p \).
- In general, we can write each lottery as:
  - \( L = (\text{outcome 1, outcome 2, 1 - p, p}) \)
- **Example**: in a lottery you win $10 with probability 0.5, and nothing with probability 0.5.
  - \( L = ($10, 0, 0.5, 0.5) \)
Lotteries: Expected Value

- On average, how much should you expect to win in the lottery $L = (15, 5, 0.5, 0.5)$?
  - You should expect (on average) to win: $0.5 \times 15 + 0.5 \times 5 = 10$
  - This is the expected value of the lottery. Denote it by $EL$.
- Another example: What is the expected value of the lottery $L = (15, 5, 0.8, 0.2)$?
  - $EL = 0.8 \times 15 + 0.2 \times 5 = 13$
- Is expected value a good decision criterion?

The St Petersburg Paradox

- Should you base your decisions under uncertainty on the expected value of the prospects (lotteries) you have to choose between?
- Consider the following lottery:
  - “I toss a coin. If it comes up heads (H), I pay you $1; if it comes up tails (T), I toss again. If it then comes up H, I pay you $2; if it comes up T, I toss again. If it the comes up H, I pay you $4; if it comes up T, I toss again ...(and so on)”
The St Petersburg Paradox, cont’d

- **Coin**: Probability: Payoff:
  - H  1/2  $1
  - TH  1/2 x 1/2 = 1/4  $2
  - TTH  1/2 x 1/2 x 1/2 = 1/8  $4
  - TTTH … = 1/16  $8
  - … … …

- Expected value: 1/2 $1 + 1/4 $2 + 1/8 $4 + 1/16 $8 … = $0.5 + $0.5 + $0.5 + … = “infinity”

- Should you therefore be willing to give up all your wealth to partake in this lottery?

Preferences over Lotteries

- Clearly, expected value is not a good decision criterion. It does not take into account your attitude to risk.

- We need a theory of preferences over risky prospects: preferences over lotteries.
  - This must take into account that you may dislike risk.

- We can then use the apparatus we already have: individuals maximize utility (they choose the lottery they most prefer).
**Expected Utility**

- **Example**: Suppose you are indifferent between having $7 and the lottery \( L = (15, 5, 0.5, 0.5) \).
  - Getting $7 with certainty is really also a “degenerate” lottery: \( L' = (7, 7, 0.5, 0.5) \).
- **Indifference**: \( L \sim L' \). Suppose we can represent preferences (over lotteries) by some utility function \( v \).
  - We know that indifference implies: \( v(L) = v(L') \).
- **What is the utility** \( v(L) \) **that you get from lottery** \( L \)?
- **This utility is uncertain**. Therefore: *expected utility*.
  - With probability 0.5 you get the certain utility \( u(15) \), and with probability 0.5 you get the certain utility \( u(5) \).
  - So expected utility \( v(L) = 0.5 u(15) + 0.5 u(5) \).

**Expected Utility, cont’d**

- So we know: \( v(L) = 0.5 u(15) + 0.5 u(5) \).
- Analogously: \( v(L') = 0.5 u(7) + 0.5 u(7) \).
- Since \( L \sim L' \), we know that \( v(L) = v(L') \):
  - \( 0.5 u(15) + 0.5 u(5) = 0.5 u(7) + 0.5 u(7) \)
  - \( 0.5 u(15) - 0.5 u(7) = 0.5 u(7) - 0.5 u(5) \)
  - \( u(15) - u(7) = u(7) - u(5) \)
  - The difference between the utility of getting $15 and the utility of getting $7 is the same as the difference between the utility of getting $7 and the utility of getting $5. *Utility differences matter!*
Cardinal Utility

1. \( u(15) - u(7) = u(7) - u(5) \)
2. From \( L \sim L' \) we know the shape of the utility function: it is concave (for the specified preferences).

Cardinal Utility, cont’d

1. Introducing preferences over lotteries gives utility more structure: utility differences matter.
2. Differences are a *cardinal* property of utility.
3. A utility function \( u \) with this property is called a *von Neumann-Morgenstern* (or *vNM*) utility function.
4. Implication: the shape of the utility function matters.
5. And: the shape tells us about attitudes to risk.
Cardinal Utility, cont’d

- Cardinal properties of functions are those preserved only under positive affine transformation.
- **Definition**: A *positive affine* transformation is a function \( f(\bullet) \) of the form:
  - \( f(x) = ax + b \) (where \( a > 0 \))
  - (Sometimes functions of this form are called “increasing linear” functions: this is slightly incorrect. Strictly speaking, linear functions are functions of the form \( f(x) = ax \).)
- Differences (for instance: “\( u(7) - u(5) \) is as large as \( u(15) - u(7) \)”) are a cardinal property.

Risk Aversion and Concavity

- **Definition**: An individual is *risk averse* if she always prefers receiving (for certain) the expected value of the lottery to playing the lottery \( \Rightarrow \) concave utility function.
A Function $f(x)$ is concave if its slope is always declining.

That is, a function $f(x)$ is concave if $f'(x)$ is everywhere declining.

That is, a function $f(x)$ is concave if $f''(x)$ is negative.

- $f''(x)$ is the second derivative of $f(x)$.

**Risk Loving and Convexity**

Definition: An individual is risk loving if she always prefers playing the lottery to receiving (for certain) the expected value of the lottery $\Rightarrow$ convex utility function.
Risk Neutrality and Linearity

Definition: An individual is risk neutral if she is always indifferent between receiving the expected value of the lottery and playing the lottery ⇒ linear utility function.

Marginal Utility of Wealth

The slope of the utility function v over wealth is the marginal utility of wealth:

- risk aversion ≡ diminishing marginal utility of wealth;
- risk loving ≡ increasing marginal utility of wealth;
- risk neutrality ≡ constant marginal utility of wealth.
Suppose we know that J. Alfred Prufrock is risk averse. He is offered a gamble in which with probability $1/4$ he loses £1000, while with probability $3/4$ he wins £600. If he says no, he gets nothing. Can we say in general whether he will take the gamble?

Now suppose we know that his utility function over wealth is $u(c) = \sqrt{1000 + c}$, where $c$ is the prize in pounds. Does this make him a risk averse individual? Would he take the gamble?
The Demand for Insurance:
An Application

Actuarially Fair Insurance:
Demand for Full Insurance

The Consumption "Space"

- Two outcomes:
  - $c_g$ (consumption in the "good" state of nature)
  - $c_b$ (consumption in the "bad" state of nature)
- Endowment $(m_g, m_b)$
- The bad state occurs with probability $p$.
- Every pair (bundle) of contingent consumption is a lottery $L = (c_g, c_b, 1 - p, p)$. 

- Individuals have preferences over lotteries.
What is Insurance?

A risk-averse individual prefers any certain consumption between $c$ and $m_b$ to the lottery $L = (m_g, m_b, 1 - p, p)$. 
$c$ is the certainty equivalent of the lottery $L = (m_g, m_b, 1 - p, p)$.

Insurance

Suppose you can buy insurance at rate $\gamma$:
- If you choose to insure $1$, you pay insurance premium $\gamma$.

So:
- if the good state occurs you consume $\gamma$ less than your endowment in the good state
- if the bad state occurs you consume $(1 - \gamma)$ more than your endowment in the bad state (you pay premium $\gamma$, but you get the insured $1$).

The slope of the “budget line” is: $-(1 - \gamma) / \gamma$
Actuarially Fair Insurance

- The insurer’s expected profit when insuring $k$ is:
  - $(1 - p) \gamma k + p (\gamma k - k)$
- The premium is “actuarially fair” if the insurer makes zero (expected) profits:
  - $(1 - p) \gamma k + p (\gamma k - k) = 0$
  - $(1 - p) \gamma k = - pk (\gamma - 1)$
  - $(1 - p) \gamma k = pk (1 - \gamma)$
  - $(1 - p) \gamma = p (1 - \gamma)$
  - $(1 - p) / p = (1 - \gamma) / \gamma$

- When insurance is actuarially fair, the slope of the budget line (or “fair-odds line”) is:
  - $- (1 - p) / p$

Slopes of Indifference Curves

- Recall: slope of an indifference curve:
  - utility $u(x_1, x_2)$
  - total change in utility is zero: $du(x_1, x_2(x_1)) / dx_1 = 0$
  - $\partial u(x_1, x_2) / \partial x_1 + \partial u(x_1, x_2) / \partial x_2 • dx_2 / dx_1 = 0$
  - $\frac{dx_2}{dx_1} = - \frac{MU_1}{MU_2}$

- Slope of an indifference curve under uncertainty:
  - expected utility $v(c_g, c_b, (1-p), p) = (1-p) u(c_g) + p u(c_b)$
  - total change is zero: $dv(c_g, c_b(c_g), (1-p), p) / dc_g = 0$
  - $(1-p) \partial u(c_g) / \partial c_g + p • \partial u(c_b) / \partial c_b • dc_b / dc_g = 0$
  - $\frac{dc_b}{dc_g} = - (1-p) \frac{MU_g}{MU_b} / p • MU_b$
Choice

- Slope of the budget line ("fair-odds line"): \(- \frac{(1 - p)}{p}\)
- Slope of the indifference curves: \(- \frac{1-p}{p} \frac{MU_g}{MU_b}\)
- At the optimal choice, indifference curve and budget constraint are tangential, i.e. the slopes are equal. So:
  - \(\frac{1-p}{p} = \frac{MU_g}{MU_b}\)
  - \(1 - \frac{MU_g}{MU_b} = MU_g - MU_b\)

Full Insurance

- Whenever a risk-averse agent is offered insurance at an actuarially fair rate, she will choose to insure fully:
  - she will choose to equalize consumption in the good state and consumption in the bad state
  - she will locate on the "certainty line"
- This means that it is optimal for a risk-averse individual to insure away all risk, and for the risk-neutral insurer to take all the risk:
  - "optimal risk sharing"