Assignment 3 Solutions

3.1 Since the unit of length is $L$, the graph tells us that $L_x = 4L$ and $L_y = 2L$. We count nodes along each direction to get quantum numbers. There is one node along the $x$ axis; so, $n_x = 2$ ($n_x = 1$ is the ground state with no nodes). There are three nodes along $y$; so, $n_y = 4$. The general energy expression and our particular energy are

$$E = \frac{\hbar^2 \pi^2}{2m} \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right) = \frac{\hbar^2 \pi^2}{2m} \left( \frac{2^2}{4^2L^2} + \frac{4^2}{2^2L^2} \right) = \frac{17 \hbar^2 \pi^2}{8mL^2}.$$ 

The state $n_x = 8, n_y = 1$ is degenerate with this one; it has the same energy!

3.2 The oscillator frequency is related to the force constant and reduced mass through

$$\omega = 2\pi\nu = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{250 \text{ N m}^{-1}}{2.66 \times 10^{-26} \text{ kg}}} = 9.70 \times 10^{13} \text{ s}^{-1}.$$ 

or $\nu = 1.54 \times 10^{13} \text{ Hz}$, which is the number of oscillation cycles per second. The energy difference between the zero-point ground state ($\nu = 0$) and the $\nu = 1$ first excited state is

$$\hbar\omega = (1.054 \times 10^{-34} \text{ J s}) \times (9.70 \times 10^{13} \text{ s}^{-1}) = 1.02 \times 10^{-20} \text{ J} = 0.064 \text{ eV} = 515 \text{ cm}^{-1}.$$ 

In any state, the classical turning points are those positions where the total energy equals the potential energy. For the ground state, with energy $\hbar\omega/2$, we can write

$$\frac{\hbar\omega}{2} = \frac{1}{2} kx^2$$ 

and solve for $x$, which will represent one-half the distance the oscillator moves during one oscillation between maximum extension and maximum compression. We find

$$x = \sqrt{\frac{\hbar\omega}{k}} = \sqrt{\frac{1.02 \times 10^{-20} \text{ J}}{250 \text{ N m}^{-1}}} = 6.39 \times 10^{-12} \text{ m} = 0.0639 \text{ Å}.$$ 

This is a very small fraction of a typical bond length, which is on the order of 1 to 3 Å.
3.3 To find $m$, we write

$$L_z \Psi = m \frac{\hbar}{i} \frac{\partial \Psi}{\partial \phi}$$

$$= \left( \frac{\hbar}{i} \right) \left( -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta \right) \frac{\partial e^{-i\phi}}{\partial \phi}$$

$$= \left( \frac{\hbar}{i} \right) \left( -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta \right) (-i)e^{-i\phi}$$

$$= -\frac{\hbar}{i} \Psi$$

so that $m = -1$. To find $l$, we write

$$L^2 \Psi = l(l + 1) \frac{\hbar}{i} \frac{\partial^2 \Psi}{\partial \phi^2}$$

$$= -\frac{\hbar}{i} \left[ \sin \theta \cos \theta e^{-i\phi} \right]$$

which can be a big mathematical mess if we aren’t careful. To simplify things a bit, note that we can cancel the normalization constant for $\Psi$ from both sides of this equation and approach things step by step. First, we do the $\phi$ part:

$$\frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \left( \sin \theta \cos \theta e^{-i\phi} \right)$$

$$= \sin \theta \cos \theta \frac{\partial^2 e^{-i\phi}}{\sin \theta}$$

$$= \sin \theta \cos \theta \frac{\partial^2 e^{-i\phi}}{\partial \phi^2}$$

$$= \cos \theta \sin \theta (-i)^2 e^{-i\phi}$$

$$= \frac{-\cos \theta}{\sin \theta} e^{-i\phi}.$$

Then we do the $\theta$ part (carefully, setting aside the $e^{-i\phi}$ factor for the moment):

$$\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right] \left( \sin \theta \cos \theta e^{-i\phi} \right)$$

$$= \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \cos^2 \theta \sin \theta - \sin^3 \theta \right)$$

$$= \frac{\cos^3 \theta}{\sin \theta} - 5 \cos \theta \sin \theta.$$

Next, we put everything together, using one trig identity to simplify things:

$$L^2 \Psi = l(l + 1) \frac{\hbar}{i} \frac{\partial^2 \Psi}{\partial \phi^2}$$

$$= -\frac{\hbar}{i} \left[ \frac{(\cos^2 \theta - 5 \sin^2 \theta - 1) \cos \theta}{\sin \theta} \right]$$

$$= -\frac{\hbar}{i} \left[ \frac{(1 - \sin^2 \theta - 5 \sin^2 \theta - 1) \cos \theta}{\sin \theta} \right]$$

$$= -\frac{\hbar}{i} \left[ \frac{(-6 \sin^2 \theta) \cos \theta}{\sin \theta} \right]$$

$$= 6 \frac{\hbar}{i} \cos \theta \sin \theta e^{-i\phi} = 6 \frac{\hbar}{i} \Psi$$

We see that $l(l + 1) = 6$, or $l = 2$. 

3.4 If the angular momentum component along the \( z \) axis is zero, then \( m = 0 \) and the wavefunction doesn’t depend on \( \phi \). The total angular momentum eigenvalue yields \( l \):

\[
l(l + 1) \hat{l}^2 = (2/3)^2 \hat{l}^2 = 3 \cdot 4 \hat{l}^2 = 3(3 + 1) \hat{l}^2
\]

so \( l = 3 \). The entire function is the \( l = 3 m = 0 \) entry in Table 12.2:

\[
Y_{3,0}(\theta,\phi) = \sqrt{\frac{6\pi}{8}} \left( \frac{5}{3} \cos^3 \theta - \cos \theta \right).
\]

This function is zero when

\[
\frac{5}{3} \cos^3 \theta - \cos \theta = \left( \frac{5}{3} \cos^2 \theta - 1 \right) \cos \theta = 0
\]

which happens when \( \cos \theta = 0 \) (the \( xy \) plane, where \( \theta = \pi/2 \)) or when \( (5 \cos^2 \theta)/3 = 1 \) (which happens for two values of \( \theta \), \( \theta = \cos^{-1} (3/5)^{1/2} = 39.2^\circ \) or \( 180^\circ - 39.2^\circ = 147.8^\circ \)). The function looks like this:

and the nodal surfaces (cylindrically symmetric about the \( z \) axis!) are easy to visualize.