Power Electronics Analysis Techniques

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April 2, 2002

This document repeats the same information twice. First it is organized into what you should already know, what you should learn immediately, and what you should learn later. Then it repeats, organized by topic.

1 Stuff you should already know (and will need)

1.1 Element Laws and Energy Storage

- $i_C = Cdv_c/dt$
- $E_C = \frac{1}{2}Cv_c^2$
- $v_L = Ldi/dt$
- $E_L = \frac{1}{2}Li_L^2$

1.2 Power Calculations

- $P(t) = i(t)v(t)$
- Power in a resistor: $\overline{P} = V_{rms}^2/R = I_{rms}^2 R$

1.3 Techniques to avoid

The following techniques are used relatively infrequently for analyzing power electronics circuits. If you find yourself using them, ask if there's an easier way:

- Laplace Transform.
- LTI small signal analysis. Switching circuits are inherently time varying, and are often nonlinear. Exception: For analyzing feedback control of power circuits, linearizing a discrete time or averaged model is often useful.

1.4 Differential equations without the Laplace transform

1.4.1 Integration

If you can figure out $v_L(t)$ or $i_C(t)$ from analyzing the other parts of the circuit, or using the constant approximation, then all you need to do is integrate $i_C = Cdv_c/dt$ or $v_L = Ldi/dt$ to get $i_L(t)$ or $v_C(t)$ Often, you are just integrating a constant to get a ramp, sometimes integrating a ramp to get a parabola. So this is quite easy mathematically.

1.4.2 Known solutions

If all of the above fails, you can analyze a circuit using differential equations. Yuck! But it can be easy. It is rare that you will need to do this but if you do, much of the time it will either be a first order system or an undamped second order system (remember, we avoid resistors whenever possible).

First order systems

You know the solution will be $x(t) = k_1 e^{-t/\tau} + k_2$, which is sometimes better written as $x(t) = k_3[1 - e^{-t/\tau}] + k_4$. The time constant τ is either L/R or RC. (Be careful if there's more than one resistor—if there is the R used to calculate the time constant may be a series or parallel combination of the resistor values.) You can find the k's by matching initial and final conditions. You can find initial conditions by noting that:

- voltage on a capacitor doesn't change instantaneously
- current in an inductor doesn't change instantaneously

Final conditions are usually only hypothetical, because usually a new switching event will come along before steady state (real dead, nothing-happening steady-state, not periodic steady-state) is reached. But for the purpose of fitting a solution, we can still find the values it *would have* reached. To do this we solve the circuit, but because derivatives are zero,

- $i_C(t = \infty) = 0$
- $v_L(t = \infty) = 0$.

This can be accomplished on the circuit diagram by replacing caps with open circuits and inductors with shorts.

2 Stuff to Learn Right Away

2.1 Definitions and relationships for periodic steady-state

- Average values may be computed over a period T, i.e., if $x = x(t)$ = some variable such as current or voltage, $X_{dc} = \overline{x} = \frac{1}{T}\int_0^T x(t)dt$
- ac and dc components can be separated: $x_{ac}(t) = x(t) - X_{dc}$
- RMS values are defined by

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2(t)dt} \tag{1}$$

- ac and dc components of the rms value are related by

$$X_{rms} = \sqrt{X_{ac,rms}^2 + X_{dc,rms}^2} \tag{2}$$

  where $X_{dc,rms} = X_{dc} = \overline{x}$ and

$$X_{ac,rms} = rms\{x_{ac}(t)\} = \sqrt{\frac{1}{T}\int_0^T x_{ac}^2(t)dt} \tag{3}$$

2.2 Power Calculations

- Power loss in components:
  - With parasitic series resistance (ESR), $P = I_{rms}^2 R$

2.3 Circuit relationships for periodic steady-state

- On an inductor: $\overline{V_L} = 0$ (Because otherwise current ramps up or down over many cycles.)
- On a capacitor: $\overline{I_C} = 0$ (Because otherwise voltage ramps up or down over many cycles.)

2.4 Separation of dc and ac components

Note that KCL and KVL apply individually to ac and dc components. That is,

$$\sum_{loop} v_{ac}(t) = 0 \tag{4}$$

$$\sum_{loop} v_{dc} = 0 \tag{5}$$

$$\sum_{node} i_{ac}(t) = 0 \tag{6}$$

$$\sum_{node} i_{dc} = 0 \tag{7}$$

where currents are all defined entering a node (or all exiting) and voltages are taken in a consistent manner around a the loop.

2.5 Piecewise Analysis

Take each interval in the switching cycle and redraw the circuit with the closed switches replaced with shorts, and the open switches removed. This is then most likely a simple linear circuit that you can easily solve using one of the methods below. The values at the end of the interval become the initial conditions for the next interval.

2.6 Differential equations without the Laplace transform

2.6.1 Constant approximation

This is useful surprisingly often. If $i_C = Cdv_c/dt$, and C is really big, then dv_c/dt must be really small. So, at least for a short time period you can assume the voltage on a capacitor is constant. Likewise, if L is really big, then $v_L = Ldi/dt$ implies i is approximately constant. How do you know when L or C is big enough? This is another example of the method of assumed states. Assume it is constant, solve the circuit, then use integration to find out the actual behavior. If is is not approximately constant, then either make L or C bigger so that is is, or go on to a more exact solution.

3 Stuff to Learn Soon

3.1 Power Calculations

- If $v(t) = const$, i.e. pure dc.
$$\overline{P} = \overline{v(t)i(t)} = V_{dc}\overline{i(t)} = V_{dc}I_{dc} \tag{8}$$
- If $i(t) = const$, i.e. pure dc.
$$\overline{P} = \overline{v(t)i(t)} = I_{dc}\overline{v(t)} = V_{dc}I_{dc} \tag{9}$$
Thus, if either $v(t)$ or $i(t)$ is pure dc, the ac component of the other does not contribute to average power. (More generally only components of $v(t)$ and $i(t)$ at the same frequency contribute to average power.)
- Power loss in components:
  - With constant voltage drop, $P = VI_{dc}$

3.2 Method of Assumed States

This method is used to analyze circuits with diode switches. For a dc circuit, or for any given point in time:

1. Guess which diodes are on and which are off.
2. Analyze the circuit, and find the current in the diodes that are on, and the voltage across the diodes that are off.
3. Check that the current and voltage are in the allowed polarities. If any are not, you guessed wrong about those diodes. Go back to step one. If they are all OK, you are done.

4 Other useful stuff

These won't be needed a lot or right away in this course.

4.1 Fourier Series

Instead of just separating dc and ac components, we can separate a periodic waveform into frequency components at dc, the switching (fundamental) frequency and multiples of that frequency. See Krein (one of the books on reserve) Section 2.7 and 2.8 and Appendix D for a review of Fourier Series, and a coefficients for some common waveforms.

4.2 Power Calculations

- From Fourier Series:

$$\overline{P} = \sum_{n=0}^{\infty} |V_{rms,n}||I_{rms,n}|cos(\theta_n - \phi_n) \tag{10}$$

  (see Krein p. 71-72)

4.3 Differential equations without the Laplace transform

4.3.1 Known solutions

If all of the above fails, you can analyze a circuit using differential equations. Yuck! But it can be easy. It is rare that you will need to do this but if you do, much of the time it will either be a first order system or an undamped second order system (remember, we avoid resistors whenever possible)
Undamped second order systems Undamped second order systems oscillate with a frequency $f = \frac{1}{2\pi\sqrt{LC}}$. Another important parameter is the characteristic impedance $Z_c = \sqrt{L/C}$. This is the ratio of peak current to peak voltage. If the initial conditions are zero voltage and a known current, or vice-versa, Z_c can be used to find the peak value of the other parameter. If the initial conditions are non-zero for both, the peaks can be found by realizing that the energy is constant with no dissipation, i.e.,

$$E = const = \frac{1}{2}Cv^2(t) + \frac{1}{2}Li^2(t) \tag{11}$$

Given initial conditions ($i(t = 0)$ and $v(t = 0)$), one may calculate the initial total energy and equate that to $\frac{1}{2}Cv^2(t)$ for the peak voltage and to $\frac{1}{2}Li^2(t)$ for the peak current.

5 By Topic

All the stuff above repeated, organized by topic.

5.1 Definitions and relationships for periodic steady-state

- Average values may be computed over a period T, i.e., if $x = x(t) =$ some variable such as current or voltage, $X_{dc} = \overline{x} = \frac{1}{T}\int_0^T x(t)dt$
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- RMS values are defined by

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2(t)dt} \tag{12}$$

- ac and dc components of the rms value are related by

$$X_{rms} = \sqrt{X_{ac,rms}^2 + X_{dc,rms}^2} \tag{13}$$

where $X_{dc,rms} = X_{dc} = \overline{x}$ and

$$X_{ac,rms} = rms\{x_{ac}(t)\} = \sqrt{\frac{1}{T}\int_0^T x_{ac}^2(t)dt} \tag{14}$$

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- $P(t) = i(t)v(t)$
- From Fourier Series:

$$\overline{P} = \sum_{n=0}^{\infty} |V_{rms,n}||I_{rms,n}|cos(\theta_n - \phi_n) \tag{15}$$

(see Krein p. 71-72)

- Power in a resistor: $\overline{P} = V_{rms}^2/R = I_{rms}^2 R$
- If $v(t) = const$, i.e. pure dc.

$$\overline{P} = \overline{v(t)i(t)} = V_{dc}\overline{i(t)} = V_{dc}I_{dc} \tag{16}$$

- If $i(t) = const$, i.e. pure dc.

$$\overline{P} = \overline{v(t)i(t)} = I_{dc}\overline{v(t)} = V_{dc}I_{dc} \tag{17}$$

Thus, if either $v(t)$ or $i(t)$ is pure dc, the ac component of the other does not contribute to average power. (More generally only components of $v(t)$ and $i(t)$ at the same frequency contribute to average power. This is just one, very useful, application of (15).)

- Power loss in components:
  - With parasitic series resistance (ESR), $P = I_{rms}^2 R$
  - With constant voltage drop, $P = VI_{dc}$

5.4 Circuit relationships for periodic steady-state

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$$\sum_{loop} v_{dc} = 0 \tag{19}$$

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where currents are all defined entering a node (or all exiting) and voltages are taken in a consistent manner around a the loop.

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5.9 Differential equations without the Laplace transform

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$$E = const = \frac{1}{2}Cv^2(t) + \frac{1}{2}Li^2(t) \tag{22}$$

Given initial conditions ($i(t = 0)$ and $v(t = 0)$), one may calculate the initial total energy and equate that to $\frac{1}{2}Cv^2(t)$ for the peak voltage and to $\frac{1}{2}Li^2(t)$ for the peak current.