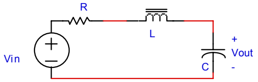
Overview of Laplace Solutions, Transfer Functions, Impedance, and Frequency Response

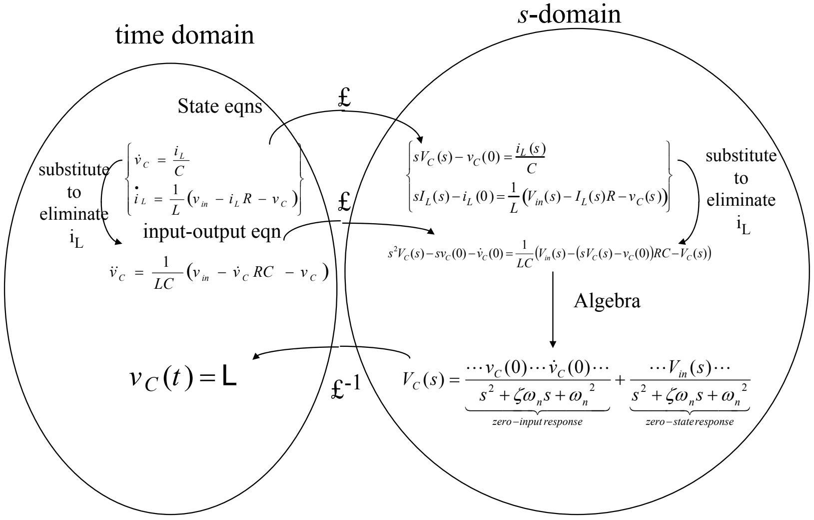
Charlie Sullivan

ENGS 22

Note: All of this works only on LTI systems!



Laplace Method



time domain

s-domain

State eqns

substitute to eliminate $\mathrm{i_L}$

$$\begin{cases}\dot{v}_C = \dfrac{i_L}{C} \\ \dot{i}_L = \dfrac{1}{L}\left(v_{in} - i_L R - v_C\right)\end{cases}$$

input-output eqn

$$\ddot{v}_C = \frac{1}{LC}\left(v_{in} - \dot{v}_C RC - v_C\right)$$

£

$$\begin{cases}sV_C(s) - v_C(0) = \dfrac{i_L(s)}{C} \\ sI_L(s) - i_L(0) = \dfrac{1}{L}\left(V_{in}(s) - I_L(s)R - v_C(s)\right)\end{cases}$$

substitute to eliminate $\mathrm{i_L}$

£

$$s^2V_C(s) - sv_C(0) - \dot{v}_C(0) = \frac{1}{LC}\left(V_{in}(s) - \left(sV_C(s) - v_C(0)\right)RC - V_C(s)\right)$$

Algebra

$$V_C(s) = \underbrace{\frac{\cdots v_C(0)\cdots \dot{v}_C(0)\cdots}{s^2 + \zeta\omega_n s + \omega_n^2}}_{zero-input\,response} + \underbrace{\frac{\cdots V_{in}(s)\cdots}{s^2 + \zeta\omega_n s + \omega_n^2}}_{zero-state\,response}$$

£-1

$$v_C(t) = \mathsf{L}$$

Two observations on Laplace Solution:

$$V_C(s) = \underbrace{\frac{\cdots v_C(0)\cdots \dot{v}_C(0)\cdots}{s^2 + \zeta\omega_n s + \omega_n^2}}_{zero\ -input\ \ response} + \underbrace{\frac{\cdots V_{in}(s)\cdots}{s^2 + \zeta\omega_n s + \omega_n^2}}_{zero\ -state\ \ response}$$

With constant, step or zero input, Responses are all

$$C + A_1e^{p_1t} + A_2e^{p_2t} + A_3e^{p_3t} + \cdots$$

Characterize by roots of denominator, $\{\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3, \ldots\}$

Real roots -> exponentials
Complex roots -> complex exponentials (damped oscillations) describe with $\underbrace{\zeta, \omega_n}$

This is total response for zero init. condits. Note that it can be written:

$$V_{out}(s) = \underbrace{\frac{\cdots}{s^2 + \zeta\omega_n s + \omega_n^2}}_{H(s)} V_{in}(s)$$

Thus, *for zero init. condits.*,

$$V_{out}(s) = H(s) \cdot V_{in}(s)$$

and $v_{out}(t) = £^{-1}\{H(s) \cdot V_{in}(s)\}$

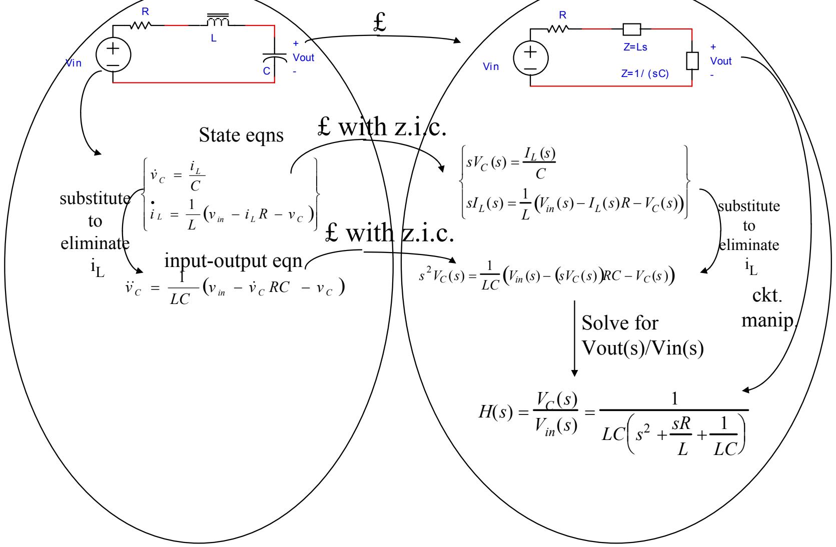
Note that this does ***not*** mean

$$v_{out}(t) = \{h(t) \cdot v_{in}(t)\}$$

Poles of $H(s) \Leftrightarrow$ response with step or constant input

---

Obtaining *H(s)* (for zero init condits)



State eqns

substitute to eliminate $\mathrm{i_L}$

$$\begin{cases}\dot{v}_C = \dfrac{i_L}{C} \\ \dot{i}_L = \dfrac{1}{L}\left(v_{in} - i_L R - v_C\right)\end{cases}$$

input-output eqn

$$\ddot{v}_C = \frac{1}{LC}\left(v_{in} - \dot{v}_C RC - v_C\right)$$

£

£ with z.i.c.

$$\begin{cases}sV_C(s) = \dfrac{I_L(s)}{C} \\ sI_L(s) = \dfrac{1}{L}\left(V_{in}(s) - I_L(s)R - V_C(s)\right)\end{cases}$$

substitute to eliminate $\mathrm{i_L}$

£ with z.i.c.

$$s^2V_C(s) = \frac{1}{LC}\left(V_{in}(s) - \left(sV_C(s)\right)RC - V_C(s)\right)$$

Solve for Vout(s)/Vin(s)

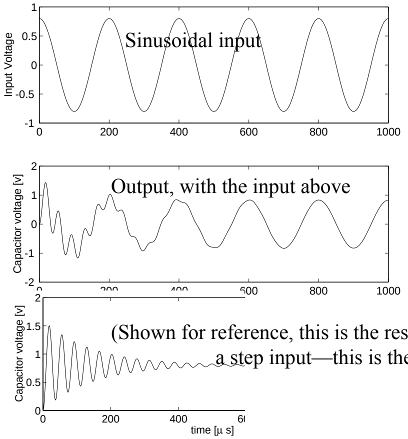
ckt. manip.

$$H(s) = \frac{V_C(s)}{V_{in}(s)} = \frac{1}{LC\left(s^2 + \dfrac{sR}{L} + \dfrac{1}{LC}\right)}$$

Response using *H(s)*

For zero init. condits., $V_{out}(s) = H(s) \cdot V_{in}(s)$ $\quad v_{out}(t) = \pounds^{-1}\{H(s) \cdot V_{in}(s)\}$

Example: H(s) from last page, plus $v_{in}(t) = u(t)\cos(\omega t)$
(cosine that turns on at time zero)



(Shown for reference, this is the response with a step input—this is the response related to roots of *H(s)*)

Result: Response reflecting roots of *H(s)* plus response reflecting input frequency. The former decays leaving sinusoidal steady state.

Calculating this complete response gets nasty, so nasty, that actually....

I used a numerical solution:

lrcplot.m

```
% Plot response of LRC circuit
% (described in lrc.m)
% to sinusoidal input.
% crs
[t,y]=ode45('lrc',[0,1e-
   3],[0;0]);
subplot(2,1,2)
plot(t*1e6,y(:,2))
big
ylabel('Capacitor voltage [v]')
xlabel('time [\mu s]')
subplot(2,1,1)
plot(t*1e6,0.8*cos(2*pi*5e3*t))
big
ylabel('Input Voltage')
```

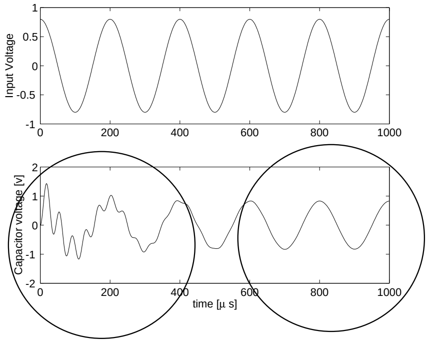
lrc.m

```
function xd = lrc(t,x)
% Derivative function for
ode45
% LRC circuit
% C Sullivan
% states iL; Vc

% Set Parameters
R = 8; % ohms
C = 0.06e-6; % farads
L = 575e-6; % Henries
v0 = 0.8; % volts
w = 2*pi*5e3;

A = [-R/L -1/L ; 1/C 0];
B = [1/L ; 0];
xd = A*x + B*v0*cos(w*t);
```

What good is H(s) with sinusoidal input?



Ignore this

Look at this:
“Sinusoidal Steady-State”

need:
G (one in this case)
ϕ (zero in this case)

Amazingly simple result:
$G = |H(j\omega)|$
$\phi = \angle(H(j\omega))$

Evaluating *H(jω)*

- Use MATLAB as a complex-number calculator.
- Get qualitative idea
- Evaluate | *H(jω)* | and< (*H(jω)*) algebraically
- Express *H(s)* as $H(s) = \dfrac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots}$

  z is a zero, a place on the *s*-plane where H –> 0 (plotted o)

  p is a pole, where H –> ∞ (plotted x)
- Now plug in *s* = *jω* $H(j\omega) = \dfrac{(j\omega - z_1)(j\omega - z_2)\cdots}{(j\omega - p_1)(j\omega - p_2)(j\omega - p_3)\cdots}$
- Each term is a distance (a complex vector distance on the *s*-plane) between *p* or *z* and *jω* , a point on the imaginary axis at a height corresponding to the input frequency we want to study.
- If we want to know | *H(jω)* | , then all we need is

$$|H(j\omega)| = \frac{|j\omega - z_1| \cdot |j\omega - z_2|\cdots}{|j\omega - p_1| \cdot |j\omega - p_2| \cdot |j\omega - p_3|\cdots}$$

Each term in this is just the length of a distance vector.