Laplace Transforms

What?

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}\,dt \qquad [1]$$

where s is a complex number (having both real and imaginary parts, $s = \sigma + j\omega$) that is constant with respect to time ($s \neq s(t)$, $ds/dt = 0$).

To a mathematician, that is what matters about the Laplace transform. But engineers almost never use that integral; in fact it is possible to use the Laplace transform on a daily basis, and actually forget equation [1]. Instead an engineer care about how the Laplace transform is useful.

Why?

The Laplace transform is a powerful tool for analyzing system models consisting of linear differential equations with constant coefficients. Uses of the Laplace transform in this context include:

1. *As a method for solving differential equations.* As we will see, the Laplace transform provides an alternative to classical time-domain methods to find the time domain solution of differential equations. Solution of differential equations via Laplace transforms involves algebra rather than dealing with differential equations.

2. *As a useful way to <u>think</u> about and analyze system behavior.* In many instances, relationships between system inputs, outputs and system models are clearly evident when considered in the *s*-domain that are less clear when considered in the time domain. Examples from among many include the existence of a transfer function, defined in the *s* domain but not defined in the time domain, which multiplies the system input to obtain the system output.

Notwithstanding the importance of the Laplace transform for solving differential equations, it is very important to realize that this is only a part of the motivation – and probably the smaller part - for using this technique. Students are encouraged to invest the effort to comfortable thinking about dynamic systems and their mathematical representation in the *s*-domain. The Laplace transform is a central feature of many courses and methodologies that build on the foundation provided by Engs 22. Among these is the design and analysis of control systems featuring feedback from the output to the input.

A Useful Analogy.

To understand the Laplace transform, use of the Laplace to solve differential equations, and the relationship between the s-domain and the time domain it is useful to consider the logarithm function.

Consider using the logarithm function to calculate the product $X \bullet Y$. As indicated in Figure 1, we can calculate $X \bullet Y$ directly using multiplication. Alternatively we can calculate this product by taking the logarithm of *X* and *Y* (transforming to the logarithm domain), adding ln*X* and ln*Y*, and then taking the inverse logarithm of the sum (exponentiating it) this sum, to go back to the original domain.

We can also use the logarithm function to visualize system behavior. Whereas it is difficult to determine whether $f(t) = e^{-at}$ and what the value of a is by visual inspection and , this is easily done from a plot of ln($f(t)$) vs t.

Figure 1. Use of the logarithm function to calculate a product.

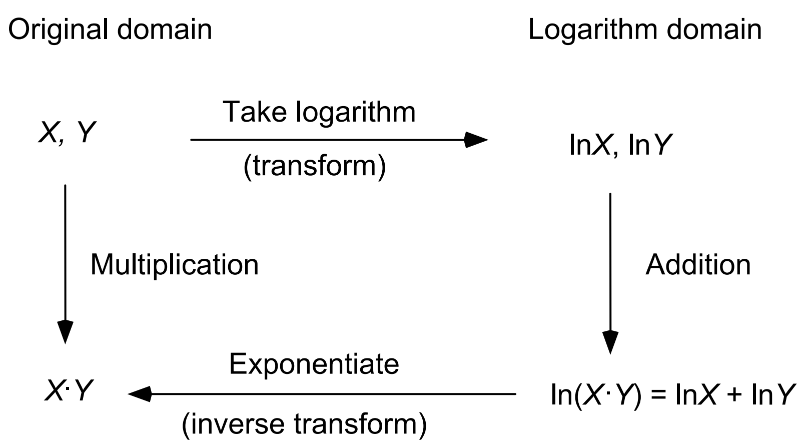
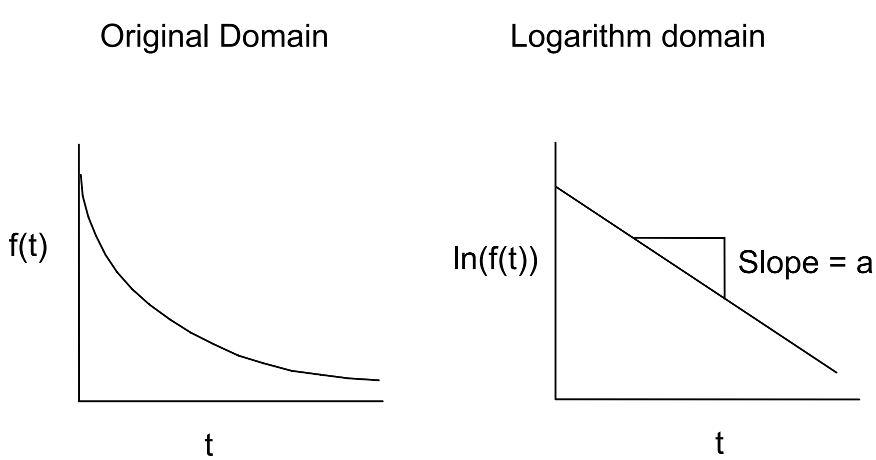


Figure 2. Visualization of system behavior for f(t) and ln(f(t)).



As suggested by Figure 1, we can calculate $X \bullet Y$ without actually multiplying—instead, we transform to the logarithm domain, perform a simpler mathematical operation (addition), and then transform back to the original domain. If multiplication were a lot harder than addition, this could be very valuable.

The idea of solving differential equations using the Laplace transform is very similar. We first transform to the *s* domain using the Laplace transform. That gets rid of all the derivatives, so solving becomes easy—it is just algebra in the *s* domain. Then we transform back to the original domain ("time domain").

An Important Limitation.

The Laplace transform and techniques related to it are only applicable to systems described by linear constant-coefficient models.

Using the Laplace Transform.

In order to apply the technique described above, it is necessary to be able to do the forward and inverse Laplace transforms. Although in principle, you could do the necessary integrals, people have been doing those integrals for centuries. They have them pretty well figured out by now, and you've got better things to do. So a better plan is to use a table of Laplace transforms. The book has a pretty good table, but we'll provide a larger table.

The table is used primarily for the inverse transform, and for transforming inputs. For other parts of your equation(s), it is only necessary to know a few properties of the Laplace transform. Some of these are derived in Appendix II, but you only need to be able to use them, not derive them.

Linearity: (both superposition and homogeneity):

$$\mathcal{L}(af_1(t) + bf_2(t)) = a\mathcal{L}(f_1(t)) + b\mathcal{L}(f_2(t)) = aF_1(s) + bF_2(s)$$

Superpositon is useful because it allows you to separate terms in your equation into simpler things that you can easily find on a table, and homogeneity is useful because it allows you to factor out multiplicative constants, so you can in fact match your terms up with table entries.

Differentiation:

$$\mathcal{L}(\dot{f}) = sL(f) - f(0) = sF(s) - f(0)$$

This is the key property that makes solutions of differential equations possible. The derivative goes away, and gets replaced by a simple multiplication by *s*. There are similar properties for second and higher derivatives. See Appendix II.

Appendix I: Complex Number Review.

As we have seen in the course of considering analytical solutions of differential equations in the time domain, taking the roots to the characteristic equations of system models often involves taking the square root of a negative number. Invariably, the solutions to such system models exhibit oscillatory behavior. By expressing the roots as complex numbers of the form $z = \sigma + j\omega$ and using Euler's equation, we are able to derive solutions of the form

$$Y_H = e^{-\sigma t}(K_1\cos\omega t + K_2\sin\omega t) \qquad [17]$$

Note that the real part of z, σ, specifies the rate of decay of the homogeneous solution whereas the imaginary part, ω, specifies the frequency of oscillation.
Complex numbers can be represented in either rectangular or polar coordinates.

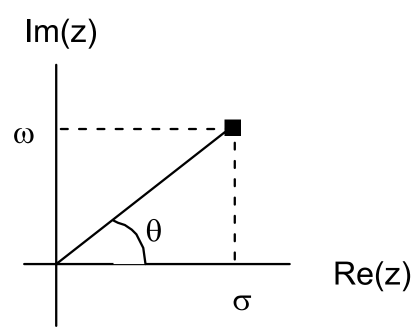
Figure 4. Complex numbers represent a point in a two-dimensional space.

Rectangular coordinates

$z = \sigma + j\omega$

$\text{Re}(z) = \sigma = A\cos\theta$

$\text{Im}(z) = \omega = A\sin\theta$



Polar coordinates

$z = Ae^{j\theta}$

$A = \text{magnitude}(z) = (\sigma^2 + \omega^2)^{1/2}$

$\theta(s) = \text{"argument of z" or } \arg(z) = \tan^{-1}(\omega/\sigma)$

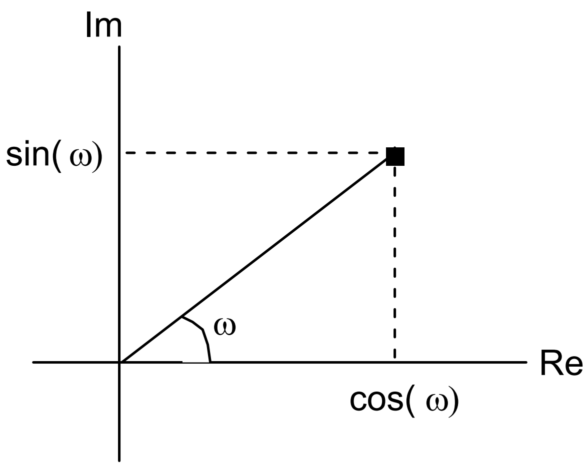
Exponents of complex numbers are also complex numbers.

$$e^z = e^{(\sigma + j\omega)} = e^{\sigma}e^{j\omega} \qquad [18]$$

$e^{j\omega}$ is a complex number with real and imaginary parts given by the Euler equation:

$$e^{j\omega} = \cos(\omega) + j\sin(\omega) \quad [19]$$

Figure 5. Real and imaginary components of $e^{j\omega}$.



The magnitude of $e^{j\omega}$ is

$$\left|e^{j\omega}\right| = \sqrt{\cos^2\omega + \sin^2\omega} = 1 \quad [20]$$

Hence the magnitude of e^z is

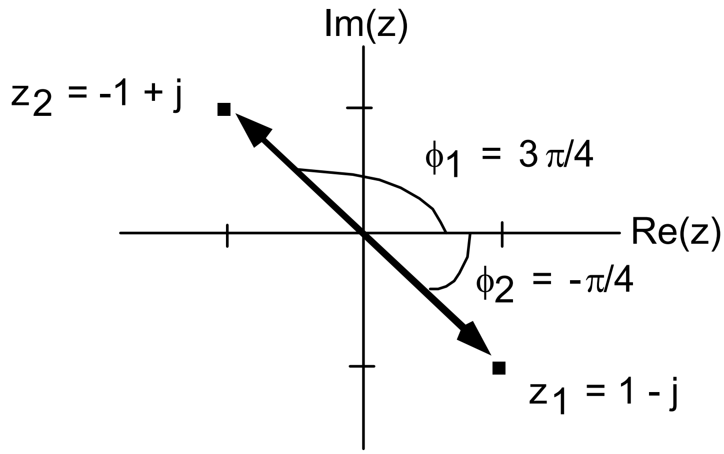
$$\left|e^{z}\right| = \left|e^{\sigma}\right|\left|e^{-j\omega}\right| = e^{\sigma} \quad [21]$$

For e^{zt}, the magnitude is $e^{\sigma t}$.
In using tables of Laplace transforms, as well as other applications involving complex numbers, it is often desired to evaluate terms such as

$$\phi = \tan^{-1}(\omega/\sigma) \quad [22]$$

For example, see entries 19, 22,23,26-31,36-38, and 39 in the attached Laplace transform table. When using this equation, the signs of ω and σ must be considered individually rather than only considering the sign of ω/σ. For example, consider complex numbers of the form $\sigma + j\omega$: $z_1 = 1 - j$ and $z_2 = -1 + j$ which are . For both of these, $\omega/\sigma = -1$. However, if we consider the signs of ω and σ individually we see that z_1 is in the fourth quadrant and z_2 is in the second quadrant and that $\phi_1 = -\pi/4$ and $\phi_2 = 3\pi/4$.

Figure 6. Location of z_1 and z_2 on the real-imaginary plane.



The MATLAB function atan2(a,b) will give you the correct value of $\tan^{-1}(a/b)$. Alternatively, you can get $\tan^{-1}(a/b)$ from an calculator and figure out based on the signs of a and b whether the answer from the calculator (which will between $-\pi$ and $+\pi$ for most calculators) needs to be corrected by adding π.

Appendix 2: Mathematical discussion of Laplace Transform and Derivation of Properties.

The Laplace transform exists whenever $f(t)e^{-st}$ is integrable. Transformable $f(t)$ include polynomials, exponentials, sinusoids, and sums and products of these. We may also observe that the Laplace transform is linear. That is,

$$\mathcal{L}(af_1(t) + bf_2(t)) = a\mathcal{L}(f_1(t)) + b\mathcal{L}(f_2(t)) = aF_1(s) + bF_2(s) \qquad [2]$$

$\mathcal{L}(f(t))$ can be found using standard methods of integration introduced in a college-level 2-term calculus course. For example the Laplace transform of the unit step function, $u(t)$ is

$$\mathcal{L}(u(t)) = \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty \qquad [3]$$

For re(s) > 0, which is true for all cases considered in Engs 22,

$$\mathcal{L}(u(t)) = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s} \qquad [4]$$

As another example, consider

$$\mathcal{L}(e^{-at}) = \int_0^\infty e^{-at}e^{-st}\,dt = \int_0^\infty e^{-(s+a)t}\,dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty \qquad [5]$$

For re(s) + a > 0, or re(s) > -a, also a condition met for all cases considered in Engs 22,

$$\mathcal{L}(e^{-at}) = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a} \qquad [6]$$

Laplace transforms for other common functions are tabulated in the attached "Laplace Transform Table" and are also discussed in your text.

In addition to functions, the Laplace transform can also be evaluated for common mathematical operations.

Differentiation.

$$\mathcal{L}(\dot{f}) = \int_0^\infty \left(\frac{df}{dt}\right)e^{-st}\,dt \qquad [7]$$

Integrating by parts:

$$\int_0^\infty u\,dv = (uv]_0^\infty - \int_0^\infty v\,du \qquad [8]$$

$$u = e^{-st}; \quad dv = \left(\frac{df}{dt}\right)dt$$

$$du = -se^{-st}\,dt; \quad v = f \qquad [9]$$

So

$$\mathcal{L}(\dot{f}) = \left(f(t)e^{-st}\right]_0^\infty - \int_0^\infty -fse^{-st}\,dt = f(\infty)e^{-\infty} - f(0) + s\int_0^\infty f(t)e^{-st}\,dt$$

or

$$\mathcal{L}(\dot{f}) = sL(f) - f(0) = sF(s) - f(0) \qquad [10]$$

Note that $f(0)$ is the initial value of the time-domain function f whose derivative is being transformed. $F(s)$, an s-domain function, is the Laplace transform of f. In general capital letters denote functions of s and lower case letters denote functions of time. Also note that

$f(0)$ and $\dot{f}(0)$ (used subsequently) can evaluated at t = 0- or at t = 0+; either way it is essential be to consistent and treat the whole problem starting at 0- or the whole problem starting at 0+.

Now consider a second derivative. For a function g equation [10] tells us

$$\mathcal{L}(\dot{g}) = sG(s) - g(0) \qquad [11].$$

If we let

$$g = \dot{f} \ \ and \ \ \dot{g} = \ddot{f}$$

then

$$\mathcal{L}(\ddot{f}) = s\mathcal{L}(\dot{f}) - \dot{f}(0) \qquad [12]$$

and substituting equation [11]

$$\mathcal{L}(\ddot{f}) = s\left[s\mathcal{L}(f) - f(0)\right] - \dot{f}(0) = s^2F(s) - sf(0) - \dot{f}(0) \qquad [13]$$

Table A-1 (from Jacquot and Long, *Introduction to Engineering Systems*, Allyn and Bacon, 1988) lists Laplace transforms for common mathematical operations. Among these are the transform of the nth derivative

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n\text{-}1} f(0) - \ldots - s^0 f^{n\text{-}1}(0) \qquad [14]$$

For example,

$$\mathcal{L}\left[\frac{d^4 f}{dt^4}\right] = s^4 F(s) - s^3 f(0) - s^2 \dot{f}(0) - s\ddot{f}(0) - \dddot{f}(0) \qquad [15]$$

Table A-1 also includes the Laplace transform of the definite integral:

$$\mathcal{L}\left[\int_0^t f(\lambda)d\lambda\right] = \frac{1}{s}F(s) \qquad [16]$$

Equation [16] is derived by integrating by parts as described by Close, Frederick & Newell.