Explanations and Discussion of Some Laplace Methods: Transfer Functions and Frequency Response

I. Anatomy of Differential Equations in the S-Domain

Parts of the s-domain solution. We will consider parts of the solution of differential equations in the s-domain with respect to a second-order ODE:

$$a_2 \ddot{y} + a_1 \dot{y} + a_o y = b_1 \dot{u} + b_o u(t) \qquad [2]$$

Note: This handout uses the convention that *U(t)* is the unit step function whereas *u(t)* is <u>any</u> input. *U(s)* is the Laplace transform of the general input function *u(t)*, and should not be confused with the unit step function (*U(t),* in this handout or *u(t)* elsewhere).

Taking the Laplace transform of [2], we obtain equation [3]:

$$a_2[s^2 Y(s) - sy(0) - \dot{y}(0)] + a_1[sY(s) - y(0)] + a_0 Y(s) = b_1[sU(s) - u(0)] + b_0 U(s)$$

which can be solved for *Y(s)* to give

$$Y(s) = \frac{y(0)[a_2 s + a_1] + \dot{y}(0)}{a_2 s^2 + a_1 s + a_o} + \frac{b_1[sU(s) - u(0)] + b_o U(s)}{a_2 s^2 + a_1 s + a_o} \qquad [4]$$

The first term is a function of initial conditions ($y(0)$, $\dot{y}(0)$, etc.) and system properties but not the input function, and goes to zero when the initial conditions are equal to zero. The second term is a function of the input function and system properties but not the initial conditions of the solution and its derivatives, and goes to zero when the input is equal to zero.

Zero input response. The response of a system when the input = 0 but initial conditions are not zero is called the zero input response. Other names for the zero input response are the "free" and "unforced" response. From equation [4] we observe that

$$Y_{\text{ZIR}}(s) = \frac{y(0)[a_2 s + a_1] + \dot{y}(0)}{a_2 s^2 + a_1 s + a_o} \qquad [5]$$

The inverse Laplace transform of this is the zero input response in the time domain, sometimes called the homogeneous solution, $y_H(t)$.

Zero state response. The response of a system when the initial conditions are = 0 but the input is not zero is called the zero state response, also called the "forced response". From equation [4] we observe that

$$Y_{\mathrm{ZSR}}(s) = \frac{b_1 s + b_o}{a_2 s^2 + a_1 s + a_o} U(s) \qquad [6]$$

For stable systems with input such that *u(t)* does not decay to zero (e.g. for constant, step, impulse, ramp, and oscillating inputs), the zero-state response is the steady-state response.* In the time domain, the zero state response is sometimes also called the particular solution, $y_P(t)$.

Transfer function. The observation, from equation [6], that the transform of the zero-state response is equal to a function of s multiplied times the transform of the input is of central importance in the context of analyzing linear dynamic systems. Accordingly we define the *transfer function*, *H(s),* as

$$H(s) = \frac{Y_{\mathrm{ZSR}}(s)}{U(s)} \qquad [7]$$

the subscript "zero state response" is often not included in writing equation [7], but is explicitly indicated here for clarity.

We have observed above that

$$y_H(\mathrm{t}) = \mathrm{L}^{-1}[Y_{\mathrm{ZIR}}(s)] \qquad [8]$$

and for stable systems with input that does not decay to zero

$$y_P(\mathrm{t}) = y_{SS} = \mathrm{L}^{-1}[Y_{\mathrm{ZSR}}(s)] \qquad [9]$$

However, <u>there is no</u> *h(t)* such that *h(t) = y(t)/u(t)*. That is, there is no time-domain analog to the transfer function *H(s)*. In light of the utility of the transfer function, examined further subsequently, this observation represents one of the main motivations to use the Laplace transform and to think in the s-domain.

The transfer function of a linear system may be written in factored form as

$$H(s) = k\frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} \qquad [10].$$

---

*The term "steady-state" response in this context is taken to mean that part of the response that remains after the transient response has decayed to zero. Note that this does *not* mean that the time derivative of the steady state response is zero. Thus for example Asin(ωτ + ϕ) is commonly called the steady-state response of a stable system with input = sin(ωτ) even though the time derivative of Asin(ωτ + ϕ) ≠ 0.

The values of s at which the numerator becomes zero ($z_1, z_2, \ldots, z_n$) are called the *zeros* of *H(s)*, and the values of s at which the denominator becomes zero ($p_1, p_2, \ldots, p_n$) are called the *poles* of *H(s)*.

II. Frequency Response

Consider a sinusoidal input of unit magnitude:

$$u(t) = A_{in}\sin(\omega t) \qquad [1],$$

As we have seen, the steady-state solution for a LTI system with a sinusoidal input is a sinusoidal output with the same frequency but potentially different magnitude and phase. Thus for *u(t)* given by equation [1],

$$y(t) = A_{out}\sin(\omega t + \phi) \qquad [2].$$

In order to describe this result, we need three numbers: The phase shift, ϕ, the amplitude, A_{out}, and the frequency ω. Actually, we only need two numbers, because we already know that the output frequency is the same as the input frequency. In addition, we know from the property of homogeneity that, for any given frequency, the output amplitude will be a constant multiple of the input amplitude. This ratio is called the gain and it is defined by $G = A_{out}/A_{in}$.

Thus, to describe the input/output relationship for a given frequency, we are interested in two numbers, the gain *G* and the phase ϕ. This pair of numbers (which can also be expressed as a single complex number, as we'll see) is called the *frequency response*. If the input and output are in the same units (e.g., both voltages, or both velocities), *G* is dimensionless. However, *G* may carry units according to the units of input and output. The phase angle of the frequency response, ϕ (radians), provides information about the temporal displacement of the output sinusoid relative to the input sinusoid.

The frequency response can be found by evaluating *H(jω)*.

One of the most powerful results stemming from consideration of system models in the s-domain is that the frequency response may be obtained by evaluating *H(s)* at $s = j\omega$. This can be shown as follows using transform inversion techniques as described in the appendix of this document.

The result from the appendix is all you need to know. It is that the complex number H(jw) lets us calculate the magnitude and phase angle of the frequency response as follows:

$$G = |H(j\omega)| \qquad [19]$$

$$\phi = \text{angle}\left(H(j\omega)\right) \qquad [20]$$

Evaluation of H(jω).

Since $H(j\omega)$ is a complex number we may write

$$H(j\omega) = G(\cos\phi + j\sin\phi) \qquad [21]$$

and

$$H(j\omega) = Ge^{j\phi} \qquad [22]$$

where

$$G = \sqrt{\{\mathrm{Re}[H(j\omega)]\}^2 + \{\mathrm{Im}[H(j\omega)]\}^2}$$

and

$$\phi = \mathrm{atan2}(\,\mathrm{Im}\{H(j\omega)\}, \mathrm{Re}\{H(j\omega)\}\,) \qquad [23]$$

When

$$\mathrm{H(s)} = \frac{\mathrm{N(s)}}{\mathrm{D(s)}} \qquad [7],$$

we may write

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{|N|e^{j\theta_N}}{|D|e^{j\theta_D}} = \frac{|N|}{|D|}e^{j(\theta_N - \theta_D)} \qquad [24]$$

$e^{j\theta_N}$ is the angle of $N(j\omega)$, denoted $\angle N(j\omega)$. Also, $e^{j\theta_D}$ is $\angle D(j\omega)$ and $e^{j(\theta_N - \theta_D)}$ is $\angle N(j\omega) - \angle D(j\omega)$. From [24], we may write

$$\mathrm{G} = |\mathrm{H}(j\omega)| = \frac{|\mathrm{N(s)}|}{|\mathrm{D(s)}|} = \frac{\sqrt{[\mathrm{Im(N(s))}]^2 + [\mathrm{Re(N(s))}]^2}}{\sqrt{[\mathrm{Im(D(s))}]^2 + [\mathrm{Re(D(s))}]^2}} \qquad [25]$$

and

$$\theta = \angle\left(\mathrm{H}(j\omega)\right) = \tan^{-1}\left[\frac{\mathrm{Im(N(s))}}{\mathrm{Re(N(s))}}\right] - \tan^{-1}\left[\frac{\mathrm{Im(D(s))}}{\mathrm{Re(D(s))}}\right] \qquad [26]$$

(where these inverse tangent functions are really atan2 functions, that get the angle in the right quadrant)

Making use of the observation that the frequency response is equal to the magnitude and angle of $H(j\omega)$ is considerably easier than deriving this result. If want to look at and understand the derivation in the appendix, great. If the derivation is opaque, you can still use the transfer function to evaluate the frequency response.

Example: Frequency Response for a First Order Systems
Consider the differential equation

$$\dot{y} + \frac{1}{\tau} y = B\sin\omega t \qquad [27]$$

Taking the Laplace transform with initial conditions = 0, corresponding to the zero-state response, we obtain

$$Y(s) = \left(\frac{1}{s + \frac{1}{\tau}} \right) B \left(\frac{\omega}{s^2 + \omega^2} \right) \qquad [28].$$

We recognize that for this case the transfer function, H(s), is

$$H(s) = \left(\frac{1}{s + \frac{1}{\tau}} \right) \qquad [29]$$

Using equations [25] and [26] with H(s) as defined in equation [29] we get

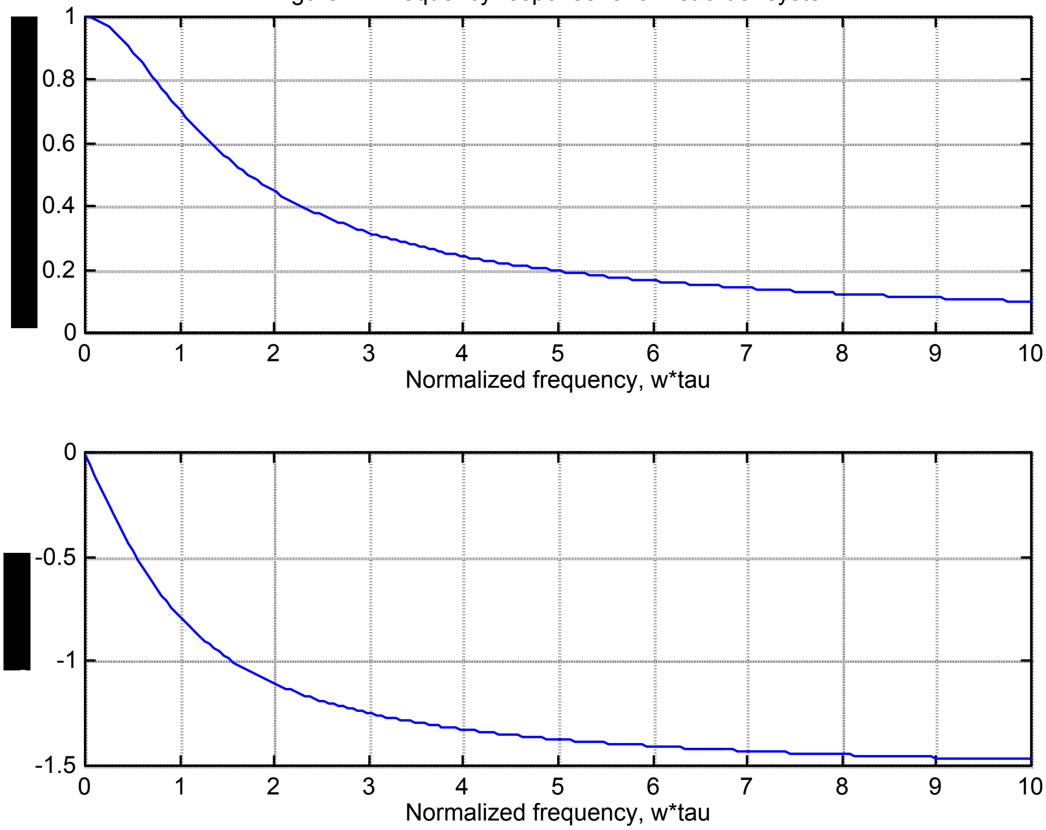
$$G = |H(j\omega)| = \frac{|1|}{\left| j\omega + \frac{1}{\tau} \right|} = \frac{1}{\sqrt{\omega^2 + \left(\frac{1}{\tau} \right)^2}} \qquad [30]$$

$$\theta = \text{angle}(H(j\omega)) = \tan^{-1}\left[\frac{0}{1} \right] - \tan^{-1}\left[\frac{\omega}{\frac{1}{\tau}} \right] \qquad [31]$$

Both the gain and the angle of the response depend on the frequency ω. When ω = 0, we see from equation [30] that G = τ which corresponds to the steady-state solution in the time domain for u(t) = cosωt. Thus we may consider a normalized magnitude equal to *G*/τ for the purpose of getting a feel for the general behavior of first order systems. Similarly, equation [31] suggests that a normalized angle be defined as ω/(1/ τ) = ωτ. The normalized magnitude and the phase angle for a first order system are plotted as a function of ωτ in Fig. 1.

It is more common to plot $20\log_{10}G$, (i.e., *G* in units of decibels), vs *f* on a log scale, and phase in degrees. This is called a “Bode plot”. See the separate handout on Bode plots for more details.

Figure 1. Frequency response for a first order system.



Appendix: Deriving the facts that G = |H(jω)| and ϕ = <H(jω).

We have seen that

$$Y_{SS}(s) = H(s)U(s) \qquad [4]$$

When *u(t)* = sin(ωt),

$$U(s) = \left[\frac{\omega}{s^2 + \omega^2}\right] \qquad [5],$$

and $Y_{ZSR}(s)$ is

$$Y_{SS}(s) = H(s)\left[\frac{\omega}{s^2 + \omega^2}\right] = H(s)\left[\frac{\omega}{(s - j\omega)(s + j\omega)}\right] \qquad [6].$$

Denoting *H(s)* as the ratio of a numerator polynomial *N(s)* and a denominator polynomial *D(s)*, we can write

$$H(s) = \frac{N(s)}{D(s)} = k\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \qquad [7].$$

Combining equations [6] and [7],

$$Y_{SS}(s) = k\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}\left[\frac{\omega}{(s-j\omega)(s+j\omega)}\right] \qquad [8]$$

or, using the partial fraction expansion theorem we can write equation [9]

$$Y_{SS}(s) = H(s)\left[\frac{\omega}{(s-j\omega)(s+j\omega)}\right] = \frac{C_1}{(s-j\omega)} + \frac{C_2}{(s+j\omega)} + \frac{C_{H1}}{(s-p_1)} + \frac{C_{H2}}{(s-p_2)} + \ldots + \frac{C_{Hn}}{(s-p_n)}$$

where C_{Hi} denotes the poles of the transfer function. All the poles of *H(s)*, whether real or complex, distinct or repeated, will result in terms that when inverted into the time domain are multiplied by e^{-pit}. Thus the only non-zero terms in equation [9] at steady state are the first two and we are left with

$$Y_{SS}(s) = H(s)\left[\frac{\omega}{(s-j\omega)(s+j\omega)}\right] = \frac{C_1}{(s-j\omega)} + \frac{C_2}{(s+j\omega)} \qquad [10].$$

Using the Heaviside method, we can find C_1 and C_2:

$$C_1 = (s-j\omega)Y_{SS}(s) = \left(H(s)\frac{\omega}{(s+j\omega)}\right]_{s=j\omega} = \frac{H(j\omega)}{2j} \qquad [11]$$

and

$$C_2 = (s+j\omega)Y_{SS}(s) = \left(H(s)\frac{\omega}{(s-j\omega)}\right]_{s=-j\omega} = \frac{-H(-j\omega)}{2j} \qquad [12]$$

In polar form

$$H(jw) = M(\omega)e^{j\theta(\omega)} \qquad [13]$$

and

$$H(-jw) = M(\omega)e^{-j\theta(\omega)} \qquad [14]$$

In equations [13] and [14] we note explicitly that M and θ are functions of ω.

Combining equations [10] through [14] we obtain

$$Y_{SS}(s) = \frac{M}{2j}\left(\frac{e^{j\theta}}{s - jw} - \frac{e^{-j\theta}}{s + jw}\right) \qquad [15]$$

Taking the inverse Laplace transform we obtain

$$y_{SS}(t) = \frac{M}{2j}\left(e^{j\theta}e^{j\omega t} - e^{-j\theta}e^{-j\omega t}\right) \qquad [16]$$

or

$$y_{SS}(t) = M\left(\frac{e^{j(\theta+\omega t)} - e^{-j(\theta+\omega t)}}{2j}\right) \qquad [17]$$

Recognizing that the term in the parentheses = sin(ωt + θ), we can write

$y_{SS}(t)$ = M sin(ωt + θ) [18].

Comparing equation [18] and equation [3], we see that θ = ϕ, the phase angle of the frequency response.