

LUMPED FLUID SYSTEMS

I. General Introduction and Applications:

Applications:

- Fluid delivery systems; plumbing. For water or other fluids, for direct use by people, or for industrial operations, such as in chemical plants.
- Fluid cooling systems.
- Fluid (hydraulic) power systems, used in industry, aerospace, and automotive applications.
- Medical: blood flow, etc.
- Hydrology; transport of nutrients, and pollutants as well as water in groundwater and in streams, rivers, and lakes.

Fluid systems either involve essentially non-compressible liquids in which case they are often called “hydraulic systems” or compressible gases in which case they are often called “pneumatic systems”. The terms hydraulic and pneumatic are most common in referring to systems that use pressurized fluid used to provide power, for the purpose of causing motion (actuation). For example, hydraulic systems are used to control flaps on airplane wings as well as apparatus on bulldozers. We will focus primarily on hydraulic systems, which are simpler to analyze than pneumatic systems.

In commenting on hydraulic power systems, Woods and Lawrence (Modeling and Simulation of Dynamic Systems, Prentice Hall, 1997) state:

“The main reason that hydraulic and pneumatic systems are so popular compared to their electro-mechanical counterparts is the power density capability of pumps and actuators. Electromagnetic motors, generators, and actuators are limited by magnetic field saturation and can produce up to about 200 pounds per square inch of actuator. In hydraulic systems, 3000 to 8000 pounds per square inch of actuator is common in aircraft applications, and 1000 pounds per square inch of actuator is common in industrial applications. Therefore, the hydraulic systems required to produce a given force output are much smaller. ...

A second advantage of fluid control systems is that the fluid circulating to and from an actuator removes the heat generated by the actuator that is doing the work. This heat follows the fluid back to a reservoir to be dissipated in a better location than inside the actuator. Electromechanical actuators and motors have limited ability to dissipate heat to the surrounding environment. Heat is the predominate damaging mechanism in electric and electronic systems.”

Although fluid systems are often taught in terms of hydraulic power systems (just as chemical dynamics is often taught in terms in industrial reactor design), the applications of lumped analysis of fluid systems is much broader (just as the principles of chemical dynamics apply to modeling resource systems). The list at the start of this section hints at some of the wide range of other applications of lumped fluid system analysis.

II. State Variables, Components, and Accounting Equations

A. State Variables

The following variables are important in characterization and analysis of fluid systems:

q = volumetric flow (m^3/s or other units)

v = volume (m^3)(e.g., volume of fluid contained in a vessel.)

h = height of fluid in a vessel (m)

$$p = \text{pressure (N/m}^2\text{)}$$

Pressure is the force per unit area exerted by a fluid. Just as in electrical systems we are typically only interested in voltage differences, in fluid systems we are typically interested in pressure differences. Thus, it is usually possible to interchange absolute pressure and *gauge pressure*, p_g , which is defined as the pressure relative to atmospheric pressure ($P_a = 1.01 \times 10^5 \text{ N/m}^2$).

B. Elements

The primary elements in fluid systems are resistance, capacitance and intertance, which are analogous to electrical resistance, capacitance, and inductance, respectively.

Resistance.

In electrical systems, the linear approximation $v = iR$, is usually so close to exact that it can be considered a physical law (Ohm's law). In fluid systems, however, linear behavior of fluid resistance, $p = qR$ is rare, and most practical valves, pipes, etc. behave nonlinearly, following a law such as $q = K\sqrt{p}$. The detailed analysis of which model is appropriate for which situation is a topic for a fluid dynamics course, not a systems course. Our goal is to figure out how to model a *system* given models of the individual elements.

For using either the linear fluid resistance relationship $p = qR$ or the nonlinear relationship $q = K\sqrt{p}$, the key thing to remember is that the pressure of interest is the pressure **difference across** the element of interest. A pipe with resistance R between two chambers each with 100 kPa pressure in them will not have any flow in it, even though the pressure in the pipe is 100 kPa. But if one has 100 kPa and the other 99 kPa, there will be a pressure difference of 1 kPa, and there will be flow (from the higher pressure to the lower pressure). If there were 1 kPa in one and 2 kPa in the other, the flow would be the same as with 99 kPa and 100 kPa.

Thus, when setting up an analysis of the system, it is important to label the pressures of interest on the diagram, and be sure to write equations using the only the defined variables.

Capacitance.

An electrical capacitor has increasing charge storage with increasing voltage. A fluid capacitor has increasing fluid storage with increasing pressure, and is defined by the equation

$$V_L = Cp. \quad [1]$$

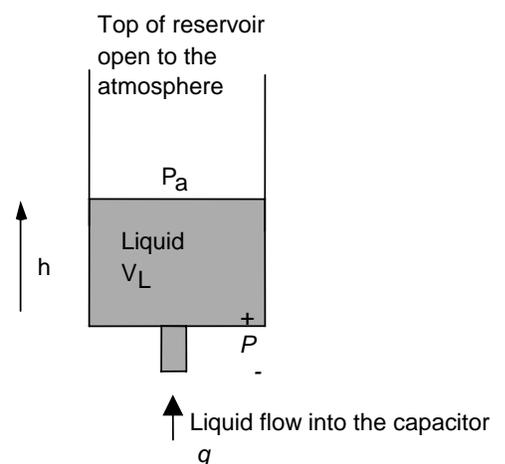
Example of a physical device that provides fluid capacitance: an open-top tank as shown in Fig. 1.

The force at the bottom of a reservoir due to the weight of the fluid is $mg = \rho V_L g$ where g is the earth's gravitational constant, and m is the mass of fluid, equal to ρV_L where ρ is the density of the fluid, and V_L is the volume of fluid in the tank, assuming the side walls of the tank are vertical. The total pressure on the inside of the bottom tank wall is a result of this pressure plus the atmospheric pressure P_a . However, we are more likely to be interested in the pressure *difference* between inside and outside, at the bottom of the tank where the pipe connects, as labeled p in Fig.1. The pressure difference is just

$$p = \frac{\rho g V_L}{A} = \rho g h \quad [2]$$

where A is the area of the base of the tank, which must be the

Figure 1. An open reservoir fluid capacitor



same all the way up since we assumed the side walls were vertical. What we see is a proportionality between pressure and volume, which is the defining characteristic of a fluid capacitor. We can write it in a more standard form as:

$$V_L = \left(\frac{A}{\rho g} \right) p \quad [3]$$

This equation is the same as [1], the definition of fluid capacitance, with $C = \left(\frac{A}{\rho g} \right)$.

Note that if the walls were not vertical, the volume/pressure relationship would not be as simple as [3], and we would have a nonlinear fluid capacitance.

When the fluid capacitance C is large, corresponding to a tank with a large area, a large increase in volume is accompanied by a small increase in pressure. The units of capacitance are volume/pressure, or $\text{m}^3/(\text{N}/\text{m}^2) = \text{m}^5/\text{N}$.

The open top tank is one way to achieve fluid capacitance. Others include a spring-loaded chamber, or the same thing with an air spring, as shown in Fig. 2. Calculations of the capacitance achieved from these devices is addressed in the linearization and nonlinear fluid elements handout.

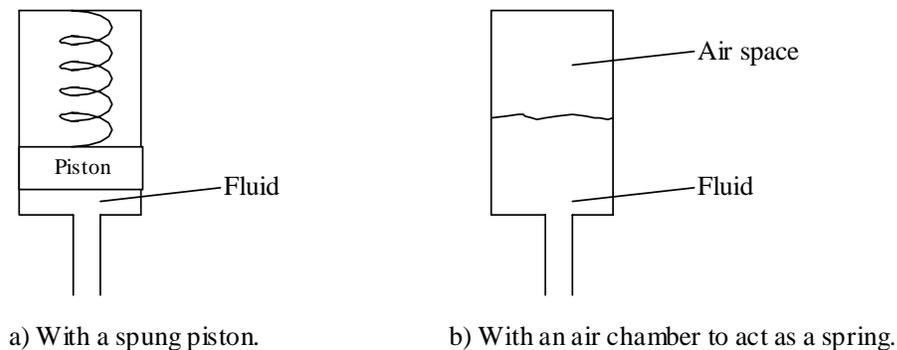


Fig. 2. Two other types of fluid capacitors.

As with electrical capacitance, differentiation of [1] provides a very useful differential equation that can be directly applied to set up a state-variable equation, in this case using pressure as a state variable:

$$dp/dt = q/C \quad [4]$$

Note that definitions of the polarity of each variable are essential; this equation is correct for the polarities shown in Fig. 1, but would be incorrect for flow defined in the opposite direction.

Inertance. Inertance arises as the result of the momentum of a flowing fluid. Consider an incompressible fluid element flowing down a pipe with constant cross sectional area A with as shown in Figure 3.

Treating this as a mass subject to two forces $p_1 A$ and $p_2 A$, we find:

$$m \dot{v} = A(p_1 - p_2) = \rho L A \dot{v} \quad [5]$$

or, noting that $q = Av$ and so $\dot{q} = A\dot{v}$, we can write,

$$\Delta p = \frac{\rho L}{A} \dot{q} \quad [6]$$

This equation is a lot like the equation for an inductor, $\Delta v = L \frac{di}{dt}$. Considering this analogy, we define inertance $I = \rho L/A$, and [6] can be written $p = I\dot{q}$. Here, the p must be understood to be the pressure difference across the inertance, as defined in Fig. 4.

Figure 3. A fluid element in a pipe.

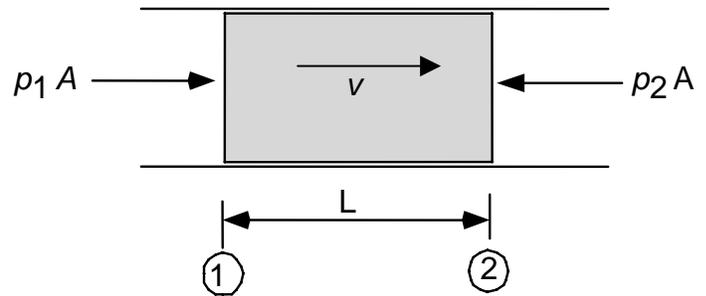
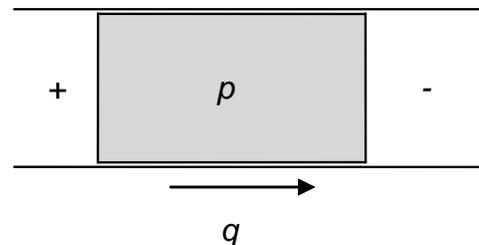


Figure 4. Variables for inertance.



Pumps. Pumps act as sources, analogous to voltage or current sources in electrical systems. Just as real power supplies don't quite maintain constant voltage, real pumps don't maintain constant pressure or flow. The deviations tend to be large for pumps. The detailed modeling of pumps is beyond the scope of this course—it is the study of an element, not of a system. However, it may be useful to know that centrifugal pumps (which use centrifugal force to generate pressure) can maintain roughly constant pressure at low flow rates, and so act approximately as pressure sources. However, their pressure drops rapidly and nonlinearly with increasing flow. “Positive displacement” pumps such as piston pumps deliver a fixed amount of fluid each cycle, and so behave like flow sources.

C. Accounting Equations.

Mass Balance.

At any given node, where pipes connect, mass balance requires that flow in equals flow out. If flows are all defined inwards, this can be written as the sum of flows equal to zero.

$$q_{in1} + q_{in2} + \dots = q_{out1} + q_{out2} + \dots \quad [7]$$

or

$$\sum q_{in} = 0 \quad [8]$$

In either case, this is directly analogous to Kirchoff's current law.

“Kirchoff's Pressure Law”

As discussed above, both fluid resistor behavior and inertance effects depend on the pressure *difference* across an element. Thus we are often interested in pressure differences rather than absolute pressures. The most commonly used pressure measurement, gauge pressure, is also defined as a pressure difference: the difference between measured pressure and atmospheric pressure.

In electrical systems, we saw that Kirchoff's voltage law was not a physical law related to pressure, but rather is just a consequence of defining voltage as a difference. Thus, if we discuss pressures as differences, we have an analogous law: the sum of pressure differences around a loop must be zero. Note that the loop may be any path through space—it need not follow a pipe.

Example: A tank emptying through a pipe.

Consider the tank shown at right emptying through a pipe. With the pressures defined as shown, we can follow the dotted line and write that the total pressure around that loop is equal to zero. Starting in the air, we go into the tank (across the wall) and the pressure increases by p_c . Next, we follow the dotted line into the pipe, and along the pipe back into the air. Thus, the total of the pressure differences around that loop is

$$p_c - p_p = 0 \quad [9]$$

This is assuming that the pressure drop in the fittings at the bottom of the tank is negligible. If not, we could include them in the resistance of the pipe and put the plus sign for the pressure drop across the pipe up in the tank.

From mass balance we have

$$q_c = -q_p \quad [10]$$

If we assume that the pipe has negligible intertance, we can consider just its resistance, and we have the two element laws:

$$dp_c/dt = q_c/C \quad q_p = p_p/R_p \quad [11], [12]$$

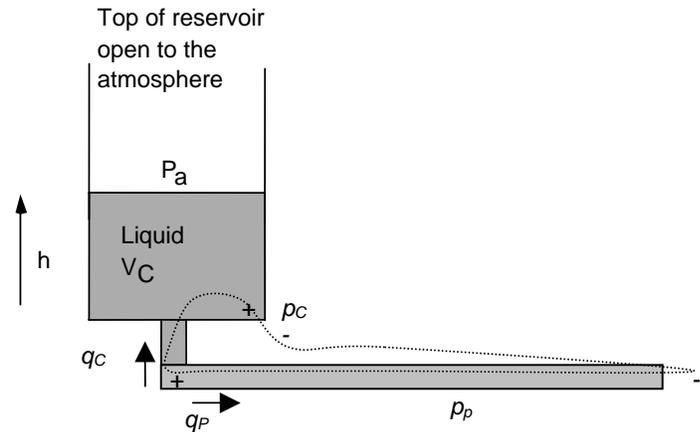
That's all the equations we need. The rest of the work is just substituting to get the RHS of the capacitor equation in terms of the state variable for the capacitor, p_c . We obtain:

$$dp_c/dt = -p_c/(R_p C) \quad [13]$$

Head

We discussed the effect of fluid height on pressure in the context of an open tank as a capacitor. Often, however, different elements may be located at different elevations, and these differences may be large compared to the elevation difference between the top and bottom of a tank. These height differences also affect pressure. These pressures are termed *head*, a term that may refer to either the height or the resulting pressure. Head may be an unavoidable consequence of the locations of system components, or it may be deliberately employed to achieve pressure, as in a water tank located atop a tower.

Figure 5. An open reservoir fluid capacitor emptying through a pipe



Head complicates pressure calculations, and it usually must be considered when writing KPL (“Kirchoffs pressure law”) equations. In principal, KPL is not changed. The sum of pressure differences is still zero around a loop. Consider the tank in Figure 6. Simply by definitions of the two pressures, the same equation [9] must hold.

$$p_C - p_P = 0 \quad [14]$$

Thus, it is tempting to conclude that the system equations are identical, and the head has no effect. However, what is different is that the pipe can no longer be simply treated as a fluid resistance, as it also includes some head. Writing the equation for that resistance with head and getting the signs right is tricky. An easier way to approach the problem is to separate the head and the resistance, and draw the model as in Fig. 7.

Now, with the signs defined in Fig. 7, it is easy to write the correct pressure equation:

$$p_C + p_h - p_R = 0 \quad [15]$$

Perhaps less obvious is the sign on the equation:

$$p_h = \rho g h_h \quad [16]$$

The sign here is can be chosen by simply considering the case in which there is nothing else going on but the head—no flow. The pressure at the + sign for p_h will clearly be larger than the pressure at the - sign, as a result of the head. Hence the signs as marked in Fig. 7 and in [16].

Now we can derive a complete system model, from [15], [16], [10], [11], and a modified version of [12], which only needs a minor modification to substitute pressure across the resistance effect of the pipe, p_R where [12] had total pressure across the pipe p_P .

$$q_p = p_R / R_P \quad [17]$$

Substituting, we obtain the new system model:

$$dp_C/dt = -(pgh_h + p_C)/(R_P C) \quad [18]$$

Both [18] and [13] can be solved by any of the methods we have developed in the course, but since they are first-order linear equations, the first-order solution method would be a good choice. Fluid systems often have nonlinear elements and must be solved numerically.

Figure 6. An open reservoir fluid capacitor emptying through a pipe

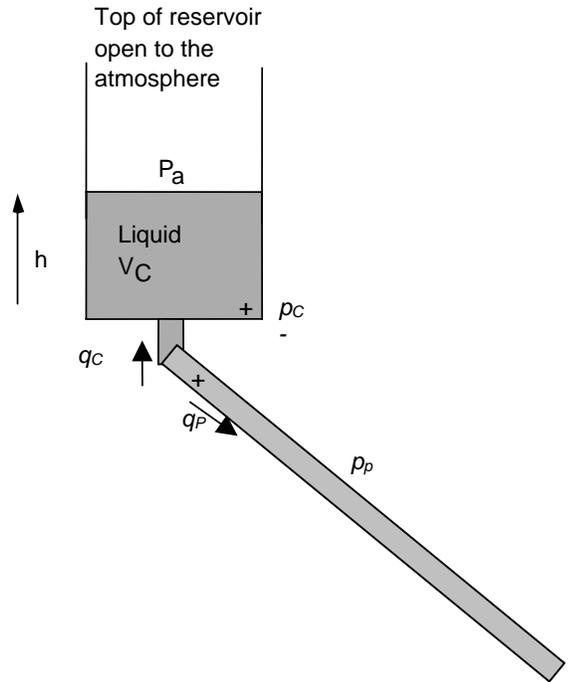
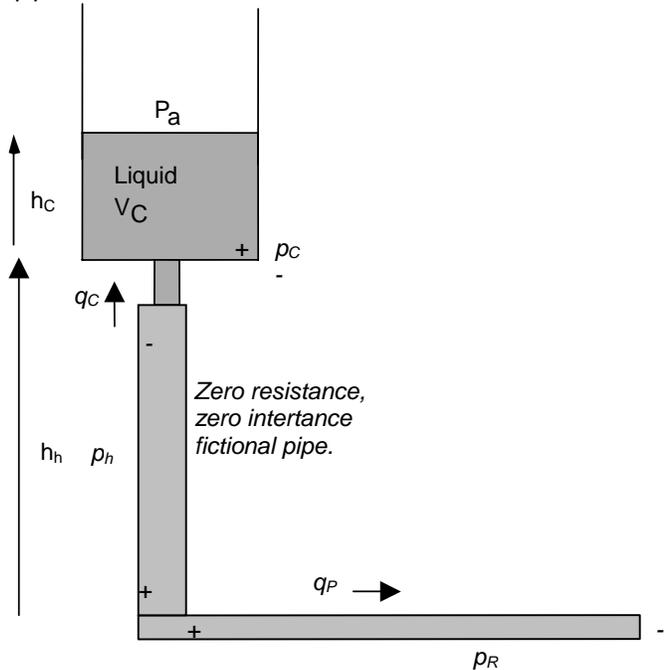
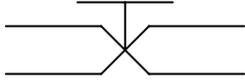


Figure 7. Model of an open reservoir fluid capacitor emptying through a pipe.



Variable or Equation	Symbol	Description	Units	Calculation	Electrical Analogy
q (flow)		A “through” variable. What goes in comes out (except for a capacitor).	m^3/s (\dot{V})		i (current)
p (pressure)		An “across” variable—defined as the difference of this quantity across the element.	$Pa = N/m^2$		v (voltage)
$p = qR$		Resistance, due to pipe or constriction. In real life, it is often nonlinear.	$R = p/q = Pa/(m^3/s)$ Can also write: $Pa \cdot s/m^3 = Ns/m^5$	Discussed separately	$v = iR$
$q^2 R = p^2 / R$	This is a valve symbol; sometimes used without the handle for a non-variable resistor. Often just a pipe is shown.	Dissipation	W		$i^2 R = v^2 / R$
$p = I \frac{dq}{dt}$		Inertance. Momentum and kinetic energy of water in pipe	$Pa/(m^3/s^2) = Pa \cdot s^2/m^3$ $kg/m^4 = Ns^2/m^5$	$I = \rho \cdot l/A$ where l and A are length and cross-sectional area of the pipe, and ρ is density of the fluid.	$v = L \frac{di}{dt}$
$E = \frac{1}{2} I q^2$		Energy	J		$E = \frac{1}{2} L i^2$
$P(s) = Q(s) I s$		Impedance	$Pa \cdot s/m^3 = Ns/m^5$ (same as resistance)		$V(s) = I(s) L s$
$q = C \frac{dp}{dt}$ (one end usually “grounded”)		Fluid capacitor, such as a tank or pressure chamber. Sometimes called an accumulator.	$(m^3/s)/(Pa/s) = m^3/Pa = m^5/N$	$C = A/(\rho g)$ For an open tank; pressure chambers discussed separately.	$i = C \frac{dv}{dt}$
$E = \frac{1}{2} C p^2$		Energy	J		$E = \frac{1}{2} C v^2$
$P(s) = Q(s) \frac{1}{sC}$		Impedance	$Pa \cdot s/m^3 = Ns/m^5$ (same as resistance)		$V(s) = I(s) \frac{1}{sC}$