Analyzing Errors

With rare exceptions, no measurement is exact. Therefore, it is essential to maintain an awareness of measurement error in laboratory work. This exercise will familiarize you with error analysis methods that you can use in labs throughout the remainder of the term. Keep this handout as a reference for future experiments.

Types of Error

There are many types of measurement error, ranging from reading the instructions wrong, to accuracy limits of instruments, to fundamental physical principles that impose limits on the maximum accuracy possible in a given measurement (e.g., the Heisenberg uncertainty principle). Here are some important distinctions between categories of errors.

1. Random error. Suppose you are timing the swing of a pendulum with a stopwatch. You can't do this very precisely, and each time you time a cycle, you get a different number. This variation is random error. A random error is one that will change each time you repeat the measurement or experiment.

2. Systematic error. Suppose the stopwatch runs fast. Each time you measure the swing, your result will be biased in the same direction. This is systematic error. A systematic error is one that consistently biases your measurement in one direction.

3. Modeling error. If you model the motion of the pendulum with the equation $\theta'' + (g/l)\ \theta = 0$, where g is the gravitational acceleration, and l is the length of the pendulum, you will find a slight discrepancy for high-amplitude swings, because the pendulum actually behaves as $\theta'' + (g/l)\sin\theta = 0$. This is a modeling error—the restoring force was not modeled accurately by the first equation. Another type of modeling error occurs when, rather than modeling an effect incorrectly, you just plain ignore it. Both of these models ignore friction and won't predict the gradual decay of the swing. There can also be *unmodeled dynamics*—additional states in the system that were not modeled. For example, if the pendulum has springy rod connected to the pivot, there can be an additional mass-spring oscillator complicating the motion. This would be an example of unmodeled dynamics.

Systematic error and random error are types of experimental error; they lead to measurements that do not represent the true behavior you are trying to measure. In the case of modeling error, however, your measurement is correct, and the error is in the prediction to which you are trying to compare the measurements. If you are looking for the source of error in an experiment, make sure you at consider all of these categories as possibilities.

Calculations Using Uncertain Measurements

In most experimental work, you make measurements (which are not exact), and then perform some calculations using them. The results of the calculations will also be uncertain, but by how much? Consider calculations using measurements x and y, with uncertainties δx and δy.

1. **Addition and subtraction**: What happens if we add measurements x and y, with uncertainties δx and δy?

$$z = x + y$$
$$z \pm \delta z = (x \pm \delta x) + (y \pm \delta y) = (x + y) \pm (\delta x + \delta y)$$
$$\Rightarrow \delta z = \delta x + \delta y$$

The bottom line is the resulting rule we can use. It says that the absolute error in the sum is the sum of the absolute errors in x and y. The same is true for subtraction. If we have the errors as percentage errors, we need to convert to absolute error first. For example, suppose we measure the mass of an empty container as 10 g ± 10%, and the total mass, full, as 320 ± 5 g. The mass of the contents is 310 ± (5 + 0.1(10)) g, or 310 ± 6 g.

2. **Multiplication and division**:

$$z = xy$$
$$z \pm \delta z = (x \pm \delta x)(y \pm \delta y) = xy \pm x\delta y \pm y\delta x \pm \delta x\delta y$$
$$\Rightarrow \delta z = \pm x\delta y \pm y\delta x \pm \delta x\delta y$$

The preceding equation looks messy and hard to remember, but if we rewrite it in terms of fractional (or percent) error it gets better:

$$\frac{\delta z}{z} = \frac{\pm\ x\delta y\ \pm\ y\delta x\ \pm\ \delta x\delta y}{xy} = \pm\frac{\delta x}{x} \pm \frac{\delta y}{y} \pm \left(\frac{\delta x}{x}\right)\left(\frac{\delta y}{y}\right)$$

If the errors are small, the last term can be neglected, and we have simply

$$\frac{\delta z}{|z|} = \frac{\delta x}{|x|} + \frac{\delta y}{|y|}$$

The absolute value is used to allow removing the ± signs, with the convention that the error is reported as a positive quantity. We are looking for the worst case in which the signs of the errors are such that they affect the result in the same direction.

The resulting rule is this: The percentage error in the product is equal to the sum of the percentage errors in *x* and *y*.

The same is true for **division**. The percentage error in the quotient is equal to the sum of the percentage errors in *x* and *y*.

For example, if x = 10 ± 0.1 and y = 5 ± 0.2, then $\delta x/|x| = 1\%$,
$\delta y/|y| = 4\%$, and x/y = 2 ± (1% + 4%) = 2 ± 5%; now 5% of 2 is 0.1, so the result is *x*/*y* = 2 ± 0.1.

3. Arbitrary functions:

Here is a general case: a function of two variables. The result applies to functions of one variable as well—see the example below.

$$z = F(x,y)$$

$$z \pm \delta z = F(x \pm \delta x, y \pm \delta y) \approx F(x,y) \pm \frac{\partial F}{\partial x}\delta x \pm \frac{\partial F}{\partial y}\delta y$$

$$\Rightarrow \delta z \approx \left|\frac{\partial F}{\partial x}\right|\delta x + \left|\frac{\partial F}{\partial y}\right|\delta y$$

The approximation in the second line is just the first-order Taylor series for *F*. The error in the output is calculated using the derivatives of the function. In the case of a function of just one variable, we just need the term for that one variable:

$$z = F(x)$$

$$\delta z \approx \left|\frac{\partial F}{\partial x}\right|\delta x$$

For example, if $F(x) = x^{0.5}$, and $x = 10 \pm 2\%$, then

$F(x) = x^{0.5} \pm (d(x^{0.5})/dx)\ \delta x = 10^{0.5} \pm 0.5\ x^{-0.5}\ \delta x = 3.162 \pm 0.158\ (0.2) = 3.16 \pm 0.03$.

Note that it would not be meaningful to report this result with any more digits than are shown in the final result, as they would not be significant. The error itself should never be reported with more than two digits.

To address experimental error rigorously, you must record an error estimate with every measurement. Then in each calculation, you propagate the error through every step as shown above, to arrive at an error estimate for the result. This level of detail will not be required for all labs. However, it is essential that you understand how to do it. This will enable you to use it when it is required, and will help you determine when it is necessary.