Introduction to Electrical Systems Modeling

Part I. DC analysis techniques

DC analysis techniques are of course important for analyzing DC circuits—circuits that are not dynamic. But why do we discuss them in a dynamic systems class? Firstly, they provide good practice and help build intuition for circuits. Secondly, we will later introduce techniques that allow analyzing many dynamic circuits as if they were simple DC circuits. That will make them really easy to solve—if you are proficient with DC circuits.

Part I of this document presents the fundamentals of circuit analysis. The information in this section is sufficient to solve any DC circuit. More importantly, it is also essential for modeling dynamic circuits, and the techniques are presented in a way that makes it directly transferable to analyzing dynamic circuits, as well as nonlinear circuits that you might encounter in an advanced electronics course.

Part II presents "circuit manipulations techniques"—shortcuts that make it faster and easier to solve DC circuits. Although the techniques are presented in terms of dc circuits, they are also applicable to dynamic circuits, as you will see in Section III of this handout, and later in the course.

Part III introduces dynamic elements and shows how to develop a state-variable model of a dynamic circuit.

What's different from what you learned in physics

Modeling of electrical systems is best approached by thinking separately about the element laws and the interaction laws. The interactions laws are Kirchoff's voltage law (KVL) and Kirchoff's current law (KCL). The element laws are relationships for individual components such as resistors (ohm's law, $V = iR$), capacitors, inductors, etc. In some textbooks and courses, KVL and the element laws are mixed together—KVL is written like

$$v_1 - iR_2 - iR_3 = 0.$$

That works for simple physics textbooks problems, but it's not really KVL. KVL is actually simpler. It is just a relationship between voltages; for example

$$v_1 - v_6 - v_5 = 0.$$

Learning to think about the interaction laws and the element laws separately is important because once you learn them separately, you are prepared to deal with any type of problem involving any kind of element configured in any kind of circuit—you won't be limited to simple physics-textbook problems.

But before we discuss KVL and KCL, we must consider variables to work with and choose sign conventions.

Variables

Current is the flow of charge. Thinking of current as flow can help you gain intuition about circuits. Charge is measured in coulombs, so its flow is in coulombs/second. Coulombs per second are also called amperes, or amps, abbreviated A.

Voltage can be thought of as the "force" that pushes the flow of charge. In an analogy to a hydraulic system, voltage would correspond to pressure. It is usually pressure differences that matter, not absolute pressure. For voltages, it is *always* just the voltage difference that matters.   Sometimes people will discuss the voltage at a point, without reference to a voltage difference. The only way that can be meaningful is with reference (implicit or explicit) to an arbitrary reference point, which is often called ground. What matters for the operation of circuit elements, however, is just the voltage difference across those elements, not the voltage with respect to an arbitrary reference point.

Voltage can also be defined in terms of potential energy of a unit charge.

Sign Conventions

As in mechanical systems we must define the sense of each variable we use, and mark that on the diagram (in electrical systems, a circuit diagram or *schematic*). Again, a given arrow or +/- polarity indication does ***not*** indicate which way things go; rather, it just indicates which direction we are going to call positive.

There is one difference, however. In electrical systems, there is a hard and fast rule about the relationship between the two main variables, current and voltage: For any element X (represented by a box here), the signs of voltage and current must be such that positive current is defined flowing from the positive voltage end to the negative voltage end. Actual current and voltage can be in any direction, but the definitions of what's positive must follow this rule.

There is one exception: Currents and voltages of sources (voltage sources and current sources) do not need to follow this rule. They may, but it is not required.
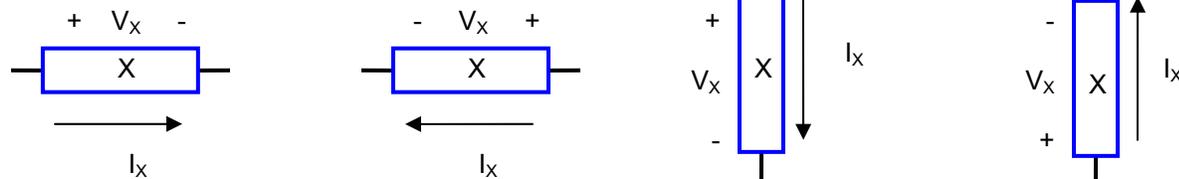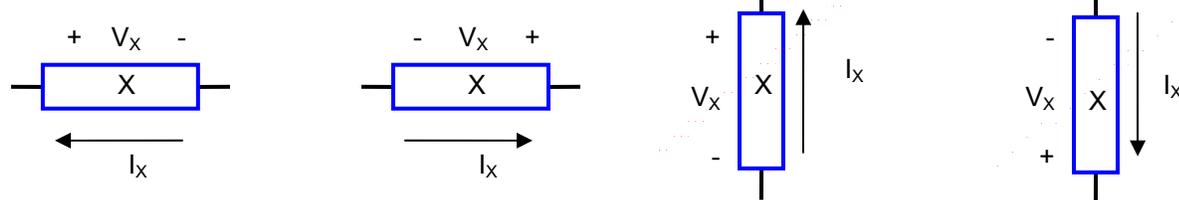

Fig. 1. Sign conventions for electrical elements.

KVL (Kirchoff's Voltage Law)

Kirchoff's voltage law (KVL) states that if you follow any closed loop around a circuit, the sum of voltage drops across each element must sum to zero.

$$\sum_{\substack{closed \\ loop}} v_n = 0$$

where v_n is the voltage across some element n. In this sum, add voltage positively, if, in going around the loop, you go up in voltage when you hit that element. That is, if you go from minus to plus in that element's voltage indication. Subtract it if you go down — that is, from plus to minus. (Actually you can follow any consistent convention you want, but the one described makes the most physical sense.)

Kirchoff's voltage law is often explained in terms of conservation of energy, but it is also very simply a consequence of the fact that the voltage for an element is defined as a voltage ***difference***; it is an inevitable consequence of that definition independent of any physics. (Where the physics comes is in showing that it is possible to uniquely define a voltage difference between two points.)

The definition above is all you need to know to apply KVL. You can follow the algorithm described and generate KVL equations for any circuit, without any further thought. However, for the reader who would like to have a better understanding of what it is really about, rather than just a rule to follow by rote, some way to visualize voltage and KVL is helpful.

One way to visualize voltage is as a height. This makes electrical potential analogous to gravitational potential, which is necessarily proportional to height (in a contant gravitational field.) For example, consider the following circuit, containing four elements. The elements are drawn as boxes to indicate that they could be any circuit elements. (The differences between circuit elements will be taken into account in their element laws; they don't matter at all for KVL.)
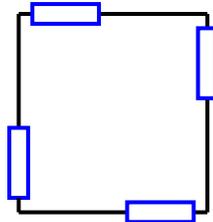
Fig. 2. Example circuit with four generic elements.

Before we go any futher, we should define variables; since we are only discussing KVL, we need only definitions of voltages. They can be defined in any polarity. Once choice is shown below.
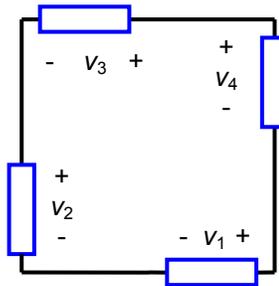
Fig. 3. Example circuit with voltage variables defined.

For simplicity, let's consider the case in which all the variables, v_1 through v_4, are positive numbers. To visualize this, first let's lay out the circuit in the first two dimensions of a three-dimensional space:
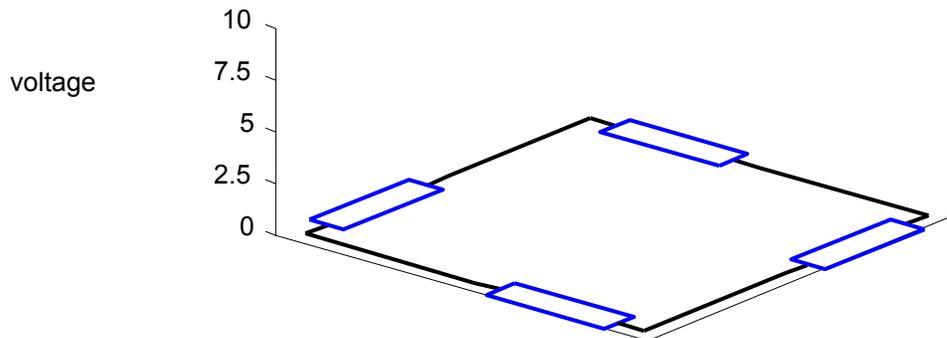
Fig. 4 Example circuit laid out in two dimensions of a 3-D space.

Now we "turn on the power" and use the third dimension to indicate voltage.
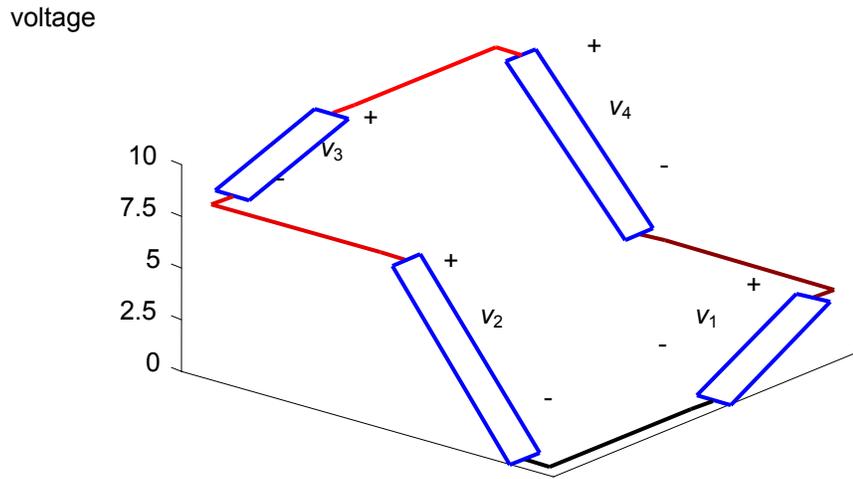
voltage

Fig. 5. Example circuit with voltage shown in the third dimension.

Consider KVL from this diagram. We can start at any point. Starting at the "base" of element 2, we go up by v_2. Then we go up by v_3. The running total becomes $v_2 + v_3$. Then we go down by v_4 and finally down by v_1. The total around the loop is

$$v_2 + v_3 - v_4 - v_1 = 0 \tag{1}$$

The sum around the loop *must* be equal to zero because we come back to the starting point, marked as zero volts.

In order to draw this example, we had to assume particular values of the voltages; these were $v_2 = 8$ V, $v_3 = 2$ V, $v_4 = 7$ V, and $v_1 = 3$ V. However, we did *not* need those assumptions in order to write KVL (1). We could have written that simply from looking at Fig. 3. Starting from the same point and going around the loop clockwise, we go – to + on element 2 (+), – to + on element 3 (+), + to – on element 4 (–), and + to – on element 1 (–), which gives the same KVL equation (1).

Since (1) can be derived independent of the numerical values for the voltages (v_1 through v_4), it ought to work just fine regardless of the values of the voltage, and regardless of their signs. Fig. 6 shows the same circuit with the same definitions of the voltage variables, but with different values for those voltages.
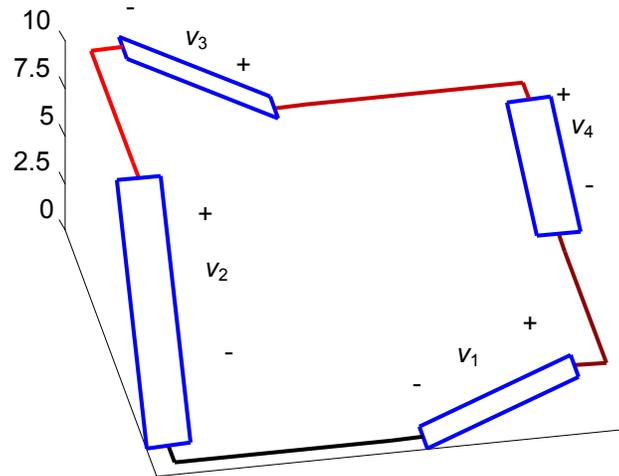
voltage

Fig. 6 The same circuit at a different point in time, when the voltages take on different values.

Here, v_3 is negative, as can be seen from the fact that its (+) end is below its (–) end. More precisely, the voltages are $v_2 = 10$ V, $v_3 = $-4 V, $v_4 = 3$ V, and $v_1 = 3$ V. To confirm that the same KVL equation (1) holds, we can plug in the numbers: 10 V – (–4 V) – 3 V – 3 V = 0

As stated under Sign Conventions, a given +/- polarity indication does ***not*** indicate the actual polarity; rather, it just indicates which direction is called positive. Here the actual state of the circuit is opposite the direction of the polarity indication. The variable v_3 is a negative number, but that doesn't matter for KVL. KVL works fine regardless.

The bottom line for using KVL is this: For writing down KVL, you don't need to know the actual polarities of the voltages in the circuit. Nor do you need to know whether your variables will contain positive or negative numbers. All you need to know is which direction you have defined them. The resulting equation will work for any values of the variables, positive or negative, as long as you stick to the original variable definitions as marked on the circuit diagram with (+) and (–).

Multiple Loops

When there are multiple possible loops in a circuit, you may write a KVL equation for each loop…but you don't need to. Not all of the equations are linearly independent, so they provide redundant information. The minimum number needed is equal to the number of "windows in the circuit. For example, in Fig. 7, there are three possible KVL equations: for the left loop, $v_1 - v_2 - v_3 = 0$; for the right loop, $v_3 - v_4 = 0$; and tracing around the outer loop one obtains $v_1 - v_2 - v_4 = 0$. All of these are correct, but you only need two of them. Any two. Any one of them can be dropped, because it can be obtained algebraically from the other two. You can tell this before writing the equations because there are two windows (a tall one on the left and a square one on the bottom right), and the number of equations needed is equal to the number of windows.
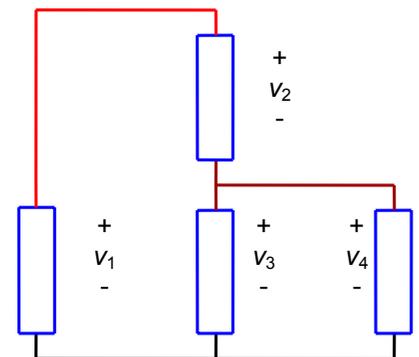

Fig. 7. A circuit with multiple KVL loops.

KCL (Kirchoff's Current Law)

Kirchoff's current law (KCL) simply says that, at some node where wires connect, the total current leaving the node must equal the total current entering the node. That must be true if a) charge is conserved (it is), and b) no charge builds up at the node. In practice, a tiny amount of charge, measured in pC or fC (10^{-12} or 10^{-15} coulombs), can build at at a node, but in the lumped-element approximation we assume this is zero[1].

For example, Fig. 8 shows three currents at a node; one in and two out. KCL says $i_1 = i_2 + i_3$.
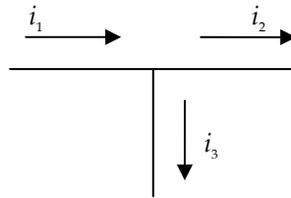

Fig. 8 Example of currents at a node.

The example in Fig. 9 might be more confusing. It shows three currents all entering the node. KCL says the sum entering equals the sum leaving; i.e., $i_1 + i_2 + i_3 = 0$. But isn't the picture wrong? Wouldn't charge be accumulating at the node if all the currents were entering the node? Yes, it would be, but remember that the arrows don't really mean that the current is going that way — they just mean that we'd call it positive if it was. So drawing it that way is fine, just as long as not all the currents are positive. Our KCL equation also says that not all the currents can be positive.

Note that the same is true in Fig. 7 — the directions of the arrows don't necessarily correspond to the directions the currents are actually going.
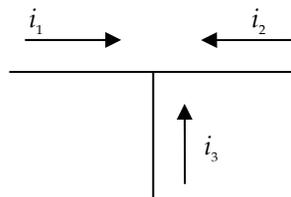

Fig. 9. Example of currents at a node.

We've been talking about nodes, but we haven't defined a node. That's because KCL really works just fine for almost any definition of a node. For example consider the circuit in Fig. 10. We could view this as two nodes — the three-way junctions like we have been looking at. The left one yields $i_1 + i_3 = i_2$, and the right one yields $i_2 = i_4 + i_5$. Alternatively, we could look it as one big node and write $i_1 + i_3 = i_4 + i_5$. This result can also be obtained by setting the first two equations equal. We see that the result is the same no matter which approach we take.
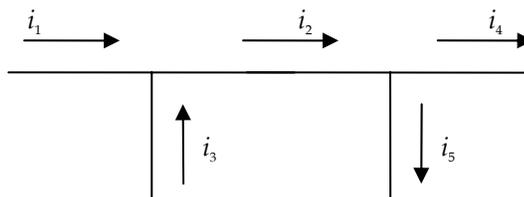

Fig. 10. How many nodes here?

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[1] If we want to model this effect, we can still use the lumped-element approximation — we simply add a lumped element to explicitly model the charge accumulation: a capacitor. The small amount of capacitance in wiring is called "stray capacitance" or "parasitic capacitance". In the new model, with the stray capacitance drawn in explicitly, KCL still holds at all nodes.

Thinking about nodes

An important rule of lumped-element circuits is that everywhere along a wire is at the same potential. Thus, it doesn't matter how the wires are arranged, as long as the *topology* of what is connected to what remains unchanged. Thus, although we could, for KCL, consider Fig. 10 to include two nodes, it is really all one big node—all at the same potential, because you can trace over the whole thing with a pen and never cross an element or lift your pen.

Fig. 11 shows an example for a complete circuit. In this circuit, it might look like there are many different nodes. But if we consider following along each wire, and shading everywhere we can go without crossing a line, we see that there is one huge node in the middle that extends from the left of element 4 to the top of element 3. We can shade that entire node in one color. Each of the three nodes is shaded a different color/pattern in Fig. 12.
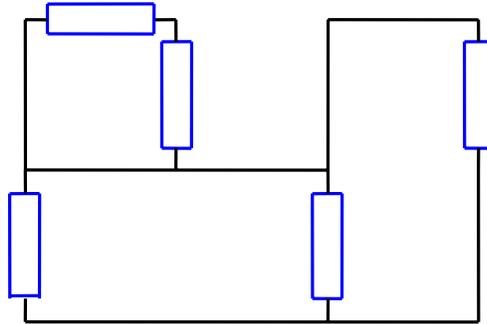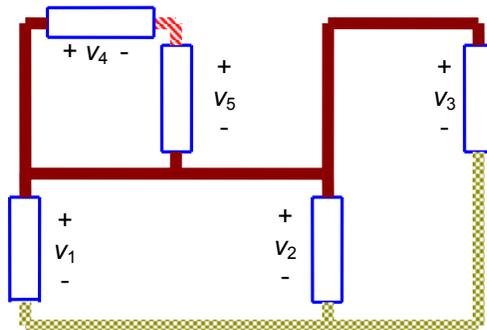

Fig. 11. Another circuit example.

Fig. 12. The circuit example in Fig. 11, with each of the three nodes shaded differently.

This technique of node shading is not necessary to derive equations, but it can be very helpful to gain insight about a circuit. You can do it either with colored pens or higlighters, or by using different patterns with a pencil (dark, heavy line; diagonal hash marks; perpendicular hashmarks, squiggles, etc.). For example, from shading nodes in Fig. 12 we see that the voltages v_1, v_2, and v_3 must all be the same—they are all the voltage difference between the solid brown line and the checked yellow line. We could have gotten that information from writing KVL for the lower left loop and the right loop, but node shading makes the result more immediately obvious. We also see from Fig. 12 that between the two series elements (4 and 5), there is a mini-node. Although writing KCL for this node might seem superfluous, KCL is in fact what tells us that the current in these two series elements must be identical. Be careful about doing this, however—you need to specify the current polarities before you can say whether they are equal and opposite or equal and in the same direction.

We can also visualize nodes with 3-D diagrams. **Fig. 13** shows the same circuit with voltage plotted on a vertical scale. Each of the three nodes is a level "plateau". If you are viewing this document in color, you will see that they are also shaded, with a color code representing voltage: Black is zero volts, red is positive voltage, and yellow is

negative voltage. The colors are shaded between black and red or yellow to indicate intermediate voltages, with smaller magnitudes than the maximum magnitude in the plot.

As you are solving problems, you will rarely, if ever, draw the circuit in three dimensions. However, the shading in Fig. 12, which is easy to do yourself, conveys much of the same information about nodes as the three-dimensional plot in **Fig. 13**. **Fig. 13** is included here to help you gain an understanding of the physical meaning of a node—a region that is all at the same potential.
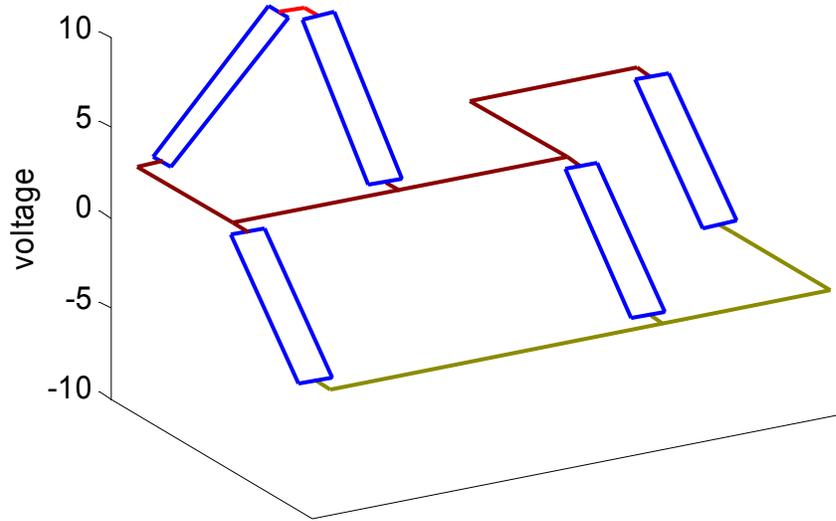

Fig. 13. The circuit example in Fig. 11, shown in three dimensions.

Element Laws

There is a chart at the end of this handout listing summarizing information for each element. Here, the dc circuit elements are discussed.
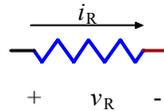

Fig. 14 A resistor.

Resistors (Fig. 14) allow current flow (through them) proportional to the voltage difference across them, according to Ohm's law:

$$v = iR. \qquad (2)$$

However, in a system, you should never write Ohm's law as in (2). Rather, you should be sure to distinguish *which* voltage equals *which* current times *which* resistance. So Ohm's law for systems is really

$$v_n = i_n R_n. \qquad (3)$$

where n indicates which element, current, and voltage we are discussing.

Voltage sources maintain a given voltage no matter what. Usually the voltage is a constant, but it might be a time-varying function. They do this regardless of current. We know everything about the voltage of a voltage source, but we know nothing about the current—the current could do anything.

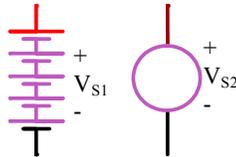Two different symbols for voltage sources are shown in Fig. 15.

Fig. 15 Two different voltage source symbols.

Batteries and power supplies are practical examples of voltage sources. Neither one behaves exactly according to the ideal law of maintaining a constant voltage, but both can come close, and a good approach to modeling systems with non ideal voltage sources is to first model them as ideal voltage sources, and then later consider any deviations from ideal behavior.

Current sources maintain a given current no matter what. Usually the current is a constant, but it might be a time-varying function. They do this regardless of voltage. We know everything about the current of a current source, but we know nothing about the voltage—the voltage could do anything. A common mistake is to assume that the voltage is zero, but it is a very bad assumption to set something equal to zero because you know nothing about it!

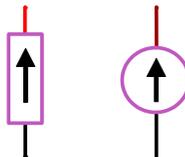Two different symbols for voltage sources are shown in Fig. 16.

Fig. 16 Two different current source symbols.

Current sources are rare in practice, but other circuit elements are approximated as current sources for certain conditions. Thus, it is important to become comfortable with solving circuits that include current sources, even though you won't see often see one in the lab.

General Step-by-Step Solution Method

There's actually nothing new to learn here—once you know KVL, KCL, and the relevant element laws, you can write down all the equations you need, and the rest of the solution process is algebra. But for convenience, here is a list of everything you need to do to solve a circuit. This approach works for any circuit, including a dynamic circuit, although we will provide a more focused step-by-step method for generating state-space models for dynamic circuits in the second half of this handout on dynamic circuits.

1. Optional: look for ways you can simplify the circuit before you start analyzing it—methods for simplification by circuit manipulation will be presented later. Using simplification is highly recommended, but you shouldn't become dependent on them—you should be able to solve any circuit without them by following the steps below.

2. Mark variable names and polarities on the circuit. Make sure that the voltage and current polarities for any given element agree, as in Fig. 1.

3. Write KVL equations. The number needed is usually the number of "windows" in the circuit. More are never needed, and fewer may be OK (see the next section).

4. Write KCL equations. The number needed is technically the number of nodes in the circuit. That includes nodes where just two elements are connected; all KCL says there is that the currents are equal in the two elements. If you are comfortable concluding that without formally writing KCL, you can just write KCL for nodes where three or more branches come together. See below for situations in which you can skip writing KCL for some nodes.

5. Write element-law equations for each element.

6. You now have all the equations you need. Proceed with algebra: strategically substitute variables to eliminate things you aren't interested in and solve for what you want to know in terms of known quantities.

When you can skip a KVL or KCL equation

1. If there are parts of the circuit you aren't interested in solving, you may be able to skip KCL or KVL for those sections.

2. KVL involving a branch with a current source in it won't normally be needed. It is still possible to write a correct KVL equation for the branch with a current source, but it will include the voltage across the current source. The element law for a current source says the voltage across it can be anything, so the KVL equation includes that unknown, and can't help you solve other parts of the circuit. However, it can be useful if the voltage across the current source is one of the things you ultimately wanted to solve for.

3. KCL for nodes for which one branch has only a voltage source in it are usually not needed, just like KVL for branches with current sources. It is of course possible to write KCL for these nodes, but it won't tell you anything very useful because the element law for a voltage source says that the current could be anything. So the only reason to write KCL would be if you wanted to find out the current through the voltage source in question.

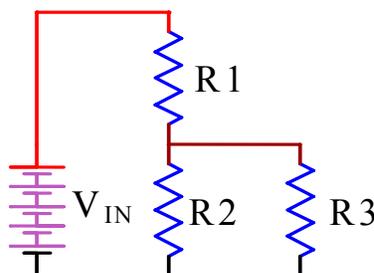## Example: Complete solution of a circuit

Fig. 17 An example circuit to solve.

Consider the circuit shown in Fig. 17. Suppose we want to find the current in resistor R3. We will follow the General Step-by-Step method presented above.

1. Optional: look for ways you can simplify the circuit before you start analyzing it.

   We'll skip this, in order to present the most general analysis.

2. Mark variable names and polarities on the circuit. Make sure that the voltage and current polarities for any given element agree, as in Fig. 1.

The circuit with polarities marked is shown in Fig. 18. Note that the polarity for each element may be chosen arbitrarily, but for a given element, once the voltage polarity is chosen, the current polarity necessarily follows, or vice versa.
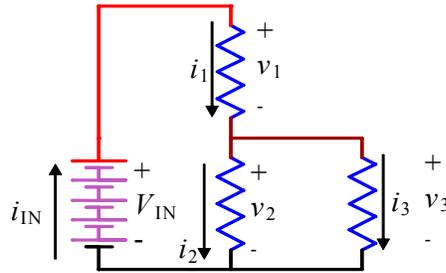

Fig. 18. Varibles defined for the circuit in Fig. 17.

3. Write KVL equations. The number needed is usually the number of "windows" in the circuit.

   We can write any two of the three possible KVL equations. The obvious ones would be around the left window and around the right window, although the outer perimeter could substitute for either of those. Since we are eventually solving for a variable related to R3, using variables related to it as much as possible is a wise strategy.

   $$V_{IN} - v_1 - v_3 = 0 \qquad (1)$$
   $$v_2 - v_3 = 0 \qquad (2)$$

   The second of these could be derived "by inspection", by noting that $v_2 - v_3$ are both equal to the voltage between the middle node and the lower node, so it must be that $v_2 = v_3$.

4. Write KCL equations. The number needed is technically the number of nodes in the circuit. That includes nodes where just two elements are connected; all KCL says there is that the currents are equal in the two elements. If you are comfortable concluding that without formally writing KCL, you can just write KCL for nodes where three or more branches come together.

   $$i_{IN} = i_1$$
   (This one is not needed for two reasons: firstly, equations involving the current of a voltage source are not useful, because we have no idea what the current is, and secondly, it may be "obvious" because V_{IN} and R1 are in series.)
   $$i_1 = i_2 + i_3 \qquad (3)$$

5. Write element-law equations for each element.

   $$v_1 = i_1 R_1 \qquad (4)$$
   $$v_2 = i_1 R_1 \qquad (5)$$
   $$v_3 = i_3 R_3 \qquad (6)$$

   The element law for the voltage source is just $V_{IN} = V_{IN}$ (a given constant)

6. You now have all the equations you need. Proceed with algebra: strategically substitute variables to eliminate things you aren't interested in and solve for what you want to know in terms of known quantities.

From here on is just algebra. Since we want to find a current eventually, we can eliminate the voltages from the KVL equations using the element-law equations, and write them as

$$V_{IN} - i_1 R_1 - i_3 R_3 = 0 \qquad (7)$$
$$i_2 R_2 = i_3 R_3 \qquad (8)$$

Using (8) and (3) (KCL), we can write

$$i_1 = i_3 R_3 / R_2 + i_3 \qquad (9)$$

(9) allows eliminating i_1 from (7) to get

$$V_{IN} - (i_3 R_3 / R_2 + i_3) R_1 - i_3 R_3 = 0 \qquad (10)$$

Solving this for i_3 yields

$$V_{IN} = i_3 \{ (R_3 / R_2 + 1) R_1 + R_3 \} \qquad (11)$$
$$i_3 = V_{IN} R_2 / \{ (R_3 + R_2) R_1 + R_2 R_3 \} \qquad (12)$$

Similar algebra could be used to solve for any of the voltages or currents, or, once a value of is found, earlier equations could be used to relatively quickly solve for values of other parameters.

Based on V_{IN} = 10 V, R_1 = 6.25 kΩ, R_2 = 10 kΩ, and R_3 = 6 kΩ, i_3 = 0.625 mA. The circuit with voltage and currents is shown in Fig. 19.
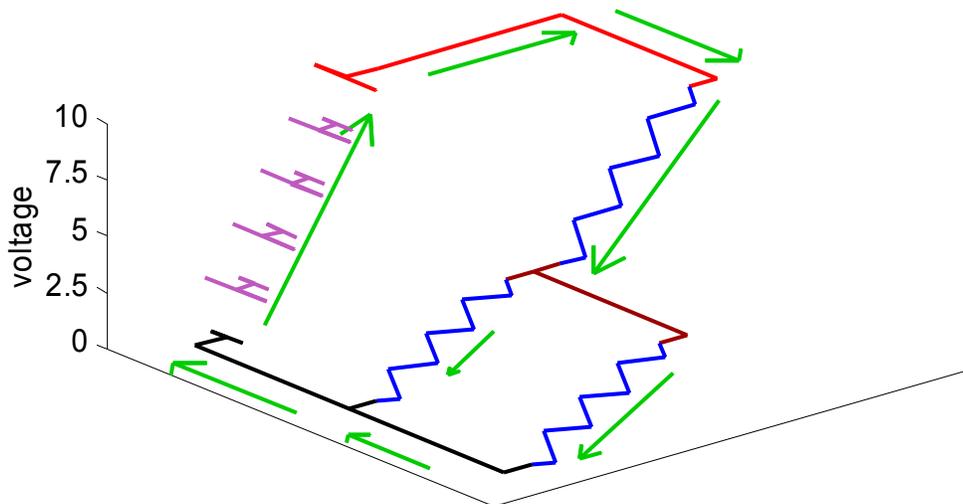

Fig. 19 Solution of example circuit. Arrows indicate the direction of actual current flow; their length is proportional to current magnitude. This is a different use of arrows from their use in defining variables.

Part II. Circuit Manipulation

The standard analysis procedure for a circuit is as follows:
2.   Define polarities of voltages and currents on all components.
3.   Write KVL equations.
4.   Write KCL equations.
5.   Write component equations.
6.   You now have a complete set of equations. Use algebra to reduce them to what you are interested in.

However, doing all that algebra provides multiple opportunities for errors, and gets you away from thinking about the physical system. It also can be tedious. Thus, electrical engineers generally only resort to that full process when they get stuck with something unusual and difficult. Most of the time, they instead use *circuit manipulation* techniques with which one can reduce a complex circuit to a simpler circuit by working with the circuit diagram rather than with algebra. The four most important equivalencies used in circuit manipulation are shown in Fig. 20.

Parallel combination

$$Rp = \frac{R1 \cdot R2}{R1 + R2} = \frac{1}{\frac{1}{R1} + \frac{1}{R2}}$$

Series combination

$$Rs = R1 + R2$$

"Current Divider": This is a way to find the current in each of two parallel resistors, given the total current.

$$I1 = Itot \frac{R2}{R1 + R2}$$

"Voltage Divider": This is a way to find the voltage across each of two series resistors, given the total voltage across the two

$$Vout = Vin \frac{R2}{R1 + R2}$$

In this third way of drawing the voltage divider, the current, Iout, must be zero.
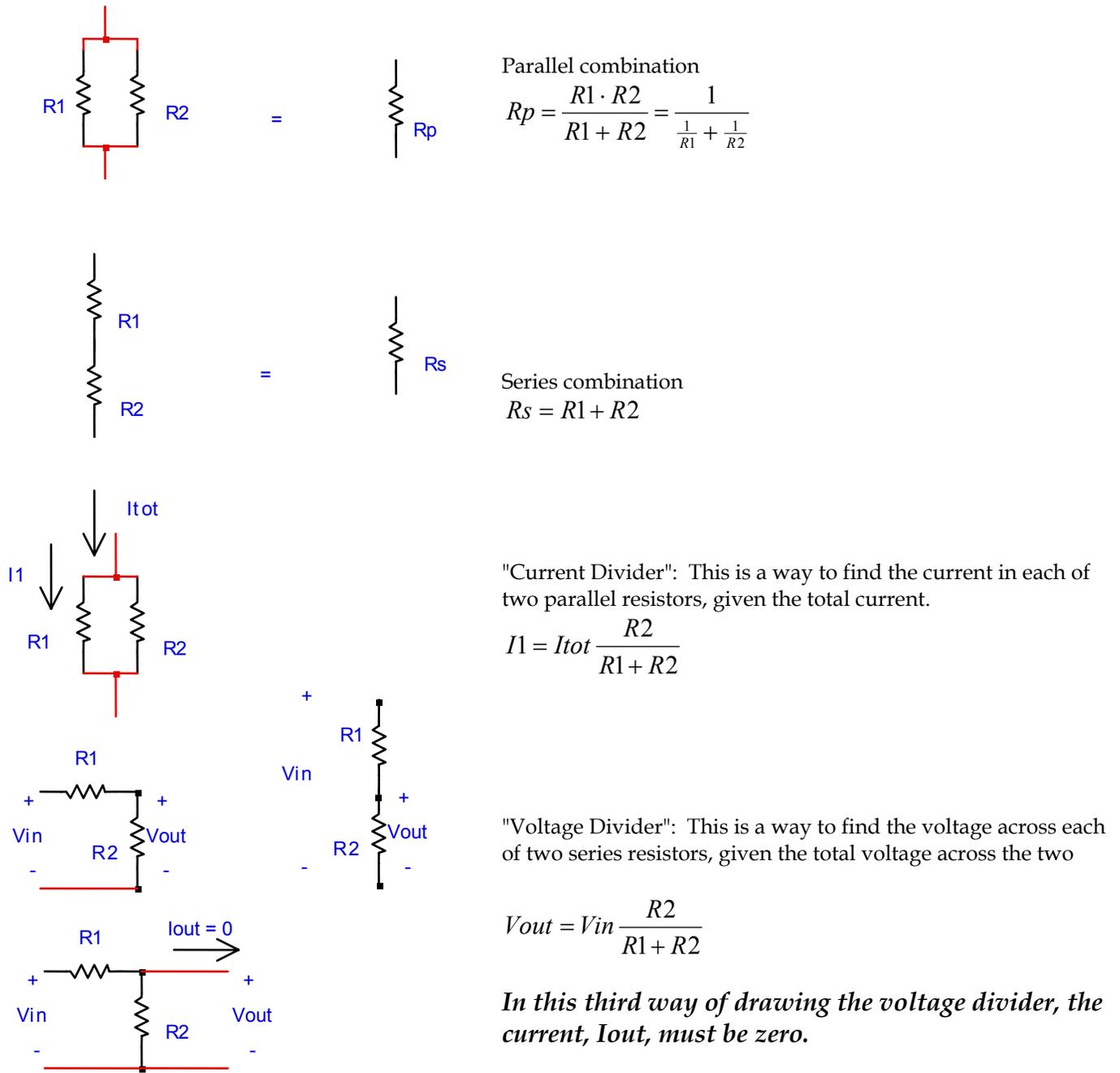
Fig. 20 Equivalencies for circuit manipulation.

Perhaps surprisingly, the aspect of applying these circuit simplifications is recognizing when two resistors are in parallel or in series (or neither). A rule that can help with that is the "circle rule". If two elements are in series (or parallel), and you are going to replace them with a single resistor equivalent to the series (or parallel) combination, the replacement has only two terminals. So you must be able to circle the original series (or parallel) combination, *and have only two wires cross the circle.* Those two wires are where you are going to connect the replacement resistor. If there is a third wire, the new replacement resistor won't work, because there won't be anywhere to connect the third

wire. Sometimes it is possible to rearrange the wires a bit, while keeping the topology (structure of the connections) and node) intact, such that it is possible to circle two elements with only two wires crossing the circle where it was not possible to do that in the original drawing.
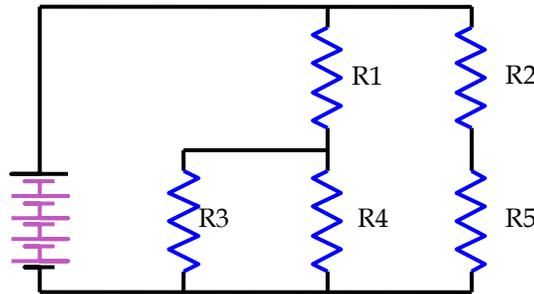

Fig. 21 Example circuit for circuit manipulation.

Consider the example in Fig. 21. It might appear that the first thing to combine would be R1 and R2. However, if we try circling them as shown in Fig. 22, there are three wires crossing, and it is not possible to drop R_P in as a replacement—if it is connected to R4, there is nothing connect to R5.

Another combination we might try is the series combination of R1 and R4. However, if we try circling them as shown in Fig. 23, there are three wires crossing, and it is not possible to drop R_S in as a replacement—there is nowhere to connect R3.
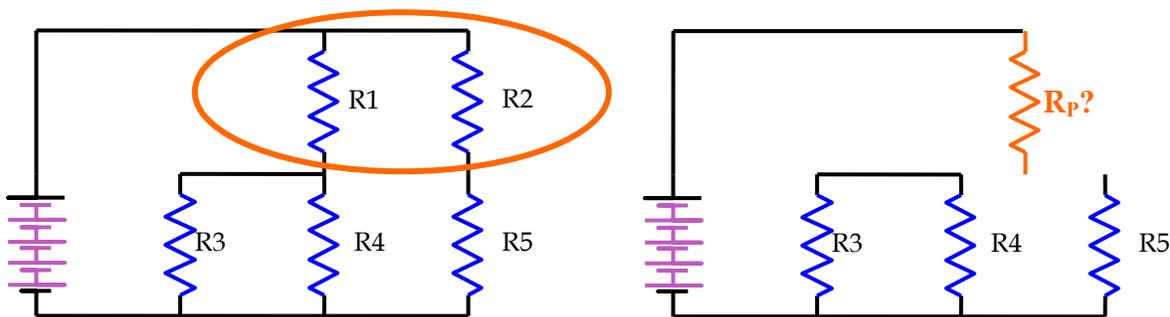

Fig. 22. Attempting to combine R1 and R2 in parallel. ***This does not work!*** The circle is crossed by three wires, and when the two "parallel" resistors are removed, there remain three terminals, but the replacement resistor has only two terminals.
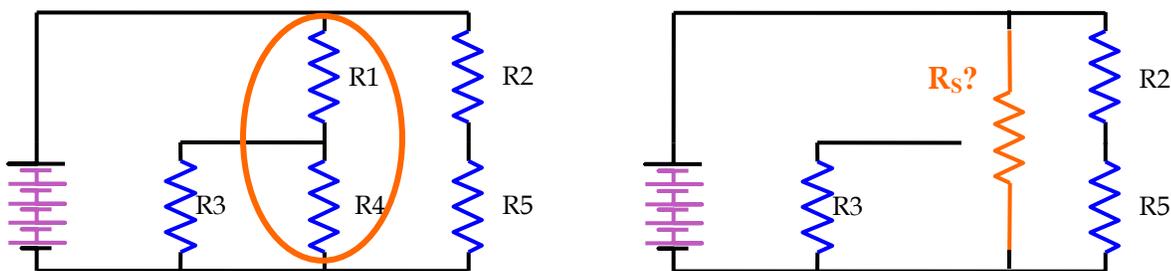
Fig. 23. Attempting to combine R1 and R4 in series. . ***This does not work!*** The circle is crossed by three wires, and when the two "series" resistors are removed, there remain three terminals, but the replacement resistor has only two terminals.

Fortunately there are valid resistor combinations that can be applied. R2 and R5 in series is the most obvious combination. This is shown in Fig. 24. Next we can consider combining R3 and R4, as shown in Fig. 25. At first there are three wires crossing, but two of them are really the same node, so the circuit can be redrawn such that only two wires cross, and it is valid to combine the two, as is shown on the left in Fig. 26. The next two simplification steps are straightforward and are described in the captions of Fig. 26 and Fig. 27.
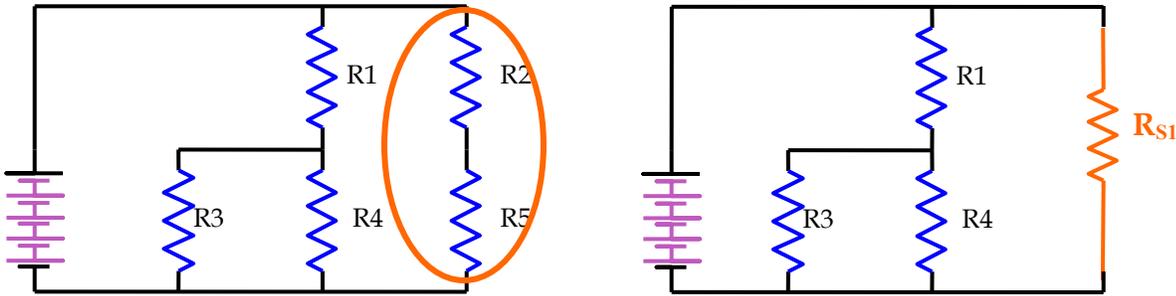

Fig. 24. Combining R2 and R5 in series. The circle is crossed by only two wires, so this is valid.
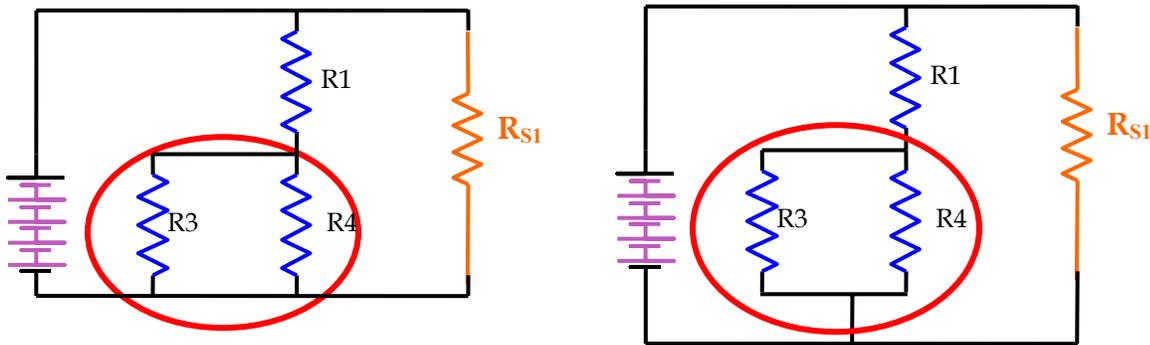

Fig. 25. Combining R3 and R4 in parallel. The circuit needs to be redrawn before we can literally have just one wire crossing the circle, but we can see that it will work from the first drawing because the two that cross at the bottom are actually the same node.
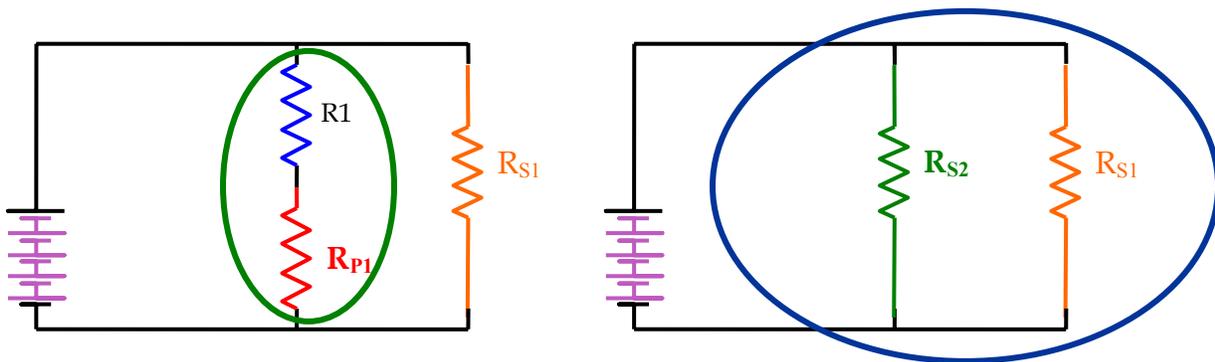
Fig. 26. Two more steps in simplifying the circuit. R_{P1} and R1 are combined in series; then R_{S1} and R_{S2} are circled in preparation for combining in parallel.

Fig. 27. Final simplified circuit. R_{S1} and R_{S2} are have been combined in parallel. The current in the resistor can now be found by simple application of Ohm's law.

Circuit solutions using circuit manipulation

The problem with the approach taken with the previous example is that, depending on what you wanted to find, the variable of interest may have disappeared from the circuit at some point during the simplification. There are two general approaches to that problem:

First approach: Reduce the circuit all the way down to a single resistor, and then work backwards to find the parameter of interest. The "voltage divider" and "current divider" rules in Fig. 20 can be particularly helpful here. This method can always be applied to dc circuits that have a single source.

or

Second approach: Only perform those circuit reduction steps that preserve the variables of interest on the schematic. Then apply the "voltage divider" and "current divider" rules in Fig. 20 to find the variables of interest. This may or may not be possible with a given circuit, for a given output variable of interest.

As an example, consider the same example solved without circuit manipulation in Section I, shown in Fig. 17. The circuit is also shown here in Fig. 28. We wish to find the current in R3. If we knew the current in R1, we could use the current divider equation to find the current in R3, but we don't know that current yet. So we have to simplify the circuit, and the only step we can take in that direction is to combine R2 and R3 in parallel. The resulting circuit does not have R3 in it, so we can't determine the current in R3 from it directly. But we can follow the first approach, and completely simplify the circuit as shown in Fig. 28.
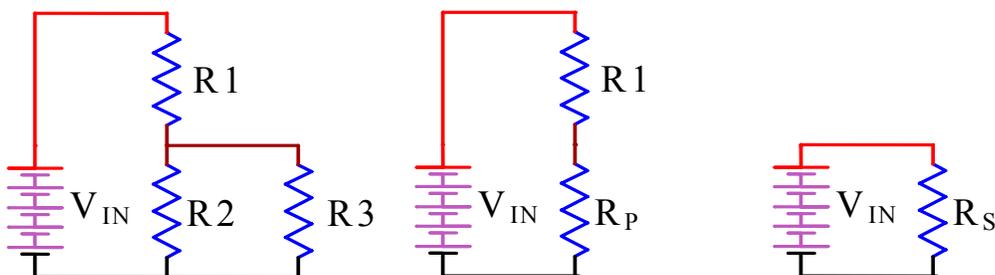

Fig. 28. Simplifying the circuit from Fig. 17.

The value of R_S in the simplified circuit in Fig. 28 can be written using " $\parallel$ " as a notation for a parallel combination:

$$R_S = R_P + R_1 = \left(R_2 \parallel R_3 \right) + R_1 = \left(10\,\text{k}\Omega \parallel 6\,\text{k}\Omega \right) + 6.35\,\text{k}\Omega = 10\,\text{k}\Omega$$

This could have also been written $R_S = \dfrac{R_2 R_3}{R_2 + R_3} + R_1$, but there's no need to bother with that form, which would

require typing the numbers for R2 and R3 into a calculator twice, when you can instead just use your MATLAB function from Lab 0 and type

```
>>rpar(6,10)+6.25
```

With the simplified circuit, it is easy to find the current out of top end of the voltage source (i_{IN} in Fig. 29) is 10 V/ 10 kΩ = 1 mA. This is equal to the current through R1, i_{R1}. We can now use the current divider equation to find the current in R3.

$$i_{R3} = \frac{R_2}{R_2 + R_3} i_{R1} = \frac{10}{6+10} 1 \, \text{mA} = 0.625 \, \text{mA}$$

which is the same result as before, except that this time we have done no algebra, only arithmetic! The circuit manipulation approach allowed us to effectively do the algebra through manipulation of the circuit diagram, and avoid doing algebra on paper.
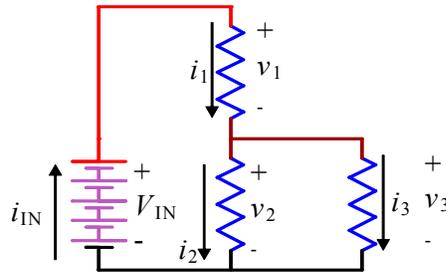

Fig. 29. Varibles defined for the circuit in Fig. 17 and Fig. 28.

Another way to solve this problem would have been to stop after doing the parallel combination of R2 and R3, and apply the voltage divider law, to obtain

$$v_P = v_3 = \frac{R_P}{R_P + R_1} V_{IN} = \frac{(R_2 \| R_3)}{(R_2 \| R_3) + R_1} V_{IN} = \frac{3.75}{3.75 + 6.25} 10 \, \text{V} = 3.75 \, \text{V}$$

The key thing that makes this possible is that the voltage across the parallel combination of resistors is the same as the voltage across each one of them. (In a series combination, the current through the series combination is the same as the current through each one.) Now a simple application of Ohm's law yields the desired current.

$$i_3 = \frac{v_3}{R_3} = \frac{3.75 \, \text{V}}{6 \, \text{k}\Omega} = 0.625 \, \text{mA}$$

Step-by-step circuit solution methods using circuit manipulation

Two approaches have been discussed. The first is to start by completely reducing the circuit:

1. Identify a pair of components that are in parallel or series. **You should be able to circle those components and have only two wires cross your circle—these wires correspond to the two wires of the new equivalent component.**
2. Replace the circled components with their equivalent.
3. Repeat 1, 2 until no more reductions are possible.
4. Solve the new reduced circuit.
5. In many cases, you will now have the result you were looking for. If not, you may need to back up step-by-step, expanding the circuit back out, and using the information about voltages or currents you obtained from the final circuit to quickly and easily find the voltages or currents you are interested in in the expanded circuit.

An alternative approach is as follows:
1. Identify the voltage or current you wish to find.
2. Look for pairs of components that are in parallel or series that you can combine while preserving the voltage or current you are interested in. Note that with parallel combinations both components and the new equivalent have the same voltage across them, and that with series combinations .both components and the new equivalent have the same current through them Reduce all the pairs you can reduce while preserving the current or voltage you want.
3. Solve the new reduced circuit. This will likely involve looking at it as a current or voltage divider.

Part III. Dynamic Circuits

Two new elements and their element laws: Inductors and Capacitors.

Inductors

An inductor is described by the element law,

$$v_L = L\frac{di_L}{dt} \qquad \text{or} \qquad \frac{di_L}{dt} = \frac{v_L}{L}.$$

The first form is more common, but the second is more useful for deriving state-variable equations, because it is already almost in the right form. The symbol for an inductor is shown in Fig. 30. The energy storage in an inductor is determined by the current:

$$E_L = \tfrac{1}{2}Li_L^{\,2}$$

Fig. 30. Inductor symbol

Parallel and series inductors can be combined just like resistors. That's everything you need to know about inductors to set up system models and solve them.

However, it is useful to also get some intuition about what inductors do. One way to do this is to think about the magnetic field in them and how magnetic induction produces a voltage. However, that is a complex, abstract, and unfamiliar phenomenon, and it is generally easier and more useful to think about the behavior of voltage and current in the inductor, rather than thinking about magnetics when you are trying to solve an electric circuit.

Analogies are another way to get intuition. As will be discussed later in the course, there are several possible analogies between electrical systems and mechanical systems. But the one that is most helpful is to consider the analogy between a mass and an inductor. A mass has momentum—once a force gets it moving it wants to keep going, unless a force is applied to stop it. Similarly, an inductor has "current momentum." Once an applied voltage gets the current going in an inductor, it likes to keep going, and will keep flowing until an opposing voltage is applied to stop it. This is illustrated in where the two curves can be interpreted as force and velocity for a mass, or voltage and current for an inductor. Voltage (force) is applied to ramp up the current (velocity). When the voltage (force) is removed, the current (velocity) continues unchanged until an opposing voltage (force) is applied to stop it.
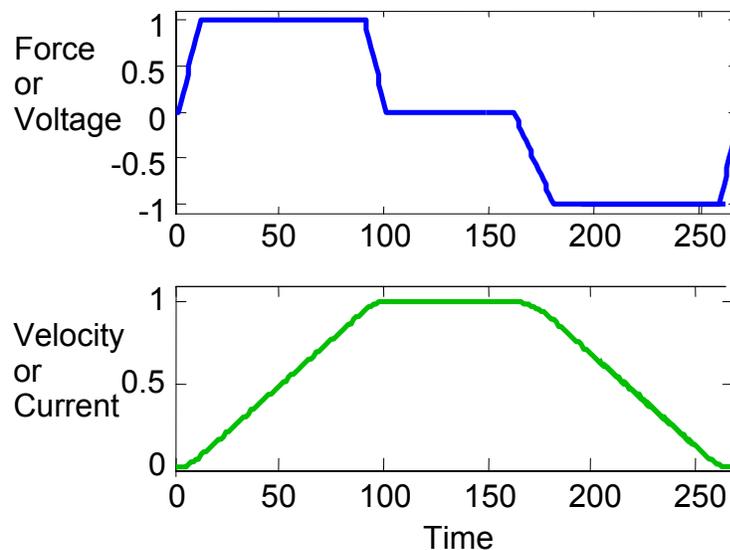

Fig. 31 Behavior of a mass or inductor. Voltage (force) is applied to ramp up the current (velocity). When the voltage (force) is removed, the current (velocity) continues unchanged until an opposing voltage (force) is applied to stop it.

Capacitors

A capacitor is described by the element law,

$$i_C = C\frac{dv_C}{dt} \qquad \text{or} \qquad \frac{dv_C}{dt} = \frac{i_C}{C}.$$

The first form is more common, but the second is more useful for deriving state-variable equations, because it is already almost in the right form. The symbol for a capacitor is shown in Fig. 32. The energy storage in a capacitor is determined by the voltage:

$$E_C = \tfrac{1}{2}Cv_C{}^2$$

Combining parallel and series capacitors works differently from combining parallel and series resistors. The equivalent capacitor for two in parallel is the sum of the values, whereas the equivalent capacitor for two in series is like the parallel resistor calculation:

$$C_S = \frac{C_1 C_2}{C_1 + C_2}$$

That's everything you need to know about capacitors to set up system models and solve them. However, some intuition is again useful.

Here, a useful analogy is the fluid analogy in which voltage corresponds to pressure and current corresponds to fluid flow. A capacitor is similar to a water balloon as shown in Fig. 33. However, this analogy doesn't quite work, because a capacitor has two terminals, and the same current flows in each terminal. And the voltage of interest is the voltage between those two terminals, not between one terminal and the ambient air as in a water balloon. However, the water balloon does share the same element law: a constant positive current will "charge" the balloon, resulting in increasing pressure, whereas a constant negative current will "discharge" the balloon, resulting in decreasing pressure.

A more precise, though perhaps less intuitive analogy is the two-terminal chamber with an elastic diaphragm shown in Fig. 34. As long as both chambers are full of water, the flow out of one chamber is equal to the flow into the other; thus we can speak of one flow, the flow through the element, even though no individual water molecules go in one side and out the other. Similarly, we can talk about the flow through a capacitor, because the charge flowing out of one side is equal to the charge flowing in the other side, even though no individual electrons make it all the way through. Also, the pressure that matters for determining the state of the diaphragm in Fig. 34 is the pressure *difference* between the two sides. If both sides are at 1000 psi, the diaphragm will be perfectly flat and centered, even though a much lower pressure *difference* would move it significantly. And finally, as with the water balloon, the same element law holds: the rate of change of pressure (difference) is determined by the flow rate.
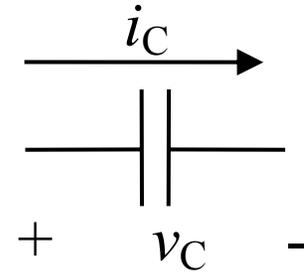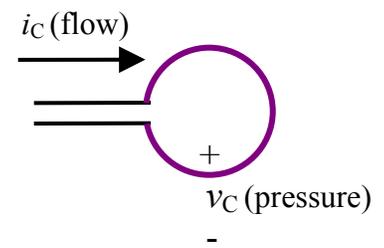

Fig. 32. Capacitor symbol

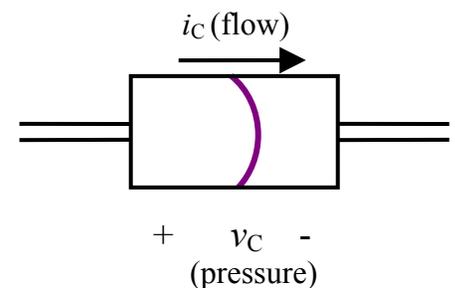Fig. 33. Water balloon analogy for a capacitor.

Fig. 34. A better fluid analogy for a capacitor.

Step-by-Step Method to Generate a Model in State-Variable Format

If you want to generate a state-space model, you can follow the general step-by-step method: write all the relevant equations, and then do algebra until you get the equations in state-variable format. But you can do it much more quickly and easily by starting with equations that are in a form close to state-variable format, and doing only the minimum amount of work necessary to put the model in the right form.

1. Optional: look for ways you can simplify the circuit before you start analyzing it, using the methods in Section II.

2. Choose state variables to match energy storage: i for inductors and v for capacitors, and label them with polarities on the circuit diagram.

3. Write the element laws for these elements in pseudo-state-variable format, with the derivative of the state variable on the left and everything else on the right.

4. Look at the right-hand side of the pseudo-state-variable equations, and identify variables that aren't allowed there in state variable format: any variables other than state variables or known quantities (constants and inputs). Derivatives of inputs are allowed, because they are (in principle) known if the inputs are known, but derivatives of state variables are not allowed.

5. Use KVL, KCL, and any element laws you haven't already used to strategically substitute and eliminate the unwanted variables you identified in step 4.