Mechanical Systems Modeling

Contents:

Linear Mechanical Elements

Description	Trans Mech
Damper (a.k.a. Dashpot or Linear Friction)	$f = \pm \boldsymbol{B}(v_1 \pm v_2)$
Power dissipation in Damper	$P = fv = f^2 \dfrac{\mathbf{1}}{\boldsymbol{B}} = v^2 \boldsymbol{B}$
Spring	$f = \pm \mathbf{K}(x_1 \pm x_2)$
Energy stored in spring	$E = \frac{1}{2}\mathbf{K}(\Delta x)^2$ or $E = \frac{1}{2}\dfrac{\mathbf{1}}{\mathbf{K}} f^2$
Mass	$f = \boldsymbol{M}\dfrac{dv}{dt}$ or $\dfrac{dv}{dt} = f / \mathbf{M}$, where f is the sum of all forces, each taken with the appropriate sign.
Energy stored in mass	$E = \frac{1}{2}\boldsymbol{M}v^2$

Step-by-step method:

1) **Choose States:** You must have at least the same number of states as energy-storage elements. Masses and springs are energy storage elements. Other choices are possible, but a safe way to go is to make the Δx for each spring a state, and the velocity of each mass a state.

2) **Free-body diagrams (FBDs):**

   How many you need: If there is something that might move in your system, but you don't know its motion yet, you need a free body diagram for it. So just about every node that can move independently needs one. Often these are masses, but they might be massless. The only ones that don't need it are nodes (or masses) you already know the motion of—for example nodes which have motion that is a given function of time, a position or velocity which is given as an input.

   What to do: Give each force on the FBD a name and an arrow defining what direction you call positive for that force (not what direction the force actually is). You don't need to give it an equation yet. Also define velocity directions on your FBD.

3) **Equations from free-body diagrams:** These are just $F = ma$. More specifically, use the total force. Each force is added to the total if the force arrow direction is the same as the velocity arrow direction, and subtracted if the directions are opposite. Also write acceleration as the derivative of velocity. To handle a massless node, just plug in $m = 0$, to find that the total force must add to zero.

4) **Spring and damper component equations:** Now write an expression, in terms of your state variables, for each of the forces in your FBD. These will generally look like $F = \pm k\,(\pm x_i \pm x_j)$ and $F = \pm b(\pm v_i \pm v_j)$ for springs and dampers connected between the *ith* and *jth* nodes. If the velocities and positions are all defined in the same direction, the equations will be of the form $F = \pm k\,(x_i - x_j)$ and $F = \pm b(v_i - v_j)$. Do a thought experiment with one velocity or position equal to zero, and the other positive, to check which sign is correct[1]. Note: For nonlinear springs and nonlinear friction, these equations would be a little different, but the general idea and the procedure are the same.

5) **Substitute and rearrange:**

   First substitute your force expressions from step 4. for each of the forces in the equations from step 3.

   Further substitute and rearrange with the objective of getting the derivative of the state variables on the left, and functions of your state variables and known inputs on the right. Derivatives of state variables aren't allowed on the right, but derivatives of inputs are allowed on the right. Be sure to eliminate all unknown functions of time on the right, except for state variables. For example, if you use Δx as a state variable, defined as $x_1 - x_2$, you are allowed to have Δx on the right hand side, but you must eliminate any references to x_1 and x_2 by rewriting them in terms of Δx.

6) **Check:**

   a) Make sure it's a valid state-variable equation. You must have a derivative of each state on the left. The RHS may contain states and things that are known (inputs, constants, derivatives of inputs) but must ***not*** have any derivatives of states.

   b) Sign check: In your final equation for $dv_i/dt = a\,v_i +$ (terms for other velocities) $+\, b\,x_i +$ (terms for other positions), both a and b should be negative. If they are not, better to check your previous steps, rather than just fix the sign here, because your error might have caused other problems as well.

Options: Many people combine steps 2 and 4, and just write the expressions for forces next to the arrows on the FBD, and never give them names. That works fine. I prefer to break it down into separate steps so that each step is simpler.

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[1] If you don't like the thought-experiment method, you can use the sign convention at the bottom of p. 21 in Cha et al. If you do this, you should be aware that that sign convention is peculiar to this book—if you later work with other engineers who didn't learn it that way, you'll be better off knowing a more general method.

Sign Conventions in Mechanical Systems

Here are a few different approaches to establishing signs of variables in mechanical systems. The method we will use is A., the "systems method." You do not need to learn about the other methods, but understanding what is different about what we are doing can help avoid confusion.

A. "Systems method"

Draw an arrow for each variable. This arrow does ***not*** represent the direction of the force/velocity/whatever, but rather represents the direction of the variable if/when the variable is positive.

Advantages:

- The variables you use are all explicitly defined, so there is reduced opportunity for confusion.
- You can sometimes simplify the problem by intelligently choosing the variables you define.
- In other types of lumped dynamic systems, particularly fluid and electrical systems, defining a coordinate system won't help, so this method is consistent with what will be required for other system types, and it is good to work in a consistent way across different system types.
- It is the most general case—all the other methods are subsets of this method, so once you learn this method you can look at any other analysis and evaluate whether or not it is done correctly.

Disadvantages:

- Many definitions are required, and if you lose track of how you defined just one variable, you get the wrong answer.
- Since different choices are possible, it is more work to compare your result with a friend's result. If you used y_1 for the position she described as $-y_1$, your equations will look different, even though the system behavior they predict will be the same.

B. Single coordinate system

Choose a coordinate system, and use vectors in that coordinate system for everything: forces, positions, and velocities. This is what many of you learned in physics.

Advantages:

- Once you define a coordinate system, you know exactly what to do—no choices are necessary.

Disadvantages:

- It's not always efficient (see the example attached).
- It doesn't work with fluid and electrical lumped dynamic systems.
- It doesn't allow you to look at the work of someone who used a different method and evaluate it.

C. Choose directions so stuff comes out positive.

Try to choose direction arrows in the directions of the actual variables. If you solve and find negative numbers, that means you drew the arrow in the wrong direction. ***This does not work for dynamic systems!*** In the course of some dynamics, variables could go in both directions, so there is no one direction you can choose that makes the numbers positive.

Advantages:

- You don't need to work with negative numbers, which makes gaining intuition easier.

Disadvantages:

- It is not, in general, possible to do this for dynamic systems. This is more than a disadvantage—it is a reason this method cannot work for dynamic systems.

D. Standardized directions.

Do method A, but pick a convention and always stick to it. This is what the book does. It has the same advantages and disadvantages as B, the single coordinate system method.

Systems Sign Convention FAQ

Q. Shouldn't the first step always be to define a coordinate system? Isn't anything we write meaningless without a coordinate system defined?

What we are doing is defining an individual coordinate system for each variable. It ***is*** true that without a definition (of the 1-D coordinate system) for each variable, equations using that variable are meaningless. That's why ***each*** variable needs a definition on the diagram.

Q. Isn't it crazy and inefficient to define a (possibly different) coordinate system for each variable?

Perhaps—that is a disadvantage of this method. But in any case, each variable does need to be defined, usually by some kind of indication on the diagram. So indicating its direction on the diagram is not necessarily more trouble, and it can be very helpful. Also, intelligent choices of variables can then allow you to streamline the solution process. For a spring, you need only one F_S, not a F_{SA} at one end and F_{SB} at the other end. See the attached example for a situation in which the systems method can cut the number of variables needed from seven down to three.

Q. If you find out F_X is negative, doesn't that mean you drew it in the wrong direction?

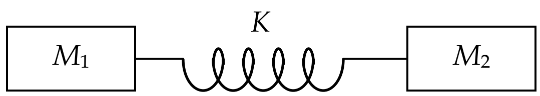
That's a luxury we don't have in dynamic systems. In a dynamic system, things often oscillate—go positive and negative—in the course of an event we are analyzing. So there is no way to draw an arrow in one direction to make the variable stay positive. This might seem like a limitation, but actually it makes life easier. We don't need to try to figure out ahead of time what the system is going to do and draw the arrows accordingly. We can draw the arrows any direction we like. (But once we draw the arrows, we are locked in to making the signs in our equations match the way we've drawn the arrows).

Q. You said you could draw the arrow in any direction, but my homework came back saying I have a sign error. Doesn't that mean I drew the arrow in the wrong direction?

It means you didn't write your equations consistent with the way you drew the arrows. Once you draw an arrow, you have no more choice—the signs in the equations are all determined by how you draw the arrows. (It is sometimes possible to work the other way, and write the equations first, and then pick arrows to match what you've said in the equations, but I think that is harder, because when you draw one arrow, it might need to be consistent with several different equations.)

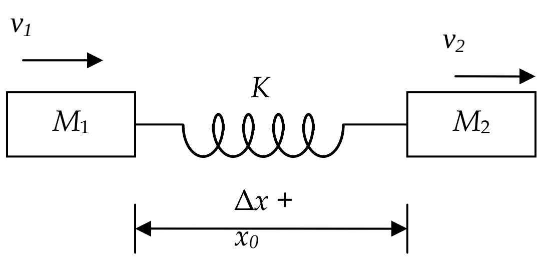
An Example

Problem statement: Find state-space equations that can be solved to find the velocities of the two masses shown below as a function of time.



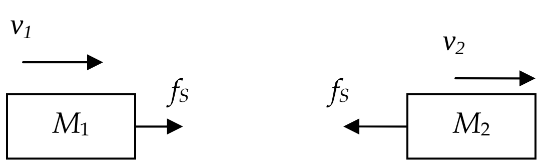
First, by the systems method

1. Pick state variables corresponding to energy stored (½ $M_1v_1^2$, ½ $M_2v_2^2$, ½ $k(\Delta x)^2$), and mark them, with directions, on the diagram.



   Because there is not really clear way to draw this, I must explain that x_0 is the unstretched spring length and Δx is the "stretch" of the spring, such that their sum is the total length.

2. Free-body diagram (FBD) for each mass or unknown node.



   Note that by Newton's third law ("for every action there is an equal and opposite reaction") the two f_S arrows must be drawn in opposite directions *if* they are given the same name.

3. Newton's second law, for each mass.

   $$\frac{f_S}{M_1} = \dot{v}_1 \qquad -\frac{f_S}{M_2} = \dot{v}_2$$

   The minus sign is not because the second force arrow is to the left. Rather, it is because the force arrow and the velocity arrow are in opposite directions.

4. Equations for forces. Start with $f_S = \pm K(\Delta x)$. Now, do a thought experiment. With Δx positive, the spring is stretched and wants to pull back in. Because that is ***the same*** as the direction we have drawn f_S on the FBD, $f_S = +K(\Delta x)$.

5. Substitute to get some of the state-space (a.k.a. state-variable) equations.

$$\dot{v}_1 = \frac{K\Delta x}{M_1} \qquad \dot{v}_2 = -\frac{K\Delta x}{M_2}$$

6. For the final state-space equation, we need to use kinematics (relationships between how things move, considered independently of why they are moving that way).

$$\dot{(\Delta x)} = v_2 - v_1$$

7. Check: First, do the equations under 5 and 6 constitute valid S.V. format?

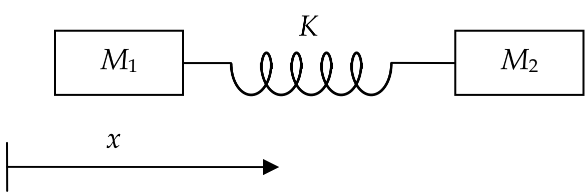
   - We have a complete set of derivatives for state variables.
   - The RHS of each equations contains only states and known stuff (constants and inputs).

   So it is valid S.V. format, and we are done.

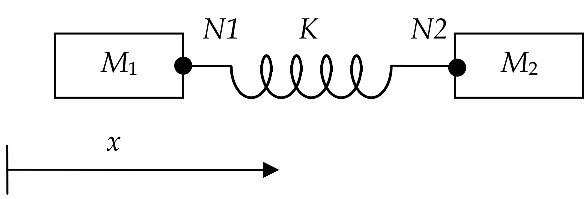
Second, by the coordinate-system method (not recommended)

NOT RECOMMENDED

1. Define coordinate system.

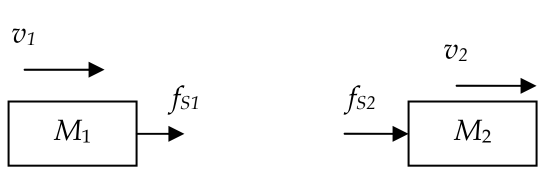


2. Define variables, using the two nodes labeled on the diagram



x_1 = position of node *N1*
x_2 = position of node *N2*
v_1 = velocity of node *N1*
v_2 = velocity of node *N2*

3. FBDs. Note that the spring force ***must*** now have two different names, since they have different values.



NOT RECOMMENDED

4. Newton's second law, for each mass.

$$\frac{f_{S1}}{M_1} = \dot{v}_1 \qquad \frac{f_{S2}}{M_2} = \dot{v}_2$$

5. Equations for forces.

$$f_{S1} = +K(x_2 - x_1 - x_0)$$
$$f_{S2} = +K(x_1 - x_2 + x_0)$$

Note that there are six signs that have to be correct in those equations. To be sure you have them all right, a lot of thought experiments are needed! As before, x_0 is the unstretched length of the spring.

6. Substitute to get some of the state-space (a.k.a. state-variable) equations.

$$\dot{v}_1 = \frac{K(x_2 - x_1 - x_0)}{M_1} \qquad \dot{v}_2 = -\frac{K(x_1 - x_2 + x_0)}{M_2}$$

7. Write kinematic equations

$$\dot{x}_1 = v_1$$
$$\dot{x}_2 = v_2$$

8. The above, in 6 and 7, constitute a non-minimum set of start variable equations. Optionally, they can now be reduced by defining:

$$\Delta x = x_2 - x_1 - x_0$$

now we can do a bunch of algebra to get the same equations as we obtained directly in the previous method.

$$\dot{v}_1 = \frac{K\Delta x}{M_1} \qquad \dot{v}_2 = -\frac{K\Delta x}{M_2}$$

$$\dot{(\Delta x)} = v_1 - v_2$$

9. Check: Do the equations under 6 and 7, or 8 constitute valid S.V. format?

   - We have a complete set of derivatives for state variables.
   - The RHS of each equations contains only states and known stuff (constants and inputs).

   So it is valid S.V. format, and we are done.

In addition to requiring more steps, this method required more variables: two forces vs. one, two node positions in addition to Δx, and x_0 (which was defined before but not used—here it appeared in the equations).