Second-Order LTI Systems

First order LTI systems with constant, step, or zero inputs have simple exponential responses that we can characterize just with a time constant. Second order systems are considerably more complicated, but are just as important, and are more interesting. We won't find a super-simple shortcut for finding solutions—the Laplace transform is usually the way to go. But we will gain a lot of insight.

Second order system

Consider a typical second-order LTI system, which we might write as

$$\ddot{y}(t) + k_1 \dot{y}(t) + k_2 y(t) = b(t)\,,$$

where $b(t)$ is some input function. (Not all second order LTI systems have exactly this same form—this is just a common example.)

1.1 Solution by Laplace transform

$$L\{\ddot{y}\} = sL\{\dot{y}\} - \dot{y}(0-) = s\left[sY(s) - y(0-)\right] - \dot{y}(0-)$$
$$= s^2 Y(s) - sy(0-) - \dot{y}(0-)$$

substituting yields

$$s^2 Y(s) - sy(0) - \dot{y}(0) + k_1 sY(s) - k_1 y(0) + k_2 Y(s) = B(s)$$

which we solve for $Y(s)$

$$Y(s)(s^2 + k_1 s + k_2) = sy(0) + \dot{y}(0) + k_1 y(0) + B(s)$$

$$Y(s) = \frac{sy(0) + \dot{y}(0) + k_1 y(0) + B(s)}{(s^2 + k_1 s + k_2)}$$

The solution to this depends on what the input $b(t)$ is, as well as the initial conditions. But in any case we can re-write the denominator in factored form:

$$Y(s) = \frac{sy(0) + \dot{y}(0) + k_1 y(0) + B(s)}{(s+a)(s+b)}$$

where $-a$ and $-b$ are the roots of the denominator polynomial, also called the "characteristic equation". The roots $-a$ and $-b$ are also called the *poles* of $Y(s)$; the values of s where $Y(s)$ goes to infinity.

Some of the common possibilities for *Y(s)* are given in table entries 10-14. (If the input is constant, zero, or a step, *Y(s)* will in fact be one of these, or a combination of them, depending on initial conditions.) We observe that all of the corresponding solutions *y(t)* can be described by

$$y(t) = C + Ae^{-at} + Be^{-bt}$$

where *A*, *B*, and *C* are constants. Thus, the solutions are like the first-order solution, but with two exponential terms, instead of just one. We will now study the behavior as a function of the values of the roots (poles), -*a* and -*b*. We can divide the possibilities into three cases: two distinct real poles, two equal real poles, or a complex conjugate pair of poles.

Case 1: Distinct Real Roots

The response usually looks like a first-order exponential response. In fact, if A and B have the same sign and the poles are close to each other, it can be nearly indistinguishable from an exponential response. However, careful examination can often show differences that are not immediately visible. In Fig. 1 we see an example of a second order response, $y(t) = 0.5e^{-t/2} + 0.5e^{-2t}$. At first it looks like a simple exponential decay. It looks like the time constant is 1, because the response is down to 37% there. But a simple first-order exponential would be down to 5% at three time constants, and at *t* = 3, it is only down to about 11%. So the general shape is similar, but it is a different shape.

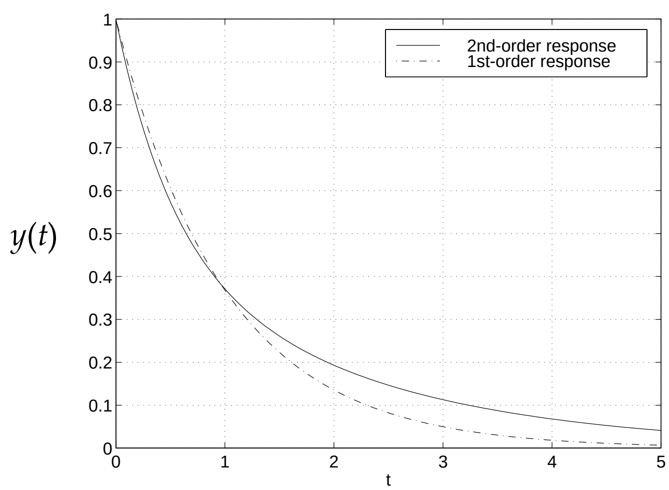
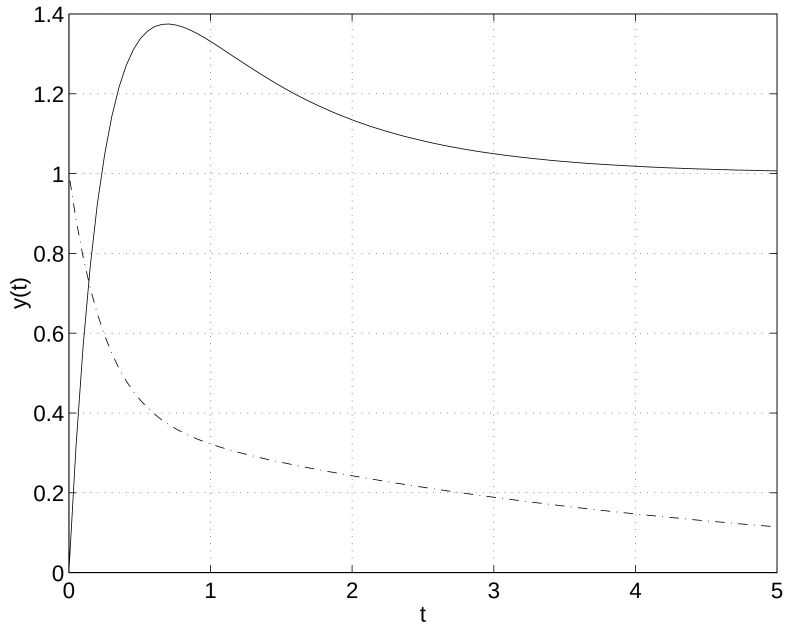


Fig. 1. First vs. second-order responses.

However, it can also look quite different. The responses with different parameter values,

$$y(t) = 1 + e^{-t} - 2e^{-4t} \qquad \text{and} \qquad y(t) = 0.4e^{-t/4} + 0.6e^{-4t}$$

are shown in Fig. 2. The first gives a response with "overshoot." The second gives a response with a distinct knee; the transition from the rapid decay of the last term to the slow decay of the first. In homework two you solved a second-order system with ode45 and saw a response very similar to this curve with a knee.



Case 2: Two Real Roots, exactly equal

This case is of little practical importance, because in practice you can't ever get the roots to match exactly. Thus, it won't be discussed further here. If you ever need to analyze this case you can use table entries 34 and 35.

Case 3: Two Complex Conjugate Roots

$y(t) = C + Ae^{-at} + Be^{-bt}$ with $b = a^*$.

Define: $\sigma \equiv \mathrm{Re}(a) = \mathrm{Re}(b) = -\mathrm{Re}(p_1) = -\mathrm{Re}(p_2)$ (1)

and: $\omega_d \equiv \mathrm{Im}(a) = -\mathrm{Im}(b)$ (2)

such that $\{a, b\} = \sigma \pm j\omega_d$

Now

$$y(t) = C + Ae^{-\sigma t - j\omega_d t} + Be^{-\sigma t + j\omega_d t}$$

$$y(t) = C + e^{-\sigma t}\left(Ae^{-j\omega_d t} + Be^{j\omega_d t}\right)$$

We can apply Euler's formula ($e^{jx} = \cos x + j \sin x$) to rewrite this as

$$y(t) = C + e^{-\sigma t}\left((A + B)\cos \omega_d t + (B - A) j \sin \omega_d t\right) \qquad (3)$$

The solution (3) is a little disconcerting, because it looks like the solution could be complex. If it was a solution for a position of an object, a complex value wouldn't be physically meaningful. For example, what if $B = -A$, so the cosine term was zero, and we had only the imaginary sine term!? Actually, B and A are complex, and are related to a and b, so it isn't so simple to see what happens, but it turns out that with a and b complex conjugates, (B-A) is always pure imaginary or zero, so the sine term always becomes real after all. (Try an example from the table to see this.)

Thus, we can write the solution as

$$y(t) = C + e^{-\sigma t}\left(D\cos\omega_d t + E\sin\omega_d t\right)$$

or

$$y(t) = C + A_0 e^{-\sigma t}\cos\left(\omega_d t + \phi\right) \qquad (4)$$

In these expressions, *C, D, E,* A_0 and ϕ are real constants that we haven't explicitly found yet, and σ and ω_d are real constants that can be determined from the poles (see eqns (1) and (2)).

Further Study of Case 3: Two Complex Conjugate Roots

From (4), we see that the response comprises a decaying sinusoidal oscillation. The frequency of this decaying or damped oscillation is ω_d radians per second, or $\omega_d/(2\pi)$ Hz. The decay rate is σ. These two numbers correspond to the horizontal and vertical positions of the poles on the complex plane. Since these are the values of *s* that make *Y(s)* go to infinity, we usually refer to this complex plane as *the s-plane.* Let's look at how the values on the s-plane and the responses correspond. On the next pages are some series of plots of pole positions and the corresponding response (for the sake of concreteness, these are all for zero initial states, and a step input). The first set is for constant ω_d of 1.5 rad/sec, and varying σ. You can observe a constant ring frequency with varying decay rate, until it gets to the point where the oscillation is damped so quickly that the oscillation isn't apparent, and the ring frequency can't be seen.

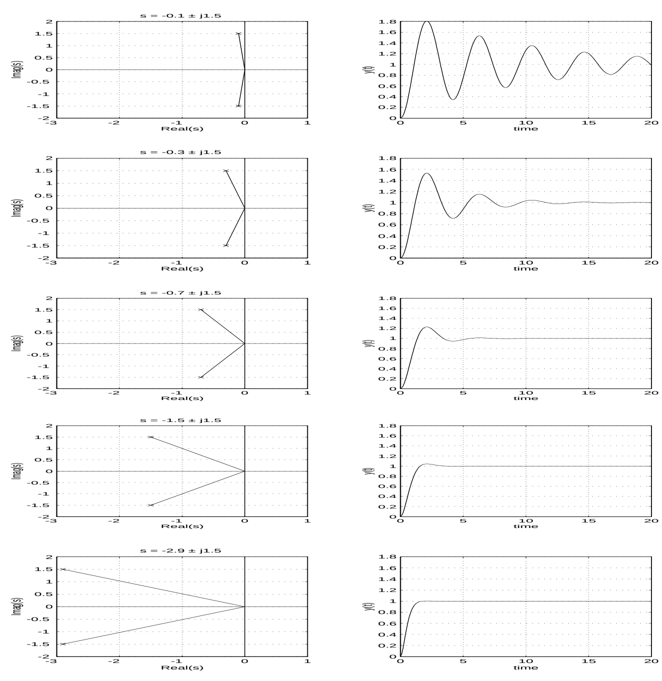


Fig. 3. Constant ring frequency with varying decay rate.

Our next series shows varying ring frequency with constant decay rate. Note the constant *envelope,* the peak amplitude of the ringing, which can be described by an exponential decay.

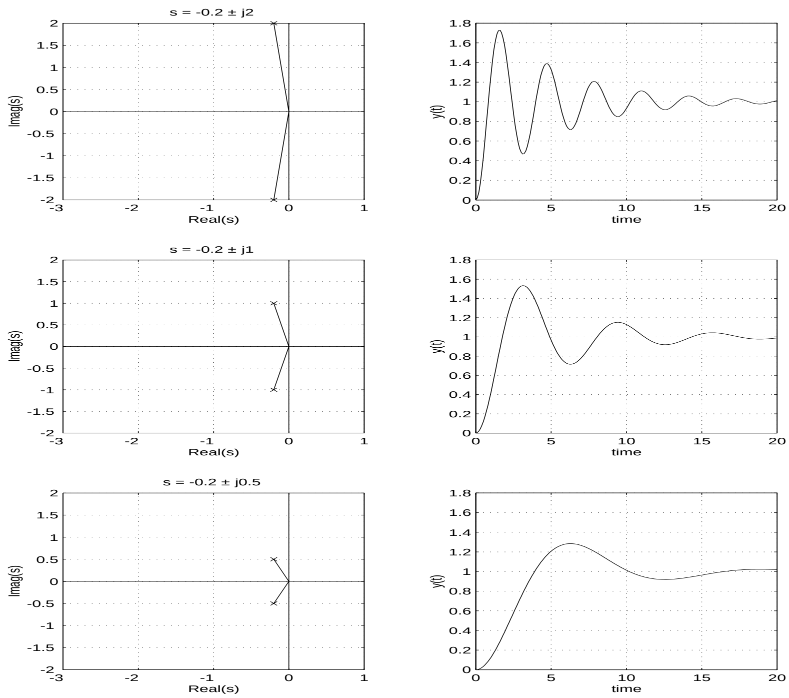
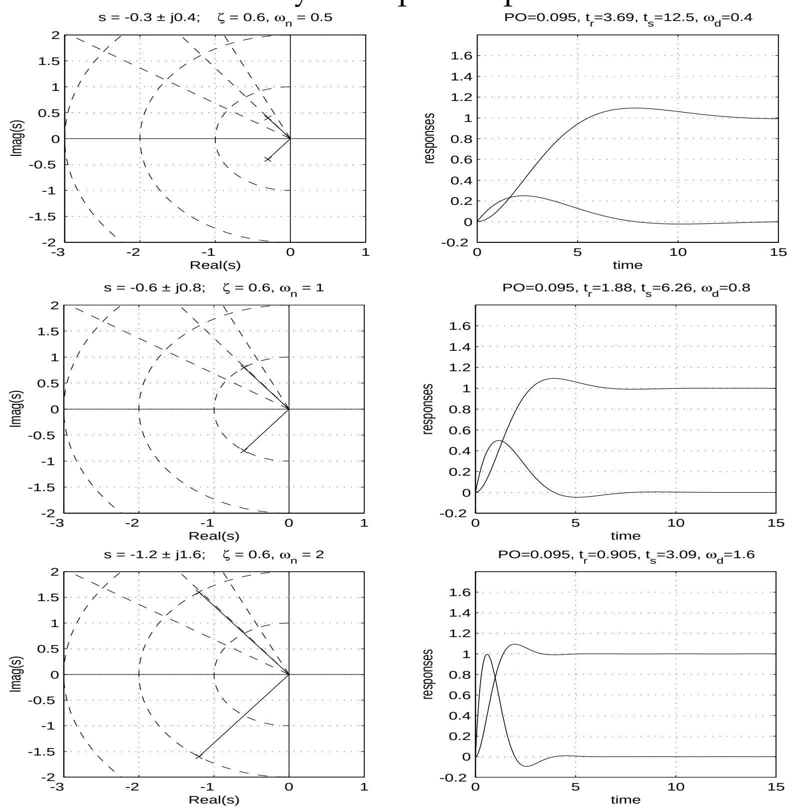


Fig. 3. Constant decay rate with varying ring frequency.

Second-Order LTI Systems—Part II

In the first part of this handout we discussed describing second-order LTI systems with complex conjugate poles in terms of σ and ω_d, the negative of the real part and the imaginary part of the poles. However, another, more commonly used description can be motivated by the three plots below, showing response of the system equations in the preceding section of this handout with a step input. For these plots we have added a curve for another response, due to some initial conditions, in order to hint at the diversity of responses possible with the same pole positions.



These responses look similar in shape, and are in fact identical, except for the time scale, and the magnitude of the initial condition response. We see that the poles all lie along a line extending out from the origin. We observe that the *shape* of the response curve is determined not be the absolute magnitude of σ or ω_d, but by their ratio. Thus, we'd like to express the characteristics of the system in terms of a unitless damping ratio rather than in terms of σ.

If we use two numbers to describe a system, a damping ratio and one other number, we would like the other number to be independent of the damping in the system. With reference to actual physical systems, we note that when the element providing the damping (e.g., a resistor in an electrical circuit, or friction in a mechanical system) is changed, not only do σ and the damping ratio change, but ω_d changes as well. This is because in the expression

$$Y(s) = \frac{sy(0) + \dot{y}(0) + k_1 y(0) + B(s)}{(s^2 + k_1 s + k_2)}$$

k_1 reflects the degree of damping, and its value affects both the real and imaginary parts of the roots. So the actual damped oscillation frequency ω_d is not independent of damping.

Thus, we choose instead the *natural frequency,* ω_n, defined as the frequency that the system *would* oscillate if it were not damped, and we define the damping ratio as $\zeta = \frac{\sigma}{\omega_n}$. This logic may or may not be convincing, but what makes it practical is that we can then re-write the system equation as

$$\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t),$$

which allows us to easily relate ω_n and ζ to k_1 and k_2 or to the actual physical parameters in the system of interest. Note that $u(t)$ is not the unit step function, but rather is just a rescaled input function,

$$u(t) = \frac{b(t)}{\omega_n^2}$$

Similar to before, the solution by the Laplace transform may be found.

$$Y(s) = \frac{\dot{y}(0) + (s + 2\zeta\omega_n)\, y(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\omega_n^2\, U(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The solutions are influenced by the roots of the polynomial $s^2 + 2\zeta\omega_n s + \omega_n^2$.

$$\begin{aligned} s^2 + 2\zeta\omega_n s + \omega_n^2 &= (s + \zeta\omega_n)^2 + (1 - \zeta^2)\,\omega_n^2 \quad \text{(completing the square)} \\ &= \left(s + \zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n\right)\left(s + \zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n\right) \quad \text{(factoring)} \end{aligned}$$

We can see in this that $\omega_d = \omega_n\sqrt{1-\zeta^2}$. It is also useful to note that the meaning of ω_n on the *s*-plane becomes the distance between the pole position and the origin. Thus, line of constant ω_n are circles around the origin at a radius ω_n.

Our previous three cases may now be described in terms of the value of ζ:

- Underdamped case, $\zeta < 1$: complex conjugate roots
- Critically damped case, $\zeta = 1$: identical real roots
- Overdamped case, $\zeta > 1$: distinct real roots

3. Solutions for a particular example: Step response (zero initial conditions, $U(s)$ = 1/s)

$$Y(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

3.1 Overdamped case ($\zeta > 1$)

Real, negative roots,

$$s = -\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

The formulation in terms of and ω_n and ζ is of most value in recognizing this case; for analyzing it it is best to use the table entries that are in terms of the negatives of the roots, such as #10, using
$a = \left(\zeta + \sqrt{\zeta^2-1}\right)\ \omega_n$, $b = \left(\zeta - \sqrt{\zeta^2-1}\right)\ \omega_n$. $ab = \omega_n{}^2$,
$a-b = -2\sqrt{\zeta^2-1}\ \ \omega_n$. Then,

$$y(t) = U_s(t) + \frac{1}{2\sqrt{\zeta^2 - 1}\,\omega_n}\left(\exp\left[-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t\right] - \exp\left[-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t\right]\right)U_s(t)$$

3.2 Critically damped case ($\zeta = 1$) Rarely important, but for reference, the result is:
Identical real, negative roots

$$s = -\omega_n$$

$$Y(s) = \frac{\omega_n^2}{s\left(s + \omega_n\right)^2}$$

By table entry 33,

$$y(t) = u(t)\left[1 - (1 + \omega_n t)e^{-\omega_n t}\right]$$

3.3 Underdamped case ($\zeta < 1$)

Complex conjugate roots

$$s = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n$$

Using the table entry 27a,

$$y(t) = \left[1 - \frac{1}{\sqrt{1-\zeta^2}}\exp\left(-\zeta\omega_n t\right)\sin\left(\omega_d t + \text{atan2}\left(\sqrt{1-\zeta^2}, \zeta\right)\right)\right] U_s(t)$$

The following properties of the step response can be calculated once you know ζ and ω_n:

3.3.1 Rise time, t_r: Time for the step response to rise from 10% to 90% of its final value.

$$t_r \approx \frac{1.8}{\omega_n} \text{ for responses with moderate overshoot.}$$

3.3.2 Settling time, t_s: Time for response to settle within ±1% of its final value. A decaying exponential reaches 1% in approximately five time constants. To be precise, solve $\exp(-\zeta\omega_n t_s) = 0.01$, to obtain:

$$t_s = \frac{4.6}{\zeta\omega_n}$$

3.3.3 Peak time, t_p: Time until the response hits its maximum overshoot. Differentiate $y(t)$ with respect to t, obtaining (after some algebra),

$$\omega_n \exp(-\zeta\omega_n t)\sin(\omega_d t) = 0$$

The sine function is zero for $\omega_d t = k\pi$, k= integer. Odd values of k correspond to peaks in the step response. The first such peak is at k=1. It follows that

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

3.3.4 Peak overshoot, M_p: The peak value of the step response, $y(t_p)$, is $(1+M_p)\,y_\infty$, where y_∞ is the final value. M_p is frequently expressed as a percentage. Knowing

$t_p = \pi/\omega_d$,

we can find

$$y(t_p) = 1 + M_p = 1 + \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right), \text{ so}$$

$$M_p = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \approx 1 - \frac{\zeta}{0.6} \text{ for } 0 \le \zeta \le 0.6$$