Analytical Solutions for First-Order LTI Systems

Scope of Application:

This handout outlines a method for finding the behavior of a first-order LTI (linear time invariant) system using a minimum of mathematics, and instead using physical reasoning regarding the system of interest. It is limited to situations in which the input is either a constant or a step.

Mathematically, it applies to any stable system modeled by a first-order LTI ODE with a constant or step input. That is, a system modeled by an equation of the form

$$\dot{x} = -kx + U(t) \tag{1}$$

where $U(t)$ is an input which may be either a constant, or a constant multiplying the unit step function $u(t)$.

Form of Solution:

A variety of mathematical methods for analyzing differential equations can be used to find that the solution to an equation such as (1) takes the form $x(t) = c_0 + c_1 e^{-kt}$, which we usually write as

$$x(t) = c_0 + c_1 e^{-t/\tau}, \tag{2}$$

where $\tau = 1/k$ is termed the time constant. It must have units of time in order to make the argument of the exponential unitless; we can also see that it has units of time from (1), where k must have units of (1/time) to match the units of $\dot{x}$.

A few important special cases of this are when $c_0 = 0$, and the result is a decaying exponential, and when $c_0 = -c_1$, in which case the solution can be written $x(t) = c(1 - e^{-t/\tau})$. This latter form is often called a *saturating exponential.* Both are plotted in Fig. 1.

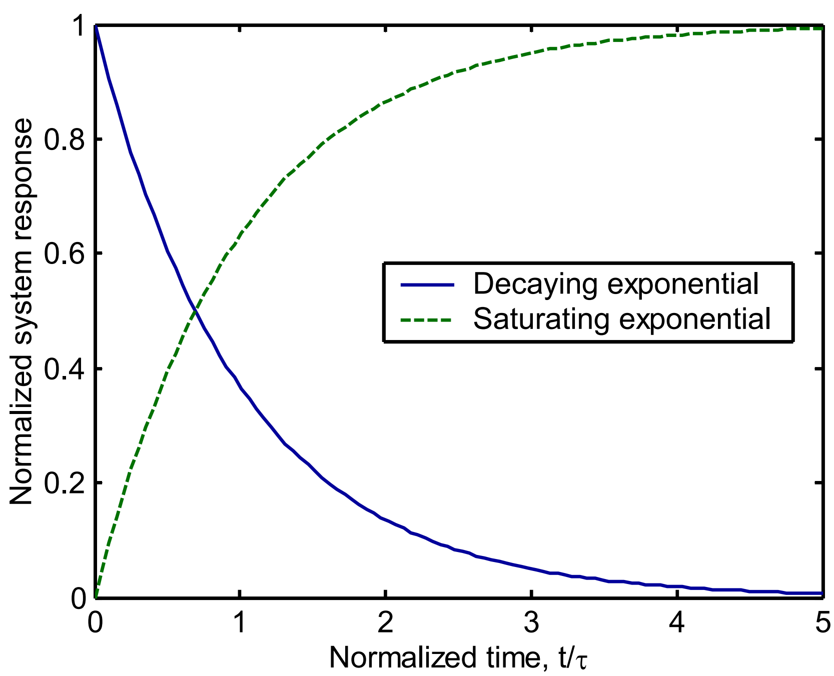


Fig. 1. Decaying and Saturating Exponentials

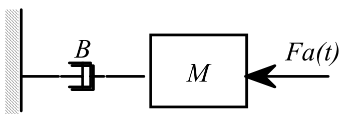
Finding the Constants:

The solution for a particular problem consists of finding the time constant, τ, and the constants c_0 and c_1 in

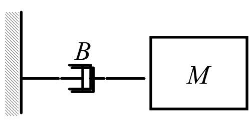
$$x(t) = c_0 + c_1 e^{-t/\tau} \qquad (2)$$

The time constant is found from the form of the differential equation. This is often the only purpose you need the actual differential equation for. A very useful simplification for this purpose is to note that the time constant is the same regardless of the input. Thus, we can set the input equal to zero for the purpose of finding the time constant. We can do this in the physical system, before writing the equation; this simplifies the process of writing the equation. However it takes some care: Setting something to zero is not the same as ignoring it. For example, if the input is a velocity, setting it equal to zero means zero velocity. That means holding the point of the input fixed, at zero velocity, not leaving it loose. On the other hand, setting a force input equal to zero means no force, which is the same as having nothing connected.

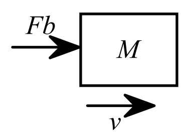
Example 1



This is a mass and a damper, with a force as an input. To find the time constant, we set the input, *Fa*(*t*), equal to zero, and the system becomes:

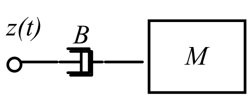


, and so with a free body diagram

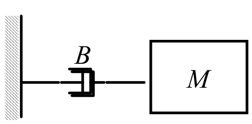


and with the expression for the damper force $F_b = -Bv$, we find the zero-input differential equation $\dot{v} = -\frac{B}{M}v$, in which we can recognize the time constant as $\tau = \frac{M}{B}$.

Example 2



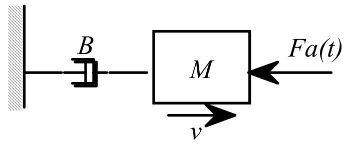
This is a similar system, but with the position of the left end of the damper as an input; a known function of time *z*(*t*). We again set this input to zero. But to set a position (or velocity) equal to zero, we must fix it in place at zero. So the system becomes:

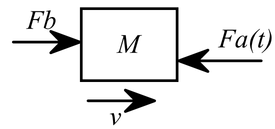


, which happens to be the same as the zero-input system for Example 1.

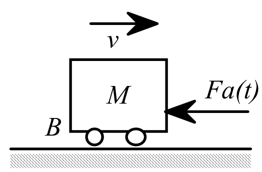
Next we need to find the constants c_0 and c_1. These can be found from the initial conditions and the final steady-state conditions. With a constant input, or a step input that becomes constant for large *t*, the final steady state solution for the state variable is always a constant, equal to c_0 (the exponential term will have decayed away completely). Thus, to find c_0, all we have to do is look at the system and determine what the final steady state behavior will be. This means finding the value of the state—which is usually best chosen as the displacement of a spring or the velocity of a mass.

Example 1, continued





To find the final steady-state value of the state, *v,* we must assume something about the input. Let us assume it is a constant force of value F_a. For steady-state, that is, for v constant, the net force on the mass must be zero. Thus, we have $F_a + Bv = 0$ (using the equation for F_b from the last page). This means the final steady-state value of v has to be $c_0 = v(t \to \infty) = -\frac{F_a}{B}$. The way the picture is drawn, this could present a problem—if the velocity is negative, and constant in steady state, the mass will eventually hit the wall. There are at least a few solutions to this problem. If the time constant is short enough, and the final steady-state velocity is small enough, it might reach steady-state very quickly, and we could use that solution to analyze a substantial part of the behavior before it hit the wall. Alternatively, we could redraw it like this:



where we have exactly the same free body diagram, but can more easily imagine a cart reaching and persisting with a steady-state velocity.

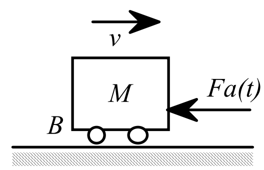
Finally, we must find c_1. This can be done using the initial conditions by plugging t = 0 into (2), to get $x(t=0) = c_0 + c_1$, which we can easily solve for c_1. This can become tricky, however, if something dramatic happens at t = 0, such as a collision, or something getting turned on. In these cases, we often separate the analysis into a discussion of the variables at t = 0-, the instant just before the event that marks t = 0, and t = 0+, the moment just after that event. A very useful property of physical systems is that energy stored can't be changed instantaneously. (To do so would require infinite power, which we know is not physically possible.) Thus, if we pick our states to correspond to energy storage, those states can't change instantaneously. So we can conclude that:

$$x(t=0-) = x(t=0+) \tag{3}$$

if x is a state corresponding to energy storage, that is, if x is velocity of a mass or position of a spring, in a mechanical system. As we move to analyzing other systems, we will find other examples of states that correspond to energy storage, and that can't change instantaneously.

Often you are interested in something in a system other than the natural states that correspond to energy storage. We will return to that case at the end of the handout.

Example 1, completed

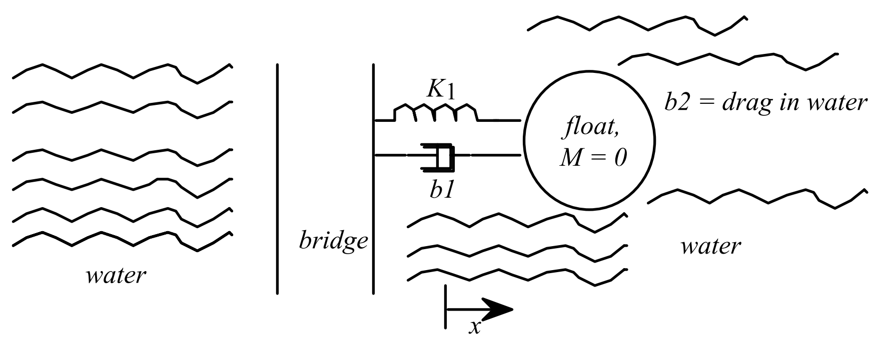


We can continue with this example, with a cart with linear friction B and an applied force $F_a\,(t)$. Previously, we assumed that the force was constant, but now let's solve the problem in which it is not constant, but is a step function, such that $F_a(t) = F_0 u(t)$. For example, this could model a force due to a motor in the cart that is turned on at $t = 0$. We will also assume that that cart is stationary before the motor is turned on. From (3), we can conclude that at $t = 0+$, the cart is still stationary. So our initial condition is $v(0+) = 0$. From before, we have $c_0 = v(t \to \infty) = -\dfrac{F_a}{B}$; to satisfy the initial condition, $c_1 = -c_0 = \dfrac{F_a}{B}$. The solution becomes

$$v(t) = c_0 + c_1 e^{-t/\tau} = -\frac{F_a}{B} + \frac{F_a}{B} e^{-t/\tau} = -\frac{F_a}{B}(1 - e^{-t/\tau})$$

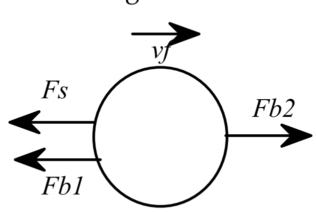
which we recognize as a saturating exponential.

Example 3



This is a somewhat contrived example, but it will serve the purpose. We have a float of negligible mass (it has to be light to float!) in the a river with flow v_r, attached to the bridge with a spring and a damper. When we first drop the float in the water, the spring is unstretched. What happens to the float when the drag of the water starts to pull it downstream?

First we want to find the time constant. We do this with the input—the water flow—set to zero. Experimentally, that could correspond to setting it in a lake. We still have a force due to water drag, but only if the float is moving. We can draw a FBD for this and derive some equations:



$$F_{b2} - F_{b1} - F_s = 0$$
$$F_s = kx$$
$$F_{b1} = b_1 v_f$$
$$F_{b2} = -b_2 v_f$$
$$-(b_1 + b_2)v_f - kx = 0$$

In this case a choice of state corresponding to energy storage is x, the stretch of the spring. Then $\dot{x} \equiv v_f$, so the last equation we got from the FBD can be re-written as

$$\dot{x} = -\frac{k}{(b_1 + b_2)} x ,$$

from which we recognize the time constant $\tau = \frac{(b_1 + b_2)}{k}$, which has units of (N/(m/s))/(N/m) = s, so it is in fact valid as a time constant.

The initial condition is that the spring is unstretched; $x = 0$. The steady state solution has the spring at a constant length. Thus, the damper is not moving and its force is zero. So the force balance equation (the first equation with the FBD) becomes

$$F_{b2} - F_s = 0$$
$$kx = b_2 v_r$$
$$x(t \to \infty) = \frac{b_2 v_r}{k}$$

From the time constant, the initial condition, and the final steady state, we can easily find an equation for position as a function of time:

$$x(t) = c_0 + c_1 e^{-t/\tau} = -\frac{b_2 v_r}{k}(1 - e^{-t/\tau})$$

Variables other than the state:

To find out what variables other than the state do, you have two choices: One is to solve for what the state is as a function of time, and then use that to find the other variables. A second method is to directly find the behavior of the other variables. This works as follows: First, find the time constant, based on the energy-storage state, as above. Then find the initial and final values, not of the state, but of the variable you are interested in. The trick here is that other variables may change instantaneously, so you need to be careful to use the values at r =0+, not 0-. The solution will be of the same form, and once again, all you need to do is find values of c_0 and c_1. This can be illustrated by continuing the above example.

We could have also been interested in velocity as a function of time. One way to get this would be just to take the derivative of the expression for position. Another way is as follows: In the same system, the time constant has to be the same. So the only difference is the initial and final conditions. The final steady state has velocity of the float equal to zero—a constant stretch in the spring. The initial condition, at t = 0- is a velocity of zero, before the float is dropped in the river. But, since the float is massless, the velocity *can* change instantaneously—there is no energy associated with the velocity. So it can have nonzero velocity just after it is dropped in the river, and we need to analyze the situation at t = 0+. At this point the spring still has zero stretch because it *can't* change instantaneously. So the forces involved are just from the two frictional forces, and the force balance equation is

$$F_{b2} = F_{b1}$$
$$b_1 v_f(t = 0+) = b_2(v_r - v_f(t = 0+))$$

We solve this to find $$v_f(t = 0+) = \frac{b_2}{b_1 + b_2} v_r$$

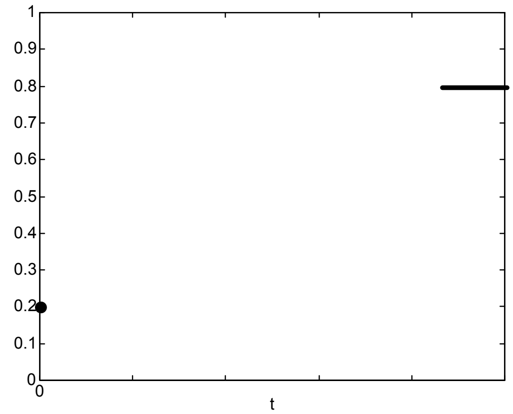
and thus the solution for velocity as a function of time is $v_f(t) = \frac{b_2}{b_1 + b_2} v_r e^{-t/\tau}$,

where once again $\tau = \frac{(b_1 + b_2)}{k}$.

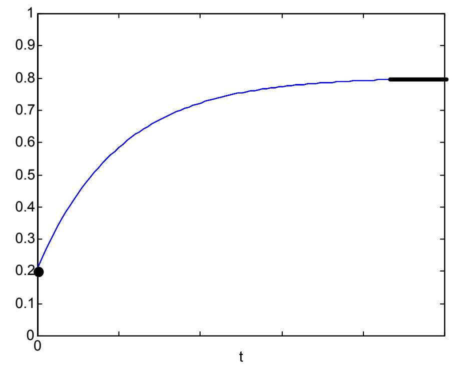
Sketching Solutions

Even with just the initial and final conditions, and the time constant, it is easy to sketch the behavior of a first order system, even without writing an explicit expression for behavior as a function of time.

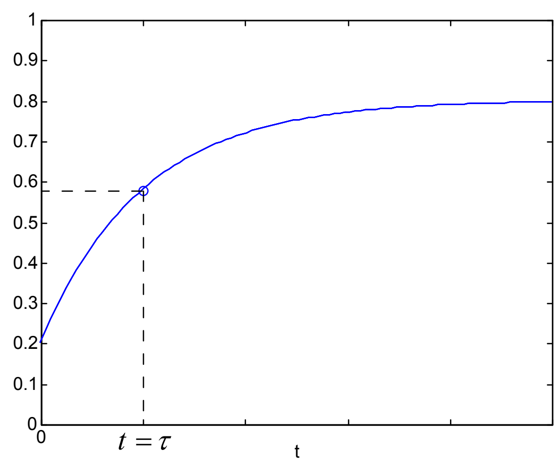
1. Mark the initial value at t =0+ (0.2 in this example), and toward the right of your graph a line for the final value (0.8 in this example).



2. Fill in an exponential curve between these.



3. Finally, fill in the time scale by labeling one point. At $t=\tau$, $e^{-t/\tau}=\frac{1}{e}=0.37$, so the curve will have gotten 63% of the way to where it is going. In this case, it is going 0.6, so 63% of that is 0.38, so it will have gone 0.38 from 0.2, and will be at 0.58. We can label that point:



The completes a basic, quantitative sketch of an exponential. Other rules of thumb for values at different points may be easily generated, but the most useful tend to be:

$t=\tau$	falls to 37%/rises to 63%
$t=3\tau$	falls to 5%/rises to 95%