NORMAL MODES OF A SYSTEM OF COUPLED HARMONIC OSCILLATORS

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Reading: Kibble, ch 11.

In this lab you will examine the motion of a system of two or more coupled oscillators driven by an external periodic driving force. The oscillators are connected in such a way that energy is transferred back and forth between them, leading to coupled oscillations. A system of \( n \) coupled one-dimensional oscillators (often said to have \( n \) "degrees of freedom": however, this phrase is used differently in different contexts, so we will not use it) is described by \( n \) coupled equations of motion. Although this motion can be quite complex, it is possible to describe the motion of the entire oscillatory system in terms of \( n \) normal coordinates, which are linear combinations of the original coordinates. Conversion to the normal coordinates results in \( n \) independent equations in which the normal coordinates are uncoupled and each varies harmonically with time. This means that each normal coordinate can be considered as a single independent oscillator, called the normal mode, or eigenmode, that resonates its characteristic frequency, usually called the eigenfrequency. If non-linearity is neglected, any arbitrary undriven motion of the system can be analyzed into its normal mode components.

In this lab you will examine systems that have from two to five normal modes.

Two Coupled Harmonic Oscillators

A simple example is two one-dimensional harmonic oscillators connected by a spring. In this lab the harmonic oscillators are two masses on an air track, each connected by a spring to a fixed point (Fig.1), and connected to each other by a third spring. In order to keep things mathematically simple we choose an arrangement which is as symmetrical as possible: the masses \( M \) are identical and the two outer springs have identical force constants \( k \). The central spring in general has a different force constant \( k' \).

![Fig. 1](image)

Applying Hooke's Law and Newton's Second Law, we find the equations of undriven motion for the two masses:

\[
M\ddot{x}_1 + (k + k')x_1 - k'x_2 = 0 \quad \text{(for mass 1)} \tag{1a}
\]

\[
M\ddot{x}_2 + (k + k')x_2 - k'x_2 = 0 \quad \text{(for mass 2)} \tag{1b}
\]

where \( x_i \) is the displacement of the \( i \)th mass, and \( \ddot{x} \) is the second time derivative of the displacement.

Since the motion is oscillatory, we look for a solution of the form

\[
x_1(t) = B_1 e^{i\omega t}, \quad x_2(t) = B_2 e^{i\omega t} \tag{2}
\]
where $\omega/2\pi$ is the frequency (to be determined) and $B_1$ and $B_2$ are the amplitudes. Substituting (2) into (1) we find

\[-M\omega^2B_1e^{i\omega t} + (k + k')B_1e^{i\omega t} - k'B_2e^{i\omega t} = 0 \quad (3a)\]
\[-M\omega^2B_2e^{i\omega t} + (k + k')B_2e^{i\omega t} - k'B_1e^{i\omega t} = 0 \quad (3b)\]

which simplify to

\[(k + k' - M\omega^2)B_1 - k'B_2 = 0 \quad (4a)\]
\[-k'B_1 + (k + k' - M\omega^2)B_2 = 0 \quad (4b)\]

For a non-trivial solution, the determinant of the coefficients of $B_1$ and $B_2$ must vanish

\[
\begin{vmatrix}
   k + k' - M\omega^2 & -k' \\
   -k' & k + k' - M\omega^2
\end{vmatrix} = 0 \quad (5)
\]

Expanding the determinant gives

\[(k + k' - M\omega^2)^2 - k'^2 = 0 \quad (6)\]

so that

\[\omega^2 = (k + k' \pm k')/M \quad (7)\]

and the characteristic frequencies for the system are:

\[\omega_1 = \sqrt{(k + 2k')/M} \quad (8a)\]
\[\omega_2 = \sqrt{k/M} \quad (8b)\]

Substituting back in (4) we find that $B_1/B_2 = -1$ for $\omega_1$ and $B_1/B_2 = +1$ for $\omega_2$. Thus the normal coordinates can be written

\[Q_1 = B(x_1 - x_2), \quad Q_2 = B(x_1 + x_2), \quad (9)\]

where $B$ is a normalization constant. It is convenient to choose $B$ so that $\Sigma B_i^2 = 1$, whence $B = 1/\sqrt{2}$.

The normal coordinates differ in the way they transform under a symmetry operation: i.e. a change of coordinates which leaves the physical system unchanged. The only such non-trivial operation in this one-dimensional system is reflection in the midplane: to do this you make the substitutions $x_1 \rightarrow -x_2, \quad x_2 \rightarrow -x_1$. If the mode is unaffected by this "inversion" operation, it is said to have even parity. If its sign changes, it has odd parity. One criterion for a normal mode is that it can be classified in this way: if we call the inversion operator $P$, a linear combination $Q$ of the coordinates is a normal mode if and only if $PQ = \pm Q$. (In more complex systems there may be many such symmetry operations).

$Q_1$ is unchanged on reflection, so its parity is even. The masses oscillate in antiphase. This mode can be excited by pulling the carts apart an equal distance and releasing them.
Q₂ changes sign on reflection, so its parity is odd. This mode occurs when the carts are pulled to the same side an equal distance. They oscillate in phase without changing the length of the coupling spring.

If you let \( k \to 0 \) this system is a model for a diatomic molecule, and Q₁ is its only mode of vibration. This is a symmetric mode of vibration of the molecule. Q₂ represents the free translation of the molecule: a diatomic molecule has no odd parity normal mode.

If \( k' << k \) we have a pair of weakly coupled oscillators. If one mass is allowed to oscillate while the other is held fixed, it will oscillate at an angular frequency \( \omega_0 = \sqrt{(k + k')/M} \). If the fixed mass is released the mode splits into two, with frequencies given approximately by \( \omega_0 (1 \pm k'/2k) \). The lower frequency mode, Q₂, is symmetric in the sense that both masses oscillate in phase, while Q₁ is antisymmetric since they oscillate in antiphase.

Note that the labels "symmetric" and "antisymmetric" depend on the physical system we are considering, while the parity is unambiguous. In the case of the weakly coupled oscillators, the mode of vibration labelled "antisymmetric" has a higher frequency than the "symmetric" one. We will use the labelling appropriate to molecules in the rest of this discussion.

Since no springs or masses are perfectly alike, the above analysis is only an approximate description of the actual physical system that you are investigating. However, small deviations from equality have little effect.

**Three Coupled Harmonic Oscillators**

This is shown in Fig.2. The central mass \( M' \) can be different from the other two (M) without destroying the reflection symmetry. There are now three normal modes.

\[
\text{Fig. 2}
\]

The secular determinant factors into two equations, one linear in \( \omega^2 \) and the other quadratic. The linear equation gives the even parity (symmetric in the molecular sense) mode

\[
Q_1 = (x_1 - x_3)/\sqrt{2}, \quad \omega_1 = \sqrt{(k + k')/M}
\]

(10).

The other two modes have odd parity and have frequencies

\[
\omega_2, \quad \omega_3 = \sqrt{[(k + k')/2M + k'/M'] \pm \sqrt{[(k+k')^2/4M^2 + k'(k'-k)/MM' + k'^2/M'^2]}}
\]

(11).

In the molecular case (a linear triatomic molecule, such as CO₂), \( k = 0 \) and we have

\[
\omega_2 = \sqrt{k'/\mu}, \quad \omega_3 = 0
\]

(12)

where the reduced mass \( \mu = (1/M + 2/M')^{-1}, \) and

\[
Q_2 = (2Mx_2 - M'(x_1 + x_3))/\sqrt{4M^2 + 2M'^2}, \quad Q_3 = (x_1 + x_2 + x_3)/\sqrt{3},
\]

(13)
the zero frequency mode corresponding as before to free translation of the molecule.

Another relatively simple case is the "chain" of equal masses and force constants: 
\( M' = M \) and \( k' = k \). In this case:

\[
\omega_1 = \sqrt{\frac{2k}{M}} \\
\omega_2, \omega_3 = \sqrt{\frac{(2\pm\sqrt{2})k}{M}}
\]

(14a) 
(14b)

giving frequencies in the ratios: \( \omega_2 : \omega_1 : \omega_3 = 1: 1.848: 2.414 \).

The normal modes \( Q_2 \) and \( Q_3 \) do not have a simple form, except in three cases: \( k = 0 \), the chain (see below), and \( k'<<k \). In the latter case we have two coupled oscillators, which only differ from the two-mass case by the addition of a mass at the center of the coupling spring.

**Chain of n masses**

We restrict ourselves to the case where all the masses and force constants are equal. The eigenfrequencies are then given by

\[
\omega_r = 2\sqrt{\frac{k}{M}} \sin \frac{\pi r}{2(n+1)}, \quad r = 1 \text{ to } n
\]

(15)

and the (unnormalized) normal modes are

\[
Q_r = \sum_{i=1}^{n} x_i \sin \frac{i\pi r}{(n+1)}
\]

(16).

The signs and relative magnitudes of the displacements of the masses are just those of a continuous string of length \( L \) vibrating at its eigenfrequency \( \omega = \pi v/L \), where \( L = (n+1)a \), \( a \) being the length of each spring, and \( v = a\sqrt{(k/M)} \) is the velocity of waves on the string (see Marion *Classical Dynamics* 2nd ed. fig 13-11, French Vibrations and Waves Fig. 5.15). The frequencies would be the same as those of the string if \( r << n \), since then the wavelength \( \lambda >> a \) and the discrete system can be treated as continuous. For larger \( r \) they are different: there is no limit to \( \omega \) on an ideal string, whereas the discrete set of masses has only \( n \) modes, with an upper limit to \( \omega \) of \( 2\sqrt{v/a} \approx 2\sqrt{v} \) for \( n >> 1 \).

Note that some of the coefficients in (16) may be zero. These occur when a node in the motion of the continuous string coincide with one of the masses in the discrete case.

The chain of \( n \) masses is a simple one-dimensional model of a crystalline solid, and the longitudinal modes of vibration of the atoms in a simple crystal are analogous to those that you will observe in this experiment. Because \( n \) is a very large number in a solid, the frequencies are essentially continuous, and are expressed as a function of the wave-vector

\( q = 2\pi/\lambda \), where \( q = \pi r/L \). The maximum possible value of \( q \) is \( q_m = \pi/a \) and the frequency vs wave-vector relation ("dispersion relation") is given by:

\[
\omega(q) = 2(v/a) \sin (qa/2).
\]

(17)

In this lab you will calculate and experimentally determine the normal modes for various oscillating systems.
Getting Started

Before starting, see the instructor or TA.

In the instructions below several calculations are called for. These can be done, and the comparison with experiment made, after you have finished in the lab.

Put the carts you need CAREFULLY on the air track. Turn the air on first, and always have it on whenever you add or remove a cart. Whenever you move a cart by hand, do it GENTLY without pressing down or tilting, so that it doesn't contact the track.

Drive system. A variable speed stepper motor turns a crank which is connected to the left hand cart by a string. The stepper motor is driven by pulses from an oscillator whose separation in time can be controlled and read on the counter. The motor makes one revolution for each 96 pulses in "mode 1" and 48 in "Mode 2". use mode 1 for frequencies up to 1.25 Hz and mode 2 for frequencies above this. Only one "mode" switch should be up at a time. The counter works best in the "period" mode, where it measure the time $t$ in milliseconds between successive pluses. Thus the driving frequency is $1000/96t$ Hz in "mode 1" and $1000/48t$ Hz in "Mode 2". Under certain circumstances the counter responds to spurious pulses and reads too short a time: if in doubt check that the readings that you are getting are reasonable by using the stop watch to get a rough measure of the rotation rate of the driver. Don't alter the other settings on the driver: use only the "mode" and on/off switches. There is a sheet with more detailed instructions by the equipment.

Parameters

The carts, with the copper strips or pick up coil included, weigh 0.216±0.002 kg. Replacing a copper strip by the brass weight increases the mass by 0.160kg.

The force constants of the numbered springs are, with an accuracy of ±2%:

#1  1.70 N/m  
#2  1.15 N/m  
#3  1.15 N/m  
#4  1.83 N/m  
#5  3.4  N/m  (not normally used in this lab)

Each spring in the matched set has a force constant of 1.77N/m ±1%.

NOTE: these measurements were made in Aug 2000. They should be rechecked at the time of the experiment.

Macscope

In order to record the motion electronically, a pickup coil is mounted on the left handcart, labelled M0. One short side of this large flat coil moves between the pole pieces of a magnet consisting of two steel L-pieces with two horseshoe magnets on top. The output of the coil is amplified and fed through an analog to digital converter to a computer under the control of "Macscope" software. In its "Scope" mode Macscope will (on command) capture a time slice of the output of the coil, and display a desired section of it. In the "Analyze" mode a chosen section is Fourier analyzed. NOTE: Before analysis you must use the mouse to "black" a section of the recorded signal, of arbitrary length $t_0$. In the Analysis mode Macscope gives a discrete spectrum of the signal, the vertical bars in the display giving the amplitude at frequency $f = n/t_0$, where $n$ runs from 1 to 128. Since the different eigenfrequencies of coupled oscillators are not harmonically related, the signal is not periodic, so the normal practice of putting $t_0$ equal to the
period won't work in this case. Instead, choose \( t_0 \) as close to a round number (say 25 or 50 seconds) as Macscope will permit, so that the \( f \)'s are simple numbers. The "Fourier spectrum" is simply the discretized frequency spectrum of the captured signal, with \( 1/t_0 \) as the unit of frequency. After analysis, the amplitude and phase at each frequency can be read by clicking on the "Data" box. Plot the amplitude and phase against frequency: note how the phase changes abruptly on passing through a normal mode. Draw a smooth curve through the points to find each eigenfrequency accurately (see next page).

To save time in the lab, it is best to analyze the data at home. Save your data files and email them to yourself. Macscope is available under "Site licensed software" on Public.

**Using Macscope**

Open the Macscope application which is in the "Macscope" folder on the desktop.

Set "Amp" to 2000 and "Time" to 1 min. The vertical scale should then be in millivolts and the horizontal scale should run from 0 to 60 sec.

Click the stop/go box: "going" should appear on the screen. Data points should start appearing on the "scope".

After about 60 sec. click stop/go to stop. The data should now be steady on the scope.

Using the mouse, black a 50 sec portion of the data. The length of time you have blacked appears in a box to the right of the scope display. Get as near to 50 sec as Macscope will permit.

Open the "analyze" menu, and click "Fourier analyze". After a few seconds, the discrete Fourier transform (FT) of your data will appear at the foot of the screen. The horizontal scale reads frequency in units of \( 1/t_0 \) (usually 0.02 Hz), the vertical scale reads amplitude. The display can be expanded horizontally by clicking "scale".

Click the FFT data box. A table will appear with columns giving: (1) the frequency in units of \( 1/t_0 \), (2) the amplitude, normalized to the maximum value, and (3) the phase of the FT. Ignore the two right hand columns (which give real and imaginary parts of the FT). At a resonance, the phase changes abruptly†. Ignore phase readings when the amplitude is very small. Find each resonance in amplitude and then plot the phase in its vicinity (4-6 points should be enough if properly chosen), draw a smooth curve like the one in Fig. 4, and interpolate to find where the phase is changing most rapidly. This will give you a more precise value for the resonant frequency. Note that the phase shift at resonance is not necessarily zero, since there is a background shift from other resonances.

![Fig. 4](Phase in the vicinity of a resonance)

† Explain why this should be so. Reminder: the FT \( A(f) \) is the (complex) response of the system to a driving force at frequency \( f \).
Two Coupled Oscillators

1. Set up according to Fig. 1, with $k' < k$. Put the shorter springs (i.e. those with the higher force constant) on the outside of the carts, and the longer spring in the middle.

2+ . From the masses and the force constants, calculate the frequencies of the normal modes for the system. Don’t forget that $f = \omega / 2\pi$.

3. Without turning on the driver, move both carts an equal distance to the right, without compressing or stretching the middle spring. Note the slow, antisymmetric pattern of oscillation. This is the odd parity normal mode. Measure its period using the stopwatch, taking a time average over at least 10 cycles, and hence obtain its frequency $f_1$. Now pull the carts apart an equal distance, note the symmetric pattern of oscillation. This is the even parity normal mode. Find its frequency $f_2$.

4. Now displace one cart only, holding the other still, and release them both. Note that the oscillations have no obvious pattern. This is created by a combination of both the normal modes, their relative amplitudes being determined by the initial condition.

5. Hold one cart in its equilibrium position while you move the other a short distance. Release both. Note how energy is transferred from one oscillator to the other. Measure the time taken for the energy to transfer from one to the other and back. Compare with your calculation+.

6. Carefully insert the pickup coil into the slots on the left hand cart (if it isn't already in place) and place the magnet over it. Make sure that one side of the coil remains within the magnet gap, and the other outside it, over the entire range of motion of the cart. Make sure that the coil moves freely, without touching the magnet, and that the connecting cable doesn't touch anything or impede the motion of the cart. Set the carts moving in arbitrary way, as in (4). Capture the coil output over a period of about 1 minute on Macscope and Fourier analyze. Plot the phase as a function of frequency (see above under ”Macscope”) and compare the eigenfrequencies that you obtain with $f_1$ and $f_2$. Why do the peaks have non-zero width?

7. Now set the carts moving in a different but arbitrary way, so as to excite both normal modes, and check with Macscope that the eigenfrequencies are unaltered, although their relative amplitudes will be. If the frequencies were altered, what would this imply?

8. Carefully thread the fishing line through the guide at one end of the air track and attach to the driver pin with a loose loop: if you make it too tight the string will wind itself up on the pin. Attach the other end of the string to the outer spring. Turn on the driver. Start with low frequency (say 0.5 Hz) and increase it slowly until the amplitude of oscillation gets large. Because the damping is very low it is easy to go right through the resonance. When you have found the peak of a resonance, turn off the driver and GENTLY stop the carts before looking for the next resonance. At the resonant frequency the amplitude of oscillation will be very large: note the pattern to the oscillation. The patterns should be identical to the patterns you observed for free oscillation in a single normal mode. Note that one will be even and the other odd parity. Measure the resonant frequencies *, not forgetting to multiply the period measured on the counter by 48 or 96 to obtain the period of rotation of the shaft. Compare these results with the predicted and previously measured values. Which method gives the most accurate result? Why? If the damping were large, which method would be best?

+ These calculations should be done before you come to the lab.
* Asterisked items may be omitted if time is short.
9*. Replace the right hand cart with the one with the heavy brass bar. Turn off the driver and let the system oscillate freely for a minute. Because you have broken the symmetry of the system, the motion will not have a clear pattern, and you may find it difficult to excite a single normal mode. Instead, use Macscope as in (6) above to obtain the resonant frequencies. Compare them with your calculation+.

**Three coupled oscillators: the linear triatomic molecule**

10. Add another cart to your system, as in Fig. 2, starting with equal masses. Choose $k' > k$ to model a one-dimensional triatomic molecule. Set up the springs so that given the force constants, the arrangement is as symmetrical as possible.

11. Try to excite the three normal modes separately, using your knowledge of the mode to start the carts of with the correct relative displacements. Measure their frequencies. Why is it much easier to excite the even mode than a single odd mode?

12. With the pickup coil on the left hand cart, use Macscope to find the three frequencies present in an arbitrary motion. Compare with (11) and with your calculation+. Now make the cart with the pickup the central cart and repeat the measurement. Is one of the frequencies missing? Which? Why?

13*. Turn on the driver and carefully set its frequency at each of the eigenfrequencies in turn. Compare the displacements qualitatively with the prediction for a free molecule. Why do the displacements in the symmetric mode agree better with theory than those in the antisymmetric mode?

14. Replace the central cart with the one with the heavy brass bar. Find the new mode frequencies with Macscope. Which modes are altered and which not? Explain. Measure the new frequencies and compare with your calculation+.

**Chain of five masses**

15. For this part only, you should use the six springs which are kept in a separate labelled box. These have been carefully selected and have force constants within 1% of each other. Their average force constant is 1.77 N/m. Please treat them with extra care and return them to their box when done.

16. Since there are now five coupled oscillators, there are five normal modes. Calculate their frequencies+. It is difficult to excite the (undriven) normal modes independently, but give it a try*.

17. With the pick-up on the left hand cart, set the carts into an arbitrary undriven motion. Use Macscope to find the five eigenfrequencies. Compare them with your calculation. If you can't find all five, try restarting the motion with different initial conditions.

18. Now swap the two left hand carts so that the pick-up is on the second cart. Swap the carts rather than moving the pick-up coil to another cart.

Which modes are missing? Why?

19*. Turn on the driver and set its frequency at each of the previously measured eigenfrequencies in turn. The higher modes are quite close in frequency and it will require some care to excite them separately. Note the pattern to the motion and the parity at each resonance.
Closing down.

When you are done, detach the springs **carefully** from the carts and put them away, with the matched set of six their own box. Remove the carts from the air track and *then* turn off the air. Don't forget to switch off the stop watch (battery operated). Turn off the counter and stepper motor driver.

**Write up**

Calculate the normal modes of two coupled oscillators with unequal masses and compare with the results of (9) above.

Set up the secular determinant for the three mass / four spring problem (Fig 2) and derive eqs. 10 and 11 for the normal mode frequencies. Show that for $k \rightarrow 0$ your results reduce to those for the triatomic molecule discussed in class, while for $k = k', m = M$, they reduce to the linear chain case (eq. 15) with $n=3$.

Find the conditions on $i$, $r$ and $n$ for the coefficient of $x_i$ in the mode $Q_r$ of a linear chain (Eq. 16) to be zero, and compare with the result of (18) above. The mode then has a node at the $i$'th particle. What would be the effect on the frequency of a mode of altering the mass of a particle at a node?

For the case $n = 5$, use eq. 16 to make a 5x5 table $(i=1-5, r = 1-5)$ of the coefficients of the $x_i$ in the $Q_r$. Note their signs, and that some of them are zero. Sketch the five $Q_r$, representing the displacement as a vector perpendicular to the chain (see Kibble, Figs. 11.5,6). Give the parity of each mode. Do you see a pattern in the parity as a function of mode number? Use your table to identify the modes observed in (17) and (19) above. Compare with the corresponding modes of a continuous string.

Answer the questions posed in the text of this write-up. Show plots of the Fourier amplitudes and phases obtained from Macscope. For each set-up, tabulate and compare the normal mode frequencies obtained by calculation, by timing the free oscillation of a single mode, by direct excitation using the driver, and from Macscope. Estimate the errors in each method and discuss the discrepancies which exceed your estimated error. If possible, suggest improvements in the experiment.