THE DRIVEN, DAMPED OSCILLATOR

I. Introduction

The driven, damped harmonic oscillator is one of the most widely useful examples encountered in introductory physics. It applies to the motion of everthing from grandfather clocks to atomic clocks with a detour through automobile suspensions, the car radio, and the bridges you drive that car over.

In this lab we will consider a very simple system: a vibrating hacksaw blade. This is a very low friction system, so if you set it in motion, it will oscillate for a very long time. To investigate the role of damping, we will introduce extra damping with an aluminum "sail" that sticks down into a puddle of glycerine. The frictional force exerted by the glycerine on the sail is proportional to the sail's velocity: $f = -bv$, as is usually the case for viscous drag.

In examining this system your objectives are to:

1. Determine the natural frequency of the undamped oscillator.
2. Determine what effect the damping force has on the value of this natural frequency, and what effect it has on the motion in general.
3. Explore the behavior of a damped, driven oscillator.
4. Compare the results of your quanitative measurements with theoretical predictions about the behavior of this system.

But first, some mathematical preliminaries. We can derive a differential equation that describes the motion of the driven, damped oscillator quite simply from Newton's second law:

$$\sum F = m\ddot{x}.$$

In this case there are three forces. The spring force, which is always opposed to the displacement, the damping force, which is always opposed to the velocity, and the driving force, which oscillates in time.

$$\sum F = F_{\text{SPRING}} + F_{\text{DAMPING}} + F_{\text{DRIVING}} = -kx - b\dot{x} + F_0 \cos \omega t$$

so that the differential equation for the motion can be written

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = F_0 \cos \omega t, \tag{1}$$

where

$$\gamma = \frac{b}{m}, \quad \text{and} \quad \omega_0^2 = \frac{k}{m}.$$

As it turns out, the solution to this equation is simple harmonic motion:

$$x(t) = A \cos(\omega t + \phi) \tag{2}$$

with the amplitude given by

$$A^2 = \left(\frac{F_0}{m}\right)^2 \frac{1}{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}. \tag{3}$$

Equation 3 will prove a bit inconvenient in lab, since you will not know the force with which you are driving your oscillator. If the damping is weak ($\gamma \ll \omega_0$) it is possible instead to express the ampltude in terms of the maximum amplitude, which we will denote A_0:

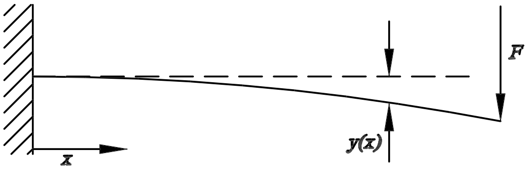
$$A^2 = A_0^2 \frac{(\gamma\omega_0)^2}{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}. \tag{4}$$

Alas, this is still not a very convenient equation. It does not have a mathematically "nice" shape. But, in the small damping limit, and for ω near ω_0, it can be approximated as

$$A^2 = A_0^2 \frac{(\gamma\omega_0)^2}{4\omega_0^2[(\omega_0 - \omega)^2 + \gamma^2/4]}. \tag{5}$$

II. Pre-lab Excercises

1. The saw blade is a "beam" clamped at one end, and with a force F applied to the other.



   If the free (un-clamped) length of the sawblade is L, then the deflection of the beam from equilibrium will be given by

   $$y(x) = \frac{F}{EI}\left(\frac{1}{6}x^3 - \frac{1}{2}Lx^2\right),$$

   where x is measured from the clamped end. This is result is too difficult for us to derive, but if you aren't shown how to solve this problem when you take Math 23, you should complain loudly!

   For now, show that the deflection of the free end (where the force is applied) obeys Hooke's law. Therefore, the beam will undergo damped simple harmonic motion, which is good, since that is what we know how to deal with!

2. Derive Eq. 4 from Eq. 3.
3. Show that if ω is close to ω_0, then $2\omega_0(\omega_0 - \omega)$ is a good approximation to $(\omega_0 - \omega)(\omega_0 + \omega)$.
4. Show that (a) in the limit of small damping, and (b) for $\omega \approx \omega_0$, Eq. 4 becomes Eq. 5.

5. Show that Eq. 5 can be re-arranged to read

$$\frac{A_0^2}{A^2} = \frac{4}{\gamma^2}(\omega_0 - \omega)^2 + 1 \tag{6}$$

.

6. Show that Eq. 6 can be re-arranged to read

$$\sqrt{A_0^2/A^2 - 1} = \frac{2}{\gamma}\omega_0 - \frac{2}{\gamma}\omega. \tag{7}$$

Show that if you measured A as a function of ω and determined A_0, then plotting the data according to Eq. 7 would give you a straight line. Find the intercept and slope.

III. Procedure

This lab will be your first encounter with an oscilloscope—at least in Dartmouth physics labs. The ones you will use have some fancy features. You will need to learn how to use these features because parts of the lab will be very difficult without them. The TA's will help you out here.

1. Check out the apparatus.

   There is a saw blade with a magnet and a funny looking aluminum sail on the end. Twang it.

   You can apply a force on the (permanent) magnet with an electromagnet. You will use a signal generator for that. Try driving the oscillator: turn up the amplitude knob on the generator and set the frequency near 6 Hz. If anything interesting happens, write it down.

   You can derive a signal proportional to the position of the blade end with a solar cell and a light bulb. See if you can figure out the basic idea. (Turning it into a useful signal is tricky and requires some electronics. . .) You should be able to observe the signal on the oscilloscope. Don't apply more than 3 volts to the bulb or you will be in need of a new one. The thing acutally works best with a faint bulb. The TA can help you set the proper illumination level.

2. Put just enough glycerine in the little cup to reach the bottom of the sail. The sail should stick no more than one mm down into the liquid.

3. Twang the blade and use the oscilloscope to measure the frequency and decay constant of the oscillator. You will need to use the scope in "single shot" mode for this. Ask the TA. If the frequency of the oscillator is not less than 10 Hz (cycles per second), then you need to lengthen the blade a bit until it vibrates at something closer to 6 or 7 Hz.

4. Use the signal generator to drive the oscillator. Find the frequency at which the amplitude of the motion is largest. Look on the scope and turn down the drive amplitude until you have a nice sine wave with no flat tops. Double check that you are at the point of maximum oscillator motion. *DO NOT MAKE FURTHER ADJUSTMENTS IN THE DRIVE AMPLITUDE FOR THE REMAINDER OF THE LAB.*

5. The oscilloscope can display two inputs at once. Set it up to show both the drive signal and the oscillator response signal. Make sure the 'scope is "triggering" off the drive signal. Figure

out how to measure the phase angle (in radians) between the drive and the response. The horizontal axis of the display is in time units, but the period of the drive can be defined to be 2π radians. Then the time delay between a peak in the drive signal and the following peak in the oscillator response can be converted to radians. This is a bit sophisticated. Your TA can help.

6. Measure the amplitude of the motion and its phase as a function of drive frequency. At each frequency, make sure all the transient behavior has died away before you record your data. Make plots of the amplitude and phase versus angular frequency (that is frequency in radians per second, not cycles per second!) as you go along. Use this plot as a guide to where you should take more points and where the behavior is boring so you don't need so many points. (Hint: where things change a lot with small changes in drive frequency, you need lots of data, and where they change very little with changes in drive frequency, you don't need very much. You should also take enough data to really nail down the amplitude at the peak.) Do as nice a job as you can here. Lousy data now will make for a lousy analysis later. Be sure to make reasonable estimates of your uncertainty in both the amplitude and the frequency.
7. Stop the drive by physically disconnecting the cable from the signal generator. Don't turn down the amplitude! Add lots more glycerine to the cup (so the sail is 1 to 2 cm into the liquid)
8. Repeat steps 3 through 6 above.
9. Clean up: use the plastic pipettes to suck the glycerine out of the cup and return it to the bottle. Wipe off the end of the sail with a paper towel.

IV. Analysis

1. Plot your two sets of amplitude data on a single graph. Determine the resonant frequency and Q. (Remember that you have measured the amplitude and that the definition of Q is in terms of *energy*. Refer to pages 427-8 of your text for help on how to get Q from the resonance curve.)

We can actually do much better. Here's how:

2. Use your data to get an accurate value for A_0, the maximum amplitude at the peak of the resonance curve. You may not have a data point at exactly this point, so make a nice plot of the data around the peak and draw a smooth, reasonable curve through the data. (Don't just "connect the dots.")
3. Manipulate your data according to the "instructions" in Eq. 7, plot it as a straight line (actually, you discover that you have two straight lines, but with some cleverness, you can make it all one). Determine ω_0 and γ from the plot. Compare your result for ω_0 to your other determinations of ω_0. Compare your result for γ with the γ you get from the Q you measured for the undriven oscillator. Eq. 7 was derived under the assumption that $\gamma \ll \omega_0$. Is this true for the data with lots of glycerin in the tub?
4. Compare both the frequency you measured in the case of undriven oscillations and the fequency of the peak in the resonance curve to your best estimate for ω_0 from the straight line fit. Is this result consistent with your expectations? (You are looking for a phenomenon known as

"frequency pulling." The more heavily damped an oscillator is, the more its actual resonant frequency deviates from ω_0, which is the natural frequency of the *undamped* oscillator.)

5. Plot your two sets of phase data on a single graph. Define the phase difference between the drive and response at very low frequencies to be zero. Compare the actual response to the prediction derived in class:

$$\tan\phi = \frac{\gamma\omega}{\omega_0^2 - \omega^2}.$$