Waves Tutorial Two

Overview:
In Wave Tutorial One, you examined how two equal amplitude waves with the same wavelength and frequency traveling in opposite directions add to form a standing wave. In this Tutorial, you will explore the effects of adding waves with different wavelengths and frequencies together.

Resources:
http://cat.sckans.edu/physics/superposition.htm . This Java applet allows you to specify functions for f(x,t) and g(x,t) and view animations of f(x,t) and g(x,t) and the sum.

Activities:
Adding two waves together:
From last week’s tutorial, you found that two waves traveling in opposite directions can produce a standing wave, provided that the amplitudes, wavelengths and frequencies of the two waves are the same. In this section, you will develop some more general intuition about what happens when any two waves traveling in the same direction interfere. After this part of the tutorial, you should be able to construct a wave with any combination of phase and group velocity and discuss the algebraic origin of phase velocity and group velocity.

1. Background: In general, you can write two different sinusoids with equal amplitude as

\begin{align*}
y_1(x,t) &= A\cos[(k_{ave}+\Delta k)x - (\omega_{ave}+\Delta\omega)t + (\varphi+\Delta\varphi)] \\
y_2(x,t) &= A\cos[(k_{ave}-\Delta k)x - (\omega_{ave}-\Delta\omega)t + (\varphi-\Delta\varphi)]
\end{align*}

Q: Use some trigonometry identities to reduce the sum of these to a single term (the result should be a product of two trig functions with different arguments). Hint: use the formulas for \( \cos(\alpha \pm \beta) \).

Q: One of the two factors in the product describes the “envelope” and the other describes the “component” waves. Which is which? If unsure, ask your TA. How fast do the component waves travel? How fast does the envelope travel? Explain the reasoning behind your answers.

2. Load Illustration 17.7 from the Physlet CD. The Physlet shows two traveling waves, \( y_1(x,t) \) and \( y_2(x,t) \) as well as their sum, \( Y(x,t) = y_1(x,t) + y_2(x,t) \).

3. Same \( \omega \), different \( k \): Initially, the two waves have identical frequencies \( \omega_1 = \omega_2 = 8.4 \text{ rad/s} \) and identical wavenumbers \( k_1 = k_2 = 8.4 \text{ rad/m} \). Change the wavenumber for one of the waves slightly, click “Set values and play.” Depending on the value of the wavenumber you chose, you may not see much. Gradually increase the difference between the wavenumbers of the two waves.
Q: What effect does increasing the difference in wavenumber have on the shape of the resulting wave? (Some sketches might be helpful here).

Q: The speed of the “envelope” of the wave is called the group velocity. What is the group velocity in this case? The speed of the component waves within the “envelope” is called the phase velocity. What is the phase velocity of the wave (approximately speaking)? (Yes, the answers to these questions are on the Physlet panel. Try not to peek).

Q: Using the result from Part 1, predict the width of the envelope $\Delta x$, based on the values of the wavenumbers of the two waves. Check the result.

4. **Same $k$, different $\omega$:** Reset the wavenumbers to the initial values: $k_1 = k_2 = 8.4 \text{ rad/m}$. Change the frequency for one of the waves slightly.
   
   Q: How does the shape of the resulting wave differ from the “Same $k$, different $k$” case?

   Q: What effect does increasing the difference in frequency have on the behavior of the resulting wave?

   Q: Does the concept of group velocity make sense for this wave? Explain.

   Q: What is the phase speed of this wave (approximately)?

   Q: Use the result from Part 1 to predict the width of the envelope, based on the values of the wavenumbers of the two waves. Check the result. (Hint: Is there an “envelope”?)
5. **Same \( v \), but different \( k \) (and \( \omega \)):** Set the parameters of wave \( y_1 (x, t) \) so that it has \( k_1 = 8.4 \text{ rad/m} \) and a speed of 1 m/s. Set up wave \( y_2 (x, t) \) with a slightly different \( k \), but make sure that wave two is traveling at the same speed as wave one (you'll have to choose an appropriate value for \( \omega_2 \)). Note: This case mimics light traveling in a vacuum (or other non-dispersive medium).

Q: Compare what you see now with what you saw in the “Same \( v \), different \( k \)” case. How are the two situations similar? How do they differ?

Q: Which moves faster: the component waves or the envelope?

6. **Designing waves:** Use what you know from Part 1 to create each of the following from two rightward traveling waves. Record the essential information of the two waves you added together to get the desired result. Note: all of these can be accomplished even while keeping \( \Delta k \ll k_{\text{ave}} \) and \( \Delta \omega \ll \omega_{\text{ave}} \).

- A wave with an envelope traveling to the right that goes slower than its components. (The usual case for free particles in quantum mechanics).

- A wave with an envelope traveling to the right that goes at the same speed as its components. (What does this correspond to?)

- A wave with an envelope traveling to the left that goes faster than its components. Notice that the envelope can go left even though the components are traveling to the right!
Adding several waves together:

Now you will look at what happens when you add a number of waves together.

7. **Modeling a light packet:** Construct a wave packet to represent light comprised of several different wavelengths added together:

\[ y = \sum A_i \cos[(k_i x - \omega_i t)] \]

Choose five or so values of \( k \) that are evenly spaced and closely spaced (setting the spacing at about 2-5% of the wavenumber seems to work well). Choose the values of \( i \) so that \( i = \text{const} \times k_i \) (the constant need not be the speed of light, but it must be the same for all wavenumbers).

Record the values you use for \( A_i, k_i \) and \( \omega_i \) here:

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Use a physlet, such as the one at [http://cat.sckans.edu/physics/superposition.htm](http://cat.sckans.edu/physics/superposition.htm), to view an animation of the wave packet you’ve made. (To use the physlet cited here, simply let \( f(x) \) equal a few of the terms and \( g(x) \) equal the rest. The physlet shows the sum of \( f(x) \) and \( g(x) \).)

Q: Measure the wavelength of the component wave. Is this what you would expect, given your values of \( k_i \)? Explain.

Q: Measure the “wavelength” of the envelope. (Note: this is the wavelength of the entire repeating pattern, not just the central peak.) Is this what you would expect, given your values of \( k_i \)? Explain.

Q: Allow the physlet to run for a while. Does the packet you’ve created spread out with time? Is this what you expect? Explain.
8. **Modeling a particle:** Choose five or so values of $k_i$ that are evenly spaced, and use values of $A_i$ that get smaller the further $k_i$ is from the average value of $k$. (You can use the same $k$’s as for the light wave if you want). However, for particles, $i = const \cdot k_i^2$.

Record the values you use for $A_i$, $k_i$ and $\omega_i$ here:

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Q: Why must $i = const \cdot k_i^2$ to model particles? (Hint: remember that, in general, $y = \sum A_i \cos\left(\frac{p_i x - E_i t}{\hbar}\right)$).

Q: Measure the wavelength of the component wave. Is this what you would expect, given your values of $k_i$? Explain.

Q: Measure the “wavelength” of the envelope. Is this what you would expect, given your values of $k_i$? Explain.

Q: Allow the physlet to run for a while. Does the packet you’ve created spread out with time? Is this what you expect? Explain. (Note: You may have to play with the parameters to get output that makes sense).

Q: In what ways does the wave packet you constructed behave like a classical particle? In what ways is it completely unlike a classical particle?