Theory

Consider a thin, flexible string (piano wire, rope, etc.) of length L, linear mass density $\mu$, under tension T, which is fixed at both ends as shown in figure 1. Two questions we might ask are whether waves can exist in such a system and if so what is the form of the function $y(x,t)$ which describes the propagation of the wave?

If a system will support waves, then the equation describing the behavior of the system will have the form of the classical wave equation,

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0 \quad . \quad (1)$$

Therefore, to answer the first question posed above, we need to derive the equation of motion for our string and compare its form to that of equation (1). To answer the second question, we need to look at the effect the fixed ends have on waves traveling down the string. In what follows we will answer these questions.

Consider a small section of the string dm which has been displaced in the vertical direction as shown in figure 1a. The displacement is $y = y(x,t)$. We will assume that the displacement is small and that $\theta$ and $\phi$ are everywhere small so that we can use the approximations $\cos \theta \approx 1$, $\theta \approx \sin \theta \approx \tan \theta$ and $\phi \approx \sin \phi \approx \tan \phi$. The element dm is acted upon by two forces, the tension T at both ends. (Since the string is thin, gravitational forces can be neglected). The forces in the horizontal and vertical directions are

$$F_x = T \cos \phi - T \cos \theta \quad \quad (2a)$$
$$F_y = T \sin \theta - T \sin \phi \quad \quad (2b)$$
Since $\cos \theta \approx \cos \phi \approx 1$, the horizontal forces cancel leaving a net force only in the y direction.

Applying the small angle approximation to equation (2b) yields

$$F_y = T(\tan \theta - \tan \phi). \quad (3)$$

But $\tan \theta = -\frac{\partial y}{\partial x} l_x$ and $\tan \phi = -\frac{\partial y}{\partial x} l_x + \Delta x$. Substituting these expressions into equation (3) gives

$$F_y = T\left(-\frac{\partial y}{\partial x} l_x \right) - T\left(-\frac{\partial y}{\partial x}_{l_x + \Delta} \right) \quad . (4)$$

To make the notation simpler, we define a function $g(x) = \frac{\partial y}{\partial x} l_x$. Substituting this into equation (4) and rearranging terms yields

$$F_y = T[g(x + \Delta x) - g(x)]. \quad (5)$$

Applying Newton's second law gives

$$m a_y = T[g(x + \Delta x) - g(x)]. \quad (6)$$

But $m = \mu \Delta x$ and $a_y = \frac{\partial^2 y}{\partial t^2}$. Substituting these expressions into equation (6) and rearranging terms yields

$$\frac{g(x + \Delta x) - g(x)}{\Delta x} - \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} = 0 \quad . (7)$$

Realizing that $\frac{g(x + \Delta x) - g(x)}{\Delta x} = \frac{\partial g(x)}{\partial x} = \frac{\partial^2 y}{\partial x^2}$, allows us to rewrite equation (7) in its final form,

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2} = 0 \quad . (8)$$

If we let $\mu/T = 1/v^2$, then we see that the equation governing the motion of the string has the same form as the classical wave equation. Therefore, waves can exist in our system. The waves will travel with velocity $v = (T/\mu)^{1/2}$ and the function $y(x,t)$ will be numerically equal to the y displacement at a time $t$ of a point on the string at position $x$.

Now that we know that waves can exist in our system, we can turn our attention to the question of the form of the function $y(x,t)$. The fact that the ends are fixed means that the y amplitude must always be zero at the ends. Therefore, only those functions for which $y(0,t) = 0 = y(L,t)$ are suitable solutions. It can be shown by substitution that functions of the form $y = A \sin (Kx \pm \omega t)$ and $y = B \cos (Kx \pm \omega t)$ [where $K$ (wave number) and $2\pi/\lambda$ and $\omega$ (angular frequency) = $2\pi v = v K$], are solutions to equation (8). However, since the functions $y = B \cos$
(Kx ± ωt) cannot always be zero when x = 0 we can eliminate that set of functions. To determine the form of the function y (x,t) for our system we must use waves of the form \( y = A \sin (Kx ± \omega t) \).

The function \( y_1 = A \sin (Kx - \omega t) \) represents a continuous sine wave traveling to the right down the string, and \( y_2 = A \sin (Kx + \omega t) \) represents one traveling to the left. If these waves are perfectly reflected at the ends, we have two waves of equal frequency, amplitude and speed traveling in opposite directions on the same string. The principle of superposition of waves states that the resulting wave will be the algebraic sum of the individual waves,

\[
y = y_1 + y_2 = A[\sin (Kx - \omega t) + \sin (Kx + \omega t)]
\]

or using the trigonometric identity for the sum of the sines of two angles (\( \sin B + \sin C = 2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(C-B) \)), we obtain

\[
y = 2A \sin Kx \cos \omega t
\]

This function obviously satisfies the boundary conditions at x = 0, but will only satisfy the boundary condition at x = L when \( K = n\pi/L \) (where \( n = 1,2,3,... \)). Limiting the values of K to only certain values also limits the wavelength, frequency and speed of the waves to certain discrete values. Therefore, unlike traveling waves on an infinite string which can have any wave-length or frequency, waves on a bounded string are quantized, restricted to only certain wavelengths and frequencies. To note this quantization, equation (9) can be rewritten as

\[
y_n = 2A_n \sin K_n x \cos \omega_n t
\]

Where \( K_n = \frac{n\pi}{L}, \omega_n = K_n v = \frac{n\pi}{L} \sqrt{T/\mu} \) and \( n = 1, 2, 3, \ldots \).

Equation (10) is the equation of a standing wave. Note that a particle at any particular point x executes simple harmonic motion as time passes and that all particles vibrate with the same frequency. Note also that the amplitude is not the same for different particles, but varies with the location x of the particle. The amplitude, \( 2A_n \sin K_n x \), has a maximum value of \( 2A_n \) at positions where \( Kx = \pi/2, 3\pi/2, 5\pi/2 \) etc. or where \( x = \lambda/4, 3\lambda/4, 5\lambda/4, \) etc. These points are called antinodes and are spaced one half wavelength apart. The amplitude has a minimum value of zero at positions where \( Kx = \pi, 2\pi, 3\pi, \) etc. or \( x = \lambda/2, 3\lambda/2, 2\lambda, \) etc. These points are called nodes and are also spaced one half wavelength apart.

Finally, it should be noted that although equation (10) is a form of wave which can exist in the bounded string system, it is not the most general form. The most general form is

\[
y_n(x, t) = \sum_{n=1}^{\alpha} (a_n \cos \omega_n t + B_n \sin \omega_n t)
\]

(11)
Now that we have determined that waves can exist in our system and how they can be represented mathematically, we might ask what we would expect to see if we tried to create the waves in an actual string. Consider then a string fixed at both ends which is being driven by a force $F \cos \omega t$. If the driving frequency is such that the distance $L$ between the ends is neither an integral or half-integral number of wavelengths, the initial and reflected waves will be "out of phase" and will destructively interfere with each other. No clear pattern will be set up. If however the string is driven with a frequency near $\omega_n$ so that $L$ is an integral or half-integral number of wavelengths, the initial and reflected waves will be "in phase" and will constructively interfere. The standing wave $y_n(x,t)$ will be produced and will attain a large amplitude. If $n = 1$ then $L = \lambda/2$ and the string is said to be vibrating at its fundamental frequency. This is the lowest frequency for which a standing wave pattern can be set up in the string. If the string is driven at a frequency which an integral multiple of the fundamental frequency, standing waves with different patterns will be set up. The patterns for the first four frequencies are shown in figure 2.

![Figure 2](image)

When the string is driven at one of its natural frequencies and its amplitude is near maximum it is said to be in resonance. A plot of $|y|^2$ vs. $\omega$ is shown in figure 3. Such a graph is called a
resonance curve. Near resonance the energy transfer (from driving mechanism to string is at its most efficient level.

Finally it should be noted that if the string is plucked rather than driven by a periodic force, then in general the response \( y(x,t) \) will not be a single natural frequency but a sum of many natural frequencies.

\[
y_n(x, t) = \sum_n (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin K_n x
\]

The observed pattern is very complicated in general. However, it is possible to pluck the string so as to have one natural frequency dominate.

![Figure 3](image)

References

The following are sections in Beiser's Concepts of Modern Physics which are pertinent to this lab. They should be read before coming to lab.

1. Chapter 3 sections 3.3. and 3.6
2. Chapter 5 sections 5.1-5.2 and 5.5 Also review Halliday and Resnick, Physics, Chapter 19.

Experimental Purpose

From the theory developed in the introduction of this lab writeup, the following predictions
can be made about a bounded string system (such as depicted in figure 1(a)) which is being driven by a periodic force $F \cos t$.

1. Stable waveforms will be seen only when the driving frequency $\omega$ is at or near one of the natural oscillating frequencies of the system $\omega_n$. The first five waveforms have the shape shown in figure 2.

2. The natural oscillating frequencies are given by the expression $\omega_n = n\pi(T/\mu)^{1/2}/L$.

3. A plot of $\omega_n^2$ vs. $T$ would yield a straight line with slope equal to $(1/\mu) (n\pi/L)^2$.

4. The speed of the wave is given by $v = (T/\mu)^{1/2}$. For a given tension the speed of the wave is independent of the oscillating frequency.

5. For a given tension, the wavelength of the waves is given by $\lambda_n = nL/2$. The nodes are found at $x = \lambda_n/2, \lambda_n, 3\lambda_n/2, 2\lambda_n$ etc. and the antinodes are found at $x = \lambda_n/4, 3\lambda_n/4, 5\lambda_n/4$, etc.

6. A plot of $|y|^2$ vs. $\omega$ for $\omega$ near a given natural oscillating frequency will have the shape shown in figure 3. The purpose of this lab is to test these predictions.

**Procedure**

The apparatus to be used in this lab is shown in figure 4. It consists of a steel wire with a linear mass density of $2.5 \times 10^{-3}$ g/cm stretched between two points with the string tension determined by the weight hung on one end. A sine oscillator generates and oscillating current $I_0\cos t$ which is amplified by the power amplifier and fed through the piano wire. Both the amplitude and frequency of the current are variable. The current in the wire has a force exerted on it due to its interaction with the magnetic field of the permanent magnet. [CAUTION: watches should not be worn near the magnets as the strength of the magnetic field can disrupt the proper functioning of a watch]. The force is proportional to $I_0 \cos \omega t$. The magnet can be moved along the wire to vary the position of the driving force. The motion of the wire can be detected by placing a dark background behind the wire and inspecting the wire visually.
1. Turn on all of the electrical equipment. Put the magnet at the middle of the wire. Set the range on the oscillator to X10. Turn the dial on the oscillator until the wire begins to vibrate in its fundamental mode. Using the fine tuning on the oscillator, adjust the frequency until the amplitude of vibration of the wire is at a maximum. Record the fundamental frequency. Note that the frequency counter gives \( \nu \) not \( \omega \). Readings on the frequency counter should be multiplied by \( 2\pi \) to get \( \omega \). Sketch the waveform. Record the position of the nodes. Record the length of the wire.

2. Find the next four harmonics by looking for a maximum amplitude response. For each harmonic:
   a. sketch the waveform
   b. record the frequency
   c. record the position of the nodes.

   The nodes can be easily found by running the tip of a sharp pencil along the wire until the vibrations felt in the pencil are eliminated or at a minimum. That is the location of the node. Note that for best results the magnet should be placed at an antinode.

3. Measure the fundamental frequency as a function of the tension in the wire. The tension is varied by changing the weights at the end of the wire. Do not put more than 4 kg & hanger of the end of the wire. Greater weight will break the wire. Take a minimum of five data points. Record your data.

4. For a given tension, measure the amplitude of the vertical displacement of the wire as a function of frequency for the fundamental frequency. Do this first by starting at frequencies below the resonance frequency and taking amplitude readings as you increase the driving
frequency. If you go past a measurement you wish to make, decrease the frequency to your starting point and increase the frequency back up to the frequency at which you wish to make the measurement. Take at least ten data points, half below and half above the resonance frequency. Repeat the measurements by starting at frequencies above the resonance frequency and taking amplitude readings as you decrease the driving frequency.

5. Pluck the wire with your finger. Try to excite single modes. Sketch a few waveforms. Describe your plucking method.

6. For a given tension find the second harmonic. Move the magnet along the wire. Does the position of the magnet affect the amplitude of the response. Record your observations.

Lab Report

Your lab notebook will be your lab report. Be sure to include the following in your notebook:

1. For the first five resonant frequencies:
   a. a comparison of the sketches of the observed waveforms with those predicted by theory.
   b. a comparison of the experimental resonant frequencies with the theoretically predicted values.
   c. a comparison of the experimental position of the nodes with the theoretically predicted value.
   d. a comparison of the experimental speeds of the waves (as computed from $v = v\lambda$) with the theoretically predicted values and a discussion of whether they are independent of as predicted by theory.

2. A plot of $\omega_1^2$ vs. $T$, with a comparison of the value of its slope with the value predicted by theory.

3. A plot of $|y|^2$ vs. $\omega$.

4. A discussion of the shapes of the waveforms sketched when plucking the wire in terms of equation (11).