**Theory**

Consider the series inductor-resistor-capacitor circuit shown in figure 1. When an alternating voltage is applied to this circuit, the current and voltage in every element (and in the overall circuit) will be of the same frequency; but only in the resistor will the current and voltage be in phase with one another. In this lab, we will look first at the relationship between the current and voltage in each element separately and then look at their combined effect on the overall functioning of the circuit.

![Figure 1](image)

**The AC Generator-Resistor Circuit**

This circuit consists of an alternating voltage generator connected in series with a resistor as shown in figure 2. From Kirchhoff’s rules we can write down these equations:

\[ V_g - V_R = 0 \]

or

\[ V_g = V_R \] (1)
Applying Ohm's law to the resistor and substituting for $V_g$ we get

$$V_o \cos (\omega t) = I(t)R$$

$$I(t) = \frac{V_o}{R} \cos (\omega t) \quad (2)$$

From this we see that the current has the same frequency as the voltage and is in phase with the voltage. This means that there is no lag between when the voltage is applied to the circuit and the circuit's response to the voltage. The current will reach its maximum at the same time as the voltage. A plot of the voltage vs time and current vs time is shown in figure 3.

This circuit consists of an alternating voltage generator connected in series with a capacitor as shown in figure 4. From Kirchhoff's rules we can write down these equations:

$$V_g - V_C = 0$$

or

$$V_g = V_C \quad (3)$$
Substituting in the appropriate voltage expressions yields

\[ V_o \cos(\omega t) = \frac{Q(t)}{C} \]

or

\[ Q(t) = CV_o \cos(\omega t) \]  \hspace{1cm} (4)

From equation (4) we see that the charge on the capacitor plates is in phase with the applied voltage. Since the voltage across the capacitor plates is proportional to the charge on the plates, the voltage across the capacitor is also in phase with the applied voltage. This means that the applied voltage, the charge on the plates and the voltage across the plates will all reach their respective maxima and minima simultaneously. Figure 5 shows a graph of \( Q(t) \) vs \( t \) and \( V(t) \) vs \( t \).

We are not, however, generally concerned with the charge on the capacitor as a function of time. A more useful function is the current in the circuit as a function of time. The current in the circuit is the rate of change of the charge on the capacitor. Graphically, the instantaneous current (the current at a particular time) is the derivative of \( Q(t) \), i.e. the slope of the \( Q(t) \) vs \( t \) graph at
the particular time. By examining figure 5, we can see that the slope (and therefore the current) is at a maximum when the charge (and voltage) is zero. Furthermore, we see that the current will reach its maximum in a particular direction (positive or negative current flow) before the charge (and voltage) reaches its maximum in that direction. We say that the current leads the charge by 90°.

We can see that our graphical analysis is confirmed mathematically by differentiating the charge with respect to time,

\[ I(t) = \frac{dQ(t)}{dt} = -\omega CV_o \sin(\omega t) \]  

(5)

Since the sine function may be considered a cosine which has been phase shifted by 90°, equation (5) may be rewritten as

\[ I(t) = \omega CV_o \cos\left(\omega t - \frac{\pi}{2}\right) \]

(6)

A plot of I(t) vs t and V(t) vs t is shown in figure 6.

![Figure 6](image)

Physically, we may think of the AC generator as placing charge on the capacitor plates during the first quarter cycle. The charging of the capacitor generates a countervoltage which will discharge the capacitor as the applied voltage becomes less positive during the second quarter cycle. The charging and discharging repeats itself (but with opposite polarity) through the same mechanisms in the third and fourth quarters of the cycle.

Although the charging of the capacitor does not remove energy from the system as a resistor does, it does by the development of a counter-voltage oppose the flow of current. This opposite
is called capacitive reactance, $X_C$. We can rewrite equation (6) as
\[
I(t) = \frac{V_o}{j\omega C} \cos \left(\omega t - \frac{\pi}{2}\right) \tag{7}
\]
Comparing equation (7) with equation (2) shows us that the term $1/\omega C$ plays the same role for the capacitor as the resistance $R$ plays for the resistor. $1/\omega C$ is the capacitive reactance. If the angular frequency is in radians/sec and the capacitance is in farads, then the units of the capacitive reactance are ohms.

**The AC Generator - Inductor Circuit**

This circuit consists of an alternating voltage generator connected in series with an inductor as shown in figure 7. From Kirchhoff’s rules, we can write down these equations
\[
V_g - V_L = 0
\]
or
\[
V_g = V_L \tag{8}
\]
Substituting the appropriate voltage expressions yield
\[
V_o \cos (\omega t) = L \frac{dI(t)}{dt} \tag{9}
\]
From equation (9), we can see that the voltage across the inductor is in phase with the rate of change of the current. In other words, when the voltage is at a maximum, the rate of change of the current will also be a maximum. For a sinusoidally varying function, the maximum rate of change occurs when the function passes through zero. Therefore, when the voltage is at a maximum, the current must be changing at its maximum rate and must have a value of zero. This is shown graphically in figure 8.
Figure 8

Figure 8 also shows us that the current will reach a maximum value after the voltage reaches its maximum. In other words, the current lags behind the voltage by 90°.

We can see that our graphical analysis is confirmed mathematically by integrating both sides of equation (9),

\[
\int L \frac{dI(t)}{dt} = \int V_0 \cos(\omega t) \, dt
\]

\[
L \cdot I(t) = \frac{V_0}{\omega} \sin(\omega t)
\]

\[
I(t) = \frac{V_0}{\omega L} \sin(\omega t)
\]  \hspace{1cm} (10)

Writing the sine as a phase shifted cosine, we get

\[
I(t) = \frac{V_0}{\omega L} \cos(\omega t - \frac{\pi}{2})
\]  \hspace{1cm} (11)

Equation (11) does indeed represent a function which has been shifted 90° to the right of the original function.

Physically, an inductor resists the change in the current in a circuit. It accomplishes this resistance through an induced voltage generated by the changing current in the coil. The induced voltage is always in a direction which will oppose the change in the current. This means that as the current in the circuit tries to increase due to the applied voltage, the induced voltage will oppose the applied voltage and partially cancel it out. The electrons will feel a smaller total voltage and the current increases at a slower rate than it would if the inductor were not in the
circircuit. As the current tries to decrease due to a decreasing applied voltage, thereby keeping the
current flowing when the applied voltage has gone to zero. In this way the inductor is the
electrical analog of the mass of an object (which opposes changes in velocity).

Although the inductor does not remove energy from the system as does the resistor, it does,
by the development of the induced voltages, oppose changes in the current. This opposition is
called inductive reactance, \( X_L \). From a comparison of equation (11) with equation (2), we can
see that the term \( L \) plays the same role for the inductor as the resistance \( R \) plays for the resistor.
If the angular frequency is in radians/sec and the inductance is in henries, then the units of \( X_L \)
are ohms.

Before going on, one point must be emphasized. In the preceding discussion, we considered
only one element at a time in series with the AC generator. Because of this, the applied volt-
age equalled the voltage across the individual element. We then derived certain phase relationships
between the current and the applied voltage. Most circuits, however, are combinations of these
elements, and in these instances the current and the applied voltage will be neither in phase nor
90° out of phase. Thus, if we continue to talk about the relationship between the current and the
applied voltages, then the phase relationships derived above will not hold. But if we talk about
the voltages across the individual elements, then the relationships derived above will hold. In
other words, in a combination circuit \( I(t) \) will be neither in phase nor 90° out of phase with \( V_g(t) \),
but \( V_c(t) \) will still be 90° out of phase with \( I(t) \), \( V_L(t) \) will still be -90° out of phase with \( I(t) \) and
\( V_R(t) \) will be in phase with \( I(t) \).

**The Series Inductor-Resistor-Capacitor Circuit**

Consider an inductor, a resistor, and a capacitor connected in series as shown in figure 9.
Kirchhoff's rules allow us to write these equations:

\[ V_g - V_C - V_R - V_L = 0 \]

or

\[ V_g = V_C + V_R + V_L \]

(12)
Substituting the appropriate voltage expressions into (12), we get

\[ L \frac{dI(t)}{dt} + R I(t) + \frac{Q(t)}{C} = V_g(t) \]

or since \( I = \frac{dQ(t)}{dt} \) we have

\[ L \frac{d^2Q(t)}{dt^2} + R \frac{dQ(t)}{dt} + \frac{1}{C} Q(t) = V_g(t) \] (13)

(Note that equation (13) is identical in form to the differential equation which governs the motion of a damped harmonic oscillator. Because of the similarity of the equations governing the systems, we would expect that under similar circumstances their behavior would be similar). In this lab we will examine the response of this circuit to two voltage patterns: a square wave voltage and a sine wave voltage.

The AC Square Wave Voltage. The square wave pattern may be thought of as consisting an "on" phase where the voltage is a constant value and an "off" phase where the voltage is zero. (Note that there are many square wave patterns. We have chosen this particular one for mathematical simplicity. The final results are the same in any case). Mathematically, this can be expressed as

\[ V = V_o \quad 0 \leq t < \frac{T}{2} \]

\[ V = 0 \quad \frac{T}{2} \leq t < T \]

During the "on" phase, provided the period of the square wave is sufficiently long, the capacitor is fully charged. During the "off" phase the system becomes a damped electric oscillator. This is similar to the situation shown in figure 10. During the "on" phase, the switch is in position A. The capacitor is being charged. During the "off" phase the switch is in position B and the system
oscillates. We will examine mathematically the system's behavior during the "off" phase ($V = 0$).

![Figure 10](image)

If we let $2\gamma = R/L$, $1/LC = \omega_0^2$ and $V = 0$, then by rearranging terms we can write equation (13) as

$$\frac{d^2 Q(t)}{dt^2} + 2\gamma \frac{dQ(t)}{dt} + \omega_0^2 Q(t) = 0 \quad (14)$$

Solving equation (14) for the charge as a function of time, we find

$$Q(t) = e^{-\gamma t} \left[ Ae^{\sqrt{\gamma^2 - \omega_0^2} t} + Be^{-\sqrt{\gamma^2 - \omega_0^2} t} \right] \quad (15)$$

From this expression we can get the current by differentiating once with respect to time. The coefficients can be determined by algebraic manipulations. By doing this and examining equation (15), we can see that there are three possible cases:

1. $\gamma^2 < \omega_0^2$ or $R < 2(L/C)^{1/2}$. This is the underdamped case and $I(t) = V_0 C \omega_0^2 e^{-\gamma_2 t} \frac{\sin \left( \frac{\omega t}{\omega_0} \right)}{\omega}$

   where $\omega = (\omega_0^2 - \gamma^2)^{1/2}$. The current will oscillate with an amplitude that decreases exponentially with time.

2. $\gamma^2 = \omega_0^2$ or $R = 2 (L/C)^{1/2}$. This is the critically damped case and $I(t) = V_0 C \omega_0^2 t e^{-\gamma_2 t}$

   The current does not oscillate, but returns to zero in the fastest manner.

3. $\gamma^2 > \omega_0^2$ or $R > 2 (L/C)^{1/2}$. This is the overdamped case and $I(t) = V_0 C \omega_0^2 e^{-\gamma_2 t} \frac{\sinh \left( \frac{\omega t}{\omega_0} \right)}{\omega}$

   where $\omega = (\gamma^2 - \omega_0^2)^{1/2}$. Again the current does not oscillate, but returns to zero at a rate slower than with critical damping.
Figure 11 shows the graphs of the three cases.

The AC Sine Wave Voltage. For the sine wave voltage, equation (13) becomes

\[ L \frac{d^2 Q(t)}{dt^2} + R \frac{dQ(t)}{dt} + \frac{Q(t)}{C} = V_o \cos(\omega t) \]  \hspace{1cm} (16)

or if we rearrange some terms and let \( 2\gamma = \frac{R}{L} \) and \( \omega_o^2 = \frac{1}{LC} \),

\[ \frac{d^2 Q(t)}{dt^2} + 2\gamma \frac{dQ(t)}{dt} + \omega_o^2 Q(t) = \frac{V_o}{L} \cos(\omega t) \]  \hspace{1cm} (17)

Solving equation (17) for the charge as a function of time yields

\[ Q(t) = D \cos(\omega t - \delta) \]  \hspace{1cm} (18)

where \( D = \frac{V_o}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 - 4\gamma^2 \omega^2}} \) and \( \delta = \tan^{-1}\left(\frac{-2\gamma \omega}{\omega_o^2 - \omega^2}\right) \). Differentiating we get the more useful expression of the current as a function of time

\[ I(t) = \frac{\omega V_o}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 - 4\gamma^2 \omega^2}} \cos(\omega t - \phi) \]  \hspace{1cm} (19)
where $\phi = \delta + \pi/2$.

Examination of equation (19) shows us that the behavior of the system will depend on the frequency at which it is driven. It is impractical to describe the system at all frequencies, but we can look at three frequency ranges to get a general idea of what is happening:

1. $\omega \ll \omega_0$. In this case, the applied voltage increases and decreases slowly. The capacitor has sufficient time to build a charge and generate a counter-voltage. As a result, the difference between the countervoltage and applied voltage is small. The total growth of the current is small. In addition, the resistor is also resisting the flow of current. The current is varying so slowly that the inductor's opposition to the changes in current is small. Overall, we would expect a small current.

2. $\omega \gg \omega_0$. Here the applied voltage increases and decreases very rapidly. There is not much time during any cycle for charge to build up on the capacitor. Little countervoltage can be built up before the polarity changes and the capacitor discharges. The capacitor therefore does not oppose greatly the flow of the current. The inductor, on the other hand, will strongly resist these rapid changes and this will keep the current small. In addition, the resistor is resisting the flow of current. Overall, we would expect a small current.

3. $\omega = \omega_0$. In this case, the capacitor will have time to change to an appreciable extent thus building up a countervoltage and opposing the current. When the counter voltage is becoming larger than the applied voltage, the capacitor is trying to drive the current back in the other direction. The inductor sees the current decreasing. The decreasing current generates an induced voltage which opposes this change, and for $\omega = \omega_0$, the two effects cancel. The only thing left is the opposition to flow caused by the resistor. We would expect a current larger than in either of the other cases. When this situation occurs, we say the system is in resonance.

Putting the above three cases together, we would expect a curve such as that in figure 12. This is called a resonance curve.
At resonance, the energy transfer (from the ac generator into the circuit) process is at its most efficient level. The ratio of the natural oscillating frequency to the full width of resonance curve at half its maximum value ($\frac{\omega_0}{\Delta\omega}$) is called the quality value (or Q value). It is a measure of the rate at which energy is lost from the system. Systems with large Q values are absorbing large amounts of energy at resonance while the damping forces are removing little energy. It can be shown from equation (19) that the Q value is also given by

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}}$$

Equation (19) also shows us that the phase relationship between the current and the driving frequency varies with the frequency. When $\omega_0^2 >> \omega^2$, $\delta = 0^\circ$ and the phase difference between the current and the driving voltage is $90^\circ$ with the current peaking before the voltage. When $\omega_0^2 << \omega^2$, $\delta = 180^\circ$ and the phase difference between the current and the driving voltage is $270^\circ$. This means the current lags the voltage by $90^\circ$. When $\omega_0^2 = \omega$, $\delta = 90^\circ$ and the phase difference is zero. The current and voltage peak simultaneously.

References

The following are the sections in Halliday and Resnick's Fundamentals of Physics that are pertinent to this lab. They should be read before coming to lab.

1. Chapter 29 section 29-7
Experimental Purpose

The purpose of this lab is to confirm the following theoretical predictions with regard to series ac circuits:

1. the current in a circuit and the voltage across the resistor are "in phase";
2. the current in a circuit leads the voltage across the capacitor by 90°;
3. the current in a circuit lags the voltage across the inductor by 90°;
4. the ratio of $V_C/(1/\omega C)$ gives the current in the circuit;
5. the ratio of $V_L/\omega L$ also gives the current in the circuit;
6. for an LRC circuit with a square wave driving voltage:
   a. when $R < 2 (L/C)^{1/2}$ the current oscillates with an angular frequency given by $\omega_o = (1/LC)^{1/2}$
   b. when $R = 2 (L/C)^{1/2}$, the current returns to zero by the fastest possible manner,
   c. when $R > 2 (L/C)^{1/2}$, the current returns to zero slower than in the underdamped case
7. for an LRC circuit with a sine wave driving voltage:
   a. when $\omega_o^2 \ll \omega$, the current leads the driving voltage by 90°.
   b. when $\omega_o^2 = \omega$, the current is in phase with the driving voltage,
   c. when $\omega_o^2 \gg \omega$, the current lags the driving voltage by 90°,
   d. the amplitude of the current varies with frequency as shown in figure 12

Procedure
1. Assemble the circuit shown in Figure 13. Note: there is one three position toggle switch on either side of the input jacks to channel A and B of your scope. The left hand switch should always be in the AC position. If the right hand switch on channel A is in the AC position, the display on that channel will show the voltage across the capacitor (\(V_C\)). If it is in the OFF position, the display will show the voltage across the entire circuit (\(V_g\)). Channel B will always show the voltage across the resistor (\(V_R\)).

![Figure 13](image.png)

The output of the generator is passed through a 10Ω resistor. The purpose of this arrangement is to isolate the signal generator from the test circuit in that only a tiny fraction of the test current actually flows through the signal generator. The source acts like an ideal EMF with a 10W internal resistance.

2. Select a \(C\) of .047 mfarads, and \(R\) of 1000Ω and a frequency of 1000 Hz.

   a. Measure the peak amplitude of \(V_C\) and \(V_R\). Compute I from the ratios \(V_R/R\) and \(V_C(1/\omega C)\). Compare your results. Does the ratio \(V_C/(1/\omega C)\) give you the current as predicted by theory?
b. Observe the relative phase of $V_C$ and $V_R$ (Remember since $V_R$ and I are always in phase with regard to phase, looking at $V_R$ is the same as looking at I). Does $V_R$ peak $90^\circ$ before $V_C$?

c. Observe the relative phase of $V_g$ and I. Measure the phase difference using Lissajous figures. Is it as expected?

d. Repeat steps 1a - 1c for $C = .0047$, keeping other quantities the same.

e. Repeat steps 1a - 1c for $C = .01$, $f = 500$ Hz.

3. Use the multimeter to measure the dc resistance of the 10 mH inductor. Then replace the capacitor in the circuit shown on Figure 13 with the inductor.

a. Select a frequency of $10^4$ Hz and an $R$ of $1000\Omega$. Measure $V_R$ and $V_L$. Compute I using the ratio $V_R/R$ and $V_L/\omega L$. Compare the two values of I. Does the ratio $V_L/\omega L$ give the current as predicted by theory?

b. Observe the relative phase of $V_L$ and I. Does I lag $V_L$ by $90^\circ$?

c. Observe the relative phase of $V_g$ and I. Measure the phase difference using Lissajous figures. Is it as expected?

d. Repeat steps 3a - 3c for $f = 10^5$ Hz, keeping the other quantities the same.

4. Assemble the circuit shown in Figure 14. Choose a value of C (there is only one L) such that the natural frequency of oscillations, $f_0$, is about $10^4$ Hz. Use an $R$ of approximately $100\Omega$. 
As before if the right switch on channel A is in the AC position, the display for channel A shows $V_C$ and channel B shows $V_R$.

a. Set the frequency of the applied square wave voltage to 500 Hz. Display $V_R$ and observe the damped oscillations of the current. Sketch what you see.

b. Display $V_R$ and measure the period of the oscillations and compare them with the theoretical value $T = 2\pi(LC)^{1/2}$.

c. Decrease the value of $C$ by a factor of 10 and observe the change in the oscillations. Measure the new period and compare with the theoretical period.

d. Reset $C$ to the value used in step 4a. Increase the value of $R$ until the overshoot in the oscillation is just removed (critical damping). Sketch what you see. Is $R = 2(L/C)^{1/2}$ as predicted by theory?

e. Increase $R$ until the overdamped case is met. Sketch what you see.
5. Use the same circuit as in step 4 except switch the input to the sine function of the oscillator. Use the same value of C as in step 4b and set R to 100W.

a. Adjust the scope so that you can simultaneously observe \( V_g \) and \( V_R \). Tune the frequency from 5000 Hz to 50,000 Hz. Describe what happens to \( V_g \) and \( V_R \) (and therefore I) as you go through the range of frequencies. Does the circuit's behavior agree with theory?

b. Measure the frequency at which resonance occurs and compare it with the value predicted by theory and with the frequency of oscillations measured in step 4b.

c. Measure the current and \( V_g \) at resonance, at 5000 Hz and at 50,000 Hz. Equation (19) can be rewritten as

\[
I(t) = \frac{V_o}{L/\omega} \sqrt{\left(\omega_o^2 - \omega^2\right)^2 - 4\gamma^2\omega^2} \cos \left(\omega t - \phi\right)
\]

Show that the factor \( \frac{L}{\omega} \sqrt{\left(\omega_o^2 - \omega^2\right)^2 - 4\gamma^2\omega^2} \) can be rewritten as

\[
\sqrt{R^2 - \left(\omega L - 1/\omega C\right)^2}
\]

This expression is called the impedance (Z) of the circuit and it plays the same role for the circuit in Ohm's law as R plays for a resistor. Compute the ratio \( V_g/Z \) for the three frequencies at which you measured I and \( V_g \). Compare the measured value of I of to the corresponding ratio of \( V_g/Z \) for each frequency.

d. Measure \( V_C \), \( V_L \) and \( V_R \). Add the three values compare to \( V_g \). Explain.

e. Measure the phase difference between \( V_g \) and I at resonance, 5000 Hz and 50,000Hz. Compare this with the theoretical predicted phase angle.

f. Starting at 5000 Hz tune the frequency until the current grows to one half its resonance value. Record the frequency at that point. Tune through resonance and when the current drops to one half its resonance value, record the frequency. Compute the difference in frequencies. This will allow you to compute a rough value for Q. Compare to the theoretical value given by equation (20).
Lab Report

Your lab notebook will be your lab report. Be sure to include the following in your notebook:

1. a discussion of whether or not the experimental results confirm the theoretical predictions given in the experimental purpose section of this writeup.

2. all calculations done.

3. answer to all the questions asked in the procedure.

4. all sketches asked for in the procedure.

5. calculations of the percent deviation between experimental results and theoretical predictions for all measured quantities.