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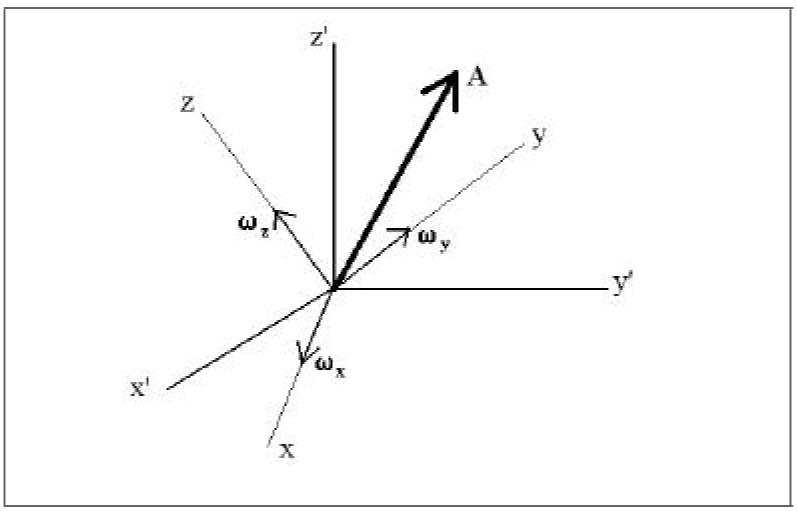
Motion in a Non-Inertial Frame

1 Time Derivatives in Fixed and Rotating Frames

Let us consider the time derivative of an arbitrary vector $\mathbf{A}$ in two reference frames. The first reference frame is called the *fixed* frame and is expressed in terms of the Cartesian coordinates $\mathbf{r}' = (x', y', z')$. The second reference frame is called the *rotating* frame and is expressed in terms of the Cartesian coordinates $\mathbf{r} = (x, y, z)$. In the Figure below, the rotating frame shares the same origin as the fixed frame and the rotation angular velocity $\boldsymbol{\omega}$ of the rotating frame (with respect to the fixed frame) has components $(\omega_x, \omega_y, \omega_z)$.



Since observations are often made in a rotating frame of reference, we decompose the vector $\mathbf{A}$ in terms of components A_i in the rotating frame (with unit vectors $\widehat{\mathsf{x}}^i$). Thus, $\mathbf{A} = A_i\,\widehat{\mathsf{x}}^i$ (using the summation rule) and the time derivative of $\mathbf{A}$ as observed in the fixed frame is

$$\frac{d\mathbf{A}}{dt} = \frac{dA_i}{dt}\,\widehat{\mathsf{x}}^i + A_i\,\frac{d\widehat{\mathsf{x}}^i}{dt}. \tag{1}$$

The interpretation of the first term is that of the time derivative of $\mathbf{A}$ as observed in the rotating frame (where the unit vectors $\widehat{\mathsf{x}}^i$ are constant) while the second term involves the time-dependence of the relation between the fixed and rotating frames. We now express $d\widehat{\mathsf{x}}^i/dt$ as a vector in the rotating frame as

$$\frac{d\widehat{\mathsf{x}}^i}{dt} = R^{ij}\,\widehat{\mathsf{x}}_j = \epsilon^{ijk}\,\omega_k\,\widehat{\mathsf{x}}_j, \tag{2}$$

where R represents the rotation matrix associated with the rotating frame of reference; this rotation matrix is anti-symmetric ($R^{ij} = -R^{ji}$) and can be written in terms of the anti-symmetric tensor ϵ^{ijk} (defined in terms of the vector product $\mathbf{A} \times \mathbf{B} = A_i B_j \, \epsilon^{ijk} \, \widehat{\mathsf{x}}_k$ for two arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$) as $R^{ij} = \epsilon^{ijk} \omega_k$, where ω_k denotes the components of the angular velocity $\boldsymbol{\omega}$ in the rotating frame. Hence, the second term in Eq. (1) becomes

$$A_i \, \frac{d\widehat{\mathsf{x}}^i}{dt} \;=\; A_i \, \epsilon^{ijk} \, \omega_k \, \widehat{\mathsf{x}}_j \;=\; \boldsymbol{\omega} \times \mathbf{A}. \tag{3}$$

The time derivative of an arbitrary rotating-frame vector $\mathbf{A}$ in a fixed frame is, therefore, expressed as

$$\left(\frac{d\mathbf{A}}{dt} \right)_f \;=\; \left(\frac{d\mathbf{A}}{dt} \right)_r \;+\; \boldsymbol{\omega} \times \mathbf{A}, \tag{4}$$

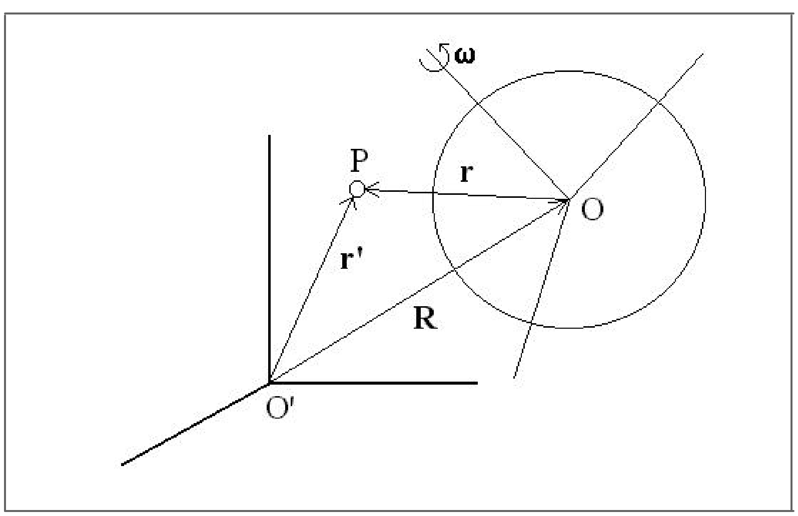
where $(d/dt)_f$ denotes the time derivative as observed in the fixed (f) frame while $(d/dt)_r$ denotes the time derivative as observed in the rotating (r) frame. An important application of this formula relates to the time derivative of the rotation angular velocity $\boldsymbol{\omega}$ itself. One can easily see that

$$\left(\frac{d\boldsymbol{\omega}}{dt} \right)_f \;=\; \dot{\boldsymbol{\omega}} \;=\; \left(\frac{d\boldsymbol{\omega}}{dt} \right)_r,$$

since the second term in Eq. (4) vanishes for $\mathbf{A} = \boldsymbol{\omega}$; the time derivative of $\boldsymbol{\omega}$ is, therefore, the same in both frames of reference and is denoted $\dot{\boldsymbol{\omega}}$ in what follows.

2 Accelerations in Rotating Frames

We now consider the general case of a rotating frame and fixed frame being related by translation and rotation.



In the Figure above, the position of a point P according to the fixed frame of reference is labeled $\mathbf{r}'$, while the position of the same point according to the rotating frame of reference is labeled $\mathbf{r}$, and

$$\mathbf{r}' \;=\; \mathbf{R} \;+\; \mathbf{r}, \tag{5}$$

where $\mathbf{R}$ denotes the position of the origin of the rotating frame according to the fixed frame. Since the velocity of the point P involves the rate of change of position, we must now be careful in defining which time-derivative operator, $(d/dt)_f$ or $(d/dt)_r$, is used.

The velocities of point P as observed in the fixed and rotating frames are defined as

$$\mathbf{v}_f \;=\; \left(\frac{d\mathbf{r}'}{dt}\right)_f \quad \text{and} \quad \mathbf{v}_r \;=\; \left(\frac{d\mathbf{r}}{dt}\right)_r, \tag{6}$$

respectively. Using Eq. (4), the relation between the fixed-frame and rotating-frame velocities is expressed as

$$\mathbf{v}_f \;=\; \left(\frac{d\mathbf{R}}{dt}\right)_f \;+\; \left(\frac{d\mathbf{r}}{dt}\right)_f \;=\; \mathbf{V} \;+\; \mathbf{v}_r \;+\; \boldsymbol{\omega}\times\mathbf{r}, \tag{7}$$

where $\mathbf{V} \;=\; (d\mathbf{R}/dt)_f$ denotes the translation velocity of the rotating-frame origin (as observed in the fixed frame).

Using Eq. (7), we are now in a position to evaluate expressions for the acceleration of point P as observed in the fixed and rotating frames of reference

$$\mathbf{a}_f \;=\; \left(\frac{d\mathbf{v}_f}{dt}\right)_f \quad \text{and} \quad \mathbf{a}_r \;=\; \left(\frac{d\mathbf{v}_r}{dt}\right)_r, \tag{8}$$

respectively. Hence, using Eq. (7), we find

$$\begin{aligned}
\mathbf{a}_f \;&=\; \left(\frac{d\mathbf{V}}{dt}\right)_f \;+\; \left(\frac{d\mathbf{v}_r}{dt}\right)_f \;+\; \left(\frac{d\boldsymbol{\omega}}{dt}\right)_f \times\mathbf{r} \;+\; \boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_f \\
&=\; \mathbf{A} \;+\; (\mathbf{a}_r \;+\; \boldsymbol{\omega}\times\mathbf{v}_r) \;+\; \dot{\boldsymbol{\omega}}\times\mathbf{r} \;+\; \boldsymbol{\omega}\times\,(\mathbf{v}_r \;+\; \boldsymbol{\omega}\times\mathbf{r}),
\end{aligned}$$

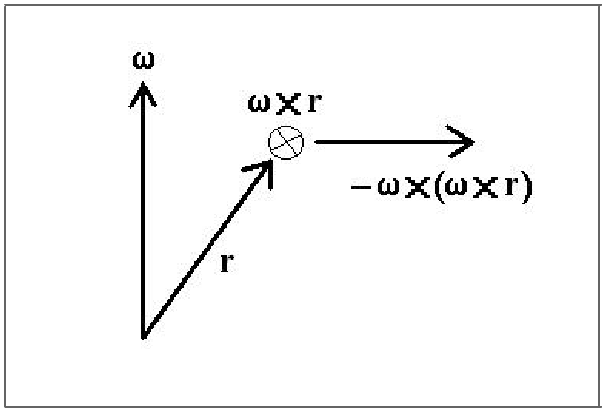
or

$$\mathbf{a}_f \;=\; \mathbf{A} \;+\; \mathbf{a}_r \;+\; 2\,\boldsymbol{\omega}\times\mathbf{v}_r \;+\; \dot{\boldsymbol{\omega}}\times\mathbf{r} \;+\; \boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{r}), \tag{9}$$

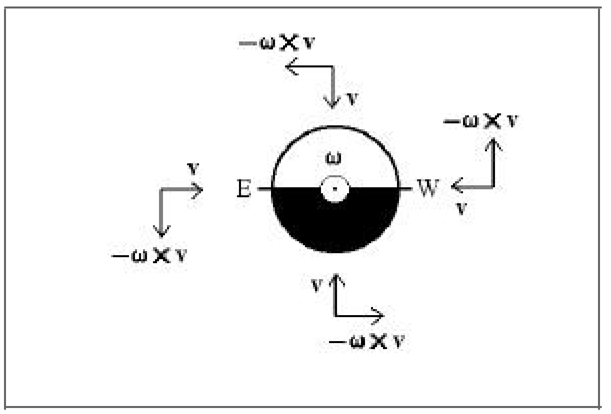
where $\mathbf{A} \;=\; (d\mathbf{V}/dt)_f$ denotes the translational acceleration of the rotating-frame origin (as observed in the fixed frame of reference). We can now write an expression for the acceleration of point P as observed in the rotating frame as

$$\mathbf{a}_r \;=\; \mathbf{a}_f \;-\; \mathbf{A} \;-\; \boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{r}) \;-\; 2\,\boldsymbol{\omega}\times\mathbf{v}_r \;-\; \dot{\boldsymbol{\omega}}\times\mathbf{r}, \tag{10}$$

which represents the sum of the net *inertial* acceleration $(\mathbf{a}_f - \mathbf{A})$, the centrifugal acceleration $-\,\boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{r})$ (see Figure below)



the *Coriolis* acceleration $-2\boldsymbol{\omega}\times\mathbf{v}_r$ (see Figure below)



and an angular acceleration term $-\dot{\boldsymbol{\omega}}\times\mathbf{r}$ which depends explicitly on the time dependence of the rotation angular velocity $\boldsymbol{\omega}$.

The centrifugal acceleration (which is directed outwardly from the rotation axis) represents a familiar *non-inertial* effect in physics. A less familiar *non-inertial* effect is the Coriolis acceleration. The Figure above shows that an object *falling* inwardly also experiences an *eastward* acceleration.

3 Lagrangian Formulation of Non-Inertial Motion

The Lagrangian for a particle of mass m moving in a non-inertial rotating frame (with its origin coinciding with the fixed-frame origin) in the presence of the potential $U(\mathbf{r})$ is expressed as

$$L(\mathbf{r},\dot{\mathbf{r}}) \;=\; \frac{m}{2}\,|\dot{\mathbf{r}} \;+\; \boldsymbol{\omega}\times\mathbf{r}|^2 \;-\; U(\mathbf{r}), \tag{11}$$

where $\boldsymbol{\omega}$ is the angular velocity vector and we use the formula

$$|\dot{\mathbf{r}} \;+\; \boldsymbol{\omega}\times\mathbf{r}|^2 \;=\; |\dot{\mathbf{r}}|^2 \;+\; 2\,\boldsymbol{\omega}\cdot(\mathbf{r}\times\dot{\mathbf{r}}) \;+\; \left[\,\omega^2\,r^2 \;-\; (\boldsymbol{\omega}\cdot\mathbf{r})^2\,\right].$$

Using the Lagrangian (11), we now derive the general Euler-Lagrange equation for $\mathbf{r}$. First, we derive an expression for the canonical momentum

$$\mathbf{p} \;=\; \frac{\partial L}{\partial \dot{\mathbf{r}}} \;=\; m\,(\dot{\mathbf{r}} \,+\, \boldsymbol{\omega}\times\mathbf{r})\,, \tag{12}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) \;=\; m\,(\ddot{\mathbf{r}} \,+\, \dot{\boldsymbol{\omega}}\times\mathbf{r} \,+\, \boldsymbol{\omega}\times\dot{\mathbf{r}})\,.$$

Next, we derive the partial derivative

$$\frac{\partial L}{\partial \mathbf{r}} \;=\; -\,\nabla U(\mathbf{r}) \;-\; m\,[\,\boldsymbol{\omega}\times\dot{\mathbf{r}} \,+\, \boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{r})\,]\,,$$

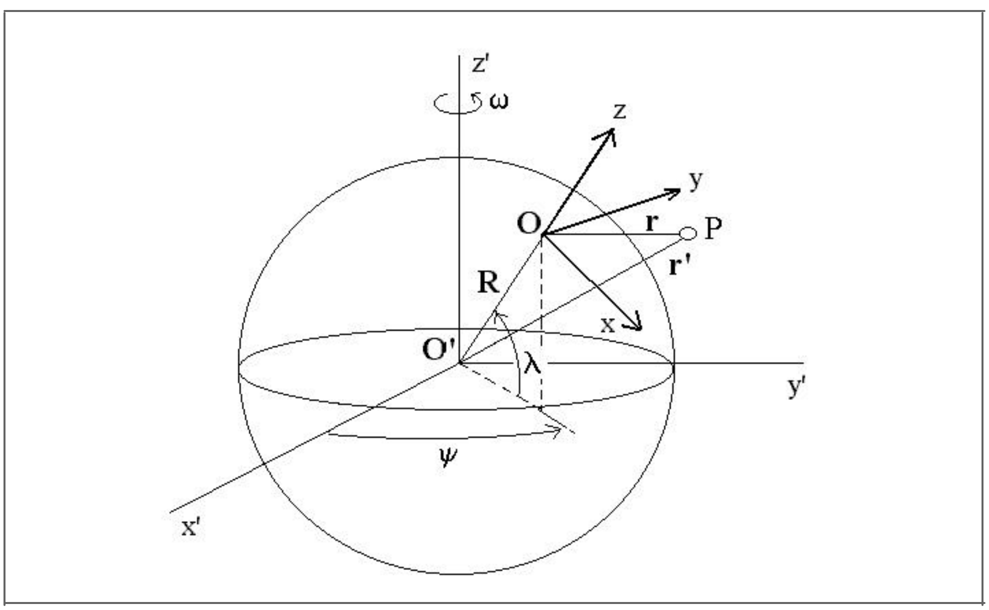
so that the Euler-Lagrange equations are

$$m\,\ddot{\mathbf{r}} \;=\; -\,\nabla U(\mathbf{r}) \;-\; m\,[\,\dot{\boldsymbol{\omega}}\times\mathbf{r} \,+\, 2\,\boldsymbol{\omega}\times\dot{\mathbf{r}} \,+\, \boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{r})\,]\,. \tag{13}$$

Here, the potential energy term generates the fixed-frame acceleration, $-\nabla U = m\,\mathbf{a}_f$, and thus the Euler-Lagrange equation (13) yields Eq. (10).

4 Motion Relative to Earth

We can now apply these expressions to the important case of the fixed frame of reference having its origin at the center of Earth (point O' in the Figure below) and the rotating frame of reference having its origin at latitude λ and longitude ψ (point O in the Figure below). We note that the rotation of the Earth is now represented as $\dot{\psi} = \omega$ and that $\dot{\boldsymbol{\omega}} = 0$.



We arrange the (x, y, z) axis of the rotating frame so that the z-axis is a continuation of the position vector $\mathbf{R}$ of the rotating-frame origin, i.e., $\mathbf{R} = R\,\widehat{\mathsf{z}}$ in the rotating frame (where $R = 6378$ km is the Earth's radius assuming a spherical Earth). When expressed in terms of the fixed-frame latitude angle λ and the azimuthal angle ψ, the unit vector $\widehat{\mathsf{z}}$ is

$$\widehat{\mathsf{z}} \;=\; \cos\lambda\;(\cos\psi\,\widehat{\mathsf{x}}' \;+\; \sin\psi\,\widehat{\mathsf{y}}') \;+\; \sin\lambda\,\widehat{\mathsf{z}}'.$$

Likewise, we choose the x-axis to be tangent to a *great circle* passing through the North and South poles, so that

$$\widehat{\mathsf{x}} \;=\; \sin\lambda\;(\cos\psi\,\widehat{\mathsf{x}}' \;+\; \sin\psi\,\widehat{\mathsf{y}}') \;-\; \cos\lambda\,\widehat{\mathsf{z}}'.$$

Lastly, the y-axis is chosen such that

$$\widehat{\mathsf{y}} \;=\; \widehat{\mathsf{z}}\times\widehat{\mathsf{x}} \;=\; -\,\sin\psi\,\widehat{\mathsf{x}}' \;+\; \cos\psi\;\widehat{\mathsf{y}}'.$$

We now consider the acceleration of a point P as observed in the rotating frame O by writing Eq. (10) as

$$\frac{d^2\mathbf{r}}{dt^2} \;=\; \mathbf{g}_0 \;-\; \ddot{\mathbf{R}}_f \;-\; \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \;-\; 2\,\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}. \tag{14}$$

The first term represents the *pure* gravitational acceleration due to the graviational pull of the Earth on point P (as observed in the fixed frame located at Earth's center)

$$\mathbf{g}_0 \;=\; -\,\frac{G\,M}{|\mathbf{r}'|^3}\,\mathbf{r}',$$

where $\mathbf{r}' = \mathbf{R} + \mathbf{r}$ is the position of point P in the fixed frame and $\mathbf{r}$ is the location of P in the rotating frame. When expressed in terms of rotating-frame spherical coordinates (r, θ, φ):

$$\mathbf{r} \;=\; r\,[\,\sin\theta\;(\cos\varphi\,\widehat{\mathsf{x}} \;+\; \sin\varphi\,\widehat{\mathsf{y}}) \;+\; \cos\theta\,\widehat{\mathsf{z}}\,],$$

the vector $\mathbf{r}'$ is written as

$$\mathbf{r}' \;=\; (R + r\,\cos\theta)\,\widehat{\mathsf{z}} \;+\; r\,\sin\theta\;(\cos\varphi\,\widehat{\mathsf{x}} + \sin\varphi\,\widehat{\mathsf{y}}),$$

and thus

$$|\mathbf{r}'|^3 \;=\; \left(R^2 \;+\; 2\,Rr\,\cos\theta \;+\; r^2\right)^{3/2}.$$

The pure gravitational acceleration is, therefore, expressed in the rotating frame of the Earth as

$$\mathbf{g}_0 \;=\; -\,g_0\left[\frac{(1+\epsilon\,\cos\theta)\,\widehat{\mathsf{z}} \;+\; \epsilon\,\sin\theta\,(\cos\varphi\,\widehat{\mathsf{x}} + \sin\varphi\,\widehat{\mathsf{y}})}{(1 + 2\,\epsilon\,\cos\theta \;+\; \epsilon^2)^{3/2}}\right], \tag{15}$$

where $g_0 = GM/R^2 = 9.789$ m/s^2 and $\epsilon = r/R \ll 1$.

The angular velocity in the fixed frame is $\boldsymbol{\omega} = \omega\,\widehat{\mathsf{z}}'$, where

$$\omega \;=\; \frac{2\pi \text{ rad}}{24 \times 3600 \text{ sec}} \;=\; 7.27 \times 10^{-5} \text{ rad/s}$$

is the rotation speed of Earth about its axis. In the rotating frame, we find

$$\boldsymbol{\omega} \;=\; \omega\;(\sin\lambda\,\widehat{\mathsf{z}} \;-\; \cos\lambda\,\widehat{\mathsf{x}}). \tag{16}$$

Because the position vector $\mathbf{R}$ rotates with the origin of the rotating frame, its time derivatives yield

$$\begin{aligned}
\dot{\mathbf{R}}_f &= \boldsymbol{\omega}\times\mathbf{R} \;=\; (\omega\,R\,\cos\lambda)\;\widehat{\mathsf{y}}, \\
\ddot{\mathbf{R}}_f &= \boldsymbol{\omega}\times\dot{\mathbf{R}}_f \;=\; \boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{R}) \;=\; -\,\omega^2\,R\cos\lambda\;(\cos\lambda\,\widehat{\mathsf{z}} \,+\, \sin\lambda\,\widehat{\mathsf{x}}),
\end{aligned}$$

and thus the centrifugal acceleration due to $\mathbf{R}$ is

$$\ddot{\mathbf{R}}_f \;=\; -\,\boldsymbol{\omega}\times\,(\boldsymbol{\omega}\times\mathbf{R}) \;=\; \alpha\,g_0\,\cos\lambda\,(\cos\lambda\,\widehat{\mathsf{z}} \,+\, \sin\lambda\,\widehat{\mathsf{x}}), \tag{17}$$

where $\omega^2\,R = 0.0337$ m/s^2 can be expressed in terms of the *pure* gravitational acceleration g_0 as $\omega^2\,R \;=\; \alpha\,g_0$, where $\alpha = 3.4 \times 10^{-3}$. We now define the physical gravitational acceleration as

$$\begin{aligned}
\mathbf{g} &= \mathbf{g}_0 \,-\, \boldsymbol{\omega}\times\left[\,\boldsymbol{\omega}\times(\mathbf{R}+\mathbf{r})\,\right] \\
&= g_0\left[\,-\,\left(1 \,-\, \alpha\,\cos^2\lambda\right)\widehat{\mathsf{z}} \,+\, (\alpha\,\cos\lambda\,\sin\lambda)\;\widehat{\mathsf{x}}\,\right],
\end{aligned} \tag{18}$$

where terms of order ϵ have been neglected. A plumb line experiences a small angular deviation $\delta(\lambda)$ from the true vertical given as

$$\tan\delta(\lambda) \;=\; \frac{g_x}{|g_z|} \;=\; \frac{\alpha\,\sin 2\lambda}{(2-\alpha)+\alpha\,\cos 2\lambda}.$$

This function exhibits a maximum at a latitude $\overline{\lambda}$ defined as $\cos 2\overline{\lambda} = -\,\alpha/(2-\alpha)$, so that

$$\tan\overline{\delta} \;=\; \frac{\alpha\,\sin 2\overline{\lambda}}{(2-\alpha)+\alpha\,\cos 2\overline{\lambda}} \;=\; \frac{\alpha}{2\sqrt{1-\alpha}} \;\simeq\; 1.7\times 10^{-3},$$

or

$$\overline{\delta} \;\simeq\; 5.86 \text{ arcmin} \quad \text{at} \quad \overline{\lambda} \;\simeq\; \left(\frac{\pi}{4} \,+\, \frac{\alpha}{4}\right) \text{ rad} \;=\; 45.05^o.$$

We now return to Eq. (14), which is written to lowest order in ϵ and α as

$$\frac{d^2\mathbf{r}}{dt^2} \;=\; -\,g\,\widehat{\mathsf{z}} \;-\; 2\,\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt}, \tag{19}$$

where

$$\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} \;=\; \omega\left[\,(\dot{x}\,\sin\lambda \,+\, \dot{z}\,\cos\lambda)\,\widehat{\mathsf{y}} \,-\, \dot{y}\;(\sin\lambda\,\widehat{\mathsf{x}} \,+\, \cos\lambda\,\widehat{\mathsf{z}})\,\right].$$

Thus, we find the three components of Eq. (19) written explicitly as

$$\left.\begin{array}{rcl} \ddot{x} & = & 2\,\omega\,\sin\lambda\,\dot{y} \\ \ddot{y} & = & -\,2\,\omega\,(\sin\lambda\,\dot{x}\;+\;\cos\lambda\,\dot{z}) \\ \ddot{z} & = & -\,g\;+\;2\,\omega\,\cos\lambda\,\dot{y} \end{array}\right\}. \tag{20}$$

A first integration of Eq. (20) yields

$$\left.\begin{array}{rcl} \dot{x} & = & 2\,\omega\,\sin\lambda\,y\;+\;C_x \\ \dot{y} & = & -\,2\,\omega\,(\sin\lambda\,x\;+\;\cos\lambda\,z)\;+\;C_y \\ \dot{z} & = & -\,g\,t\;+\;2\,\omega\,\cos\lambda\,y\;+\;C_z \end{array}\right\}, \tag{21}$$

where (C_x, C_y, C_z) are constants defined from initial conditions (x_0, y_0, z_0) and $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$:

$$\left.\begin{array}{rcl} C_x & = & \dot{x}_0\;-\;2\,\omega\,\sin\lambda\,y_0 \\ C_y & = & \dot{y}_0\;+\;2\,\omega\,(\sin\lambda\,x_0\;+\;\cos\lambda\,z_0) \\ C_z & = & \dot{z}_0\;-\;2\,\omega\,\cos\lambda\,y_0 \end{array}\right\}. \tag{22}$$

A second integration of Eq. (21) yields

$$\begin{array}{rcl} x(t) & = & x_0\;+\;C_x\,t\;+\;2\,\omega\,\sin\lambda\displaystyle\int_0^t y\,dt, \\ y(t) & = & y_0\;+\;C_y\,t\;-\;2\,\omega\,\sin\lambda\displaystyle\int_0^t x\,dt\;-\;2\,\omega\,\cos\lambda\displaystyle\int_0^t z\,dt, \\ z(t) & = & z_0\;+\;C_z\,t\;-\;\dfrac{1}{2}\,g\,t^2\;+\;2\,\omega\,\cos\lambda\displaystyle\int_0^t y\,dt, \end{array}$$

which can also be rewritten as

$$\left.\begin{array}{rcl} x(t) & = & x_0\;+\;C_x\,t\;+\;\delta x(t) \\ y(t) & = & y_0\;+\;C_y\,t\;+\;\delta y(t) \\ z(t) & = & z_0\;+\;C_z\,t\;-\;\frac{1}{2}\,g\,t^2\;+\;\delta z(t) \end{array}\right\}, \tag{23}$$

where the Coriolis *drifts* are

$$\delta x(t) \;=\; 2\,\omega\,\sin\lambda\left(\,y_0\,t\;+\;\frac{1}{2}\,C_y\,t^2\;+\;\int_0^t \delta y\,dt\right) \tag{24}$$

$$\begin{array}{rcl} \delta y(t) & = & -\,2\,\omega\,\sin\lambda\,\left(\,x_0\,t\;+\;\dfrac{1}{2}\,C_x\,t^2\;+\;\displaystyle\int_0^t \delta x\,dt\right) \\ & & -\;2\,\omega\,\cos\lambda\left(\,z_0\,t\;+\;\dfrac{1}{2}\,C_z\,t^2\;-\;\dfrac{1}{6}\,g\,t^3\;+\;\displaystyle\int_0^t \delta z\,dt\right) \end{array} \tag{25}$$

$$\delta z(t) \;=\; 2\,\omega\,\cos\lambda\left(\,y_0\,t\;+\;\frac{1}{2}\,C_y\,t^2\;+\;\int_0^t \delta y\,dt\right). \tag{26}$$

Note that each Coriolis drift can be expressed as an infinite series in powers of ω and that all Coriolis effects vanish when $\omega = 0$.

4.1 Free-Fall Problem Revisited

As an example of the importance of Coriolis effects, we consider the simple *free-fall* problem, where

$$(x_0, y_0, z_0) \;=\; (0, 0, h) \quad \text{and} \quad (\dot{x}_0, \dot{y}_0, \dot{z}_0) \;=\; (0, 0, 0),$$

so that the constants (22) are

$$C_x \;=\; 0 \;=\; C_z \quad \text{and} \quad C_y \;=\; 2\,\omega\, h\, \cos\lambda.$$

Substituting these constants into Eqs. (23) and keeping only terms up to first order in ω, we find

$$x(t) \;=\; 0, \qquad (27)$$

$$y(t) \;=\; \frac{1}{3}\, g t^3\, \omega\, \cos\lambda, \qquad (28)$$

$$z(t) \;=\; h \;-\; \frac{1}{2}\, g t^2. \qquad (29)$$

Hence, a free-falling object starting from rest touches the ground $z(T) = 0$ after a time $T = \sqrt{2h/g}$ after which time the object has drifted eastward by a distance of

$$y(T) \;=\; \frac{1}{3}\, g T^3\, \omega\, \cos\lambda \;=\; \frac{\omega\, \cos\lambda}{3}\, \sqrt{\frac{8h^3}{g}}.$$

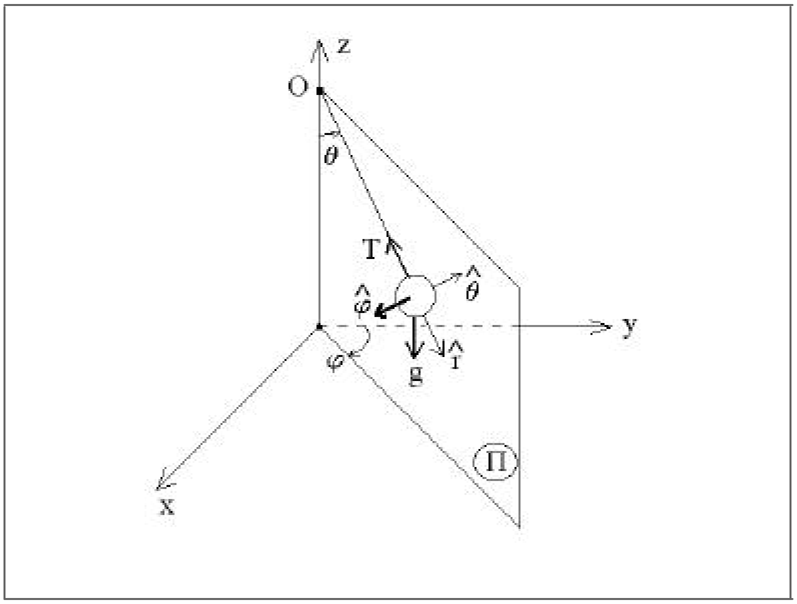
At a height of 100 m and latitude 45^o, we find an eastward drift of 1.55 cm.

4.2 Foucault Pendulum

We now consider the motion of a pendulum (of length ℓ and mass m) in the rotating frame of the Earth. The equation of motion for the pendulum is given as

$$\ddot{\mathbf{r}} \;=\; \mathbf{a}_f \;-\; 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}}, \qquad (30)$$

where $\mathbf{a}_f = \mathbf{g} + \mathbf{T}/m$ is the net fixed-frame acceleration of the pendulum expressed in terms of the gravitational acceleration $\mathbf{g}$ and the string tension $\mathbf{T}$ (see Figure below). Note that the vectors $\mathbf{g}$ and $\mathbf{T}$ span a plane Π in which the pendulum moves in the absence of the Coriolis acceleration $-\,2\boldsymbol{\omega} \times \dot{\mathbf{r}}$.



Using spherical coordinates (r, θ, φ) in the rotating frame and placing the origin O of the pendulum system at its pivot point (see Figure above), the position of the pendulum bob is

$$\mathbf{r} \;=\; \ell\;[\,\sin\theta\,(\sin\varphi\,\widehat{\mathsf{x}} \;+\; \cos\varphi\,\widehat{\mathsf{y}}) \;-\; \cos\theta\,\widehat{\mathsf{z}}\,] \;=\; \ell\,\widehat{r}(\theta,\varphi). \tag{31}$$

From this definition, we construct the unit vectors $\widehat{\theta}$ and $\widehat{\varphi}$ as

$$\widehat{\theta} \;=\; \frac{\partial\widehat{r}}{\partial\theta},\;\; \frac{\partial\widehat{r}}{\partial\varphi} \;=\; \sin\theta\,\widehat{\varphi}, \quad \text{and} \quad \frac{\partial\widehat{\theta}}{\partial\varphi} \;=\; \cos\theta\,\widehat{\varphi}. \tag{32}$$

Note that, whereas the unit vectors $\widehat{r}$ and $\widehat{\theta}$ lie on the plane Π, the unit vector $\widehat{\varphi}$ is perpendicular to it and, thus, the equation of motion of the pendulum *perpendicular* to the plane Π is

$$\ddot{\mathbf{r}}\cdot\widehat{\varphi} \;=\; -\,2\;(\boldsymbol{\omega}\times\dot{\mathbf{r}})\cdot\widehat{\varphi}. \tag{33}$$

The pendulum velocity is obtained from Eq. (31) as

$$\dot{\mathbf{r}} \;=\; \ell\left(\dot{\theta}\,\widehat{\theta} \;+\; \dot{\varphi}\sin\theta\,\widehat{\varphi}\right), \tag{34}$$

so that the azimuthal component of the Coriolis acceleration is

$$-\,2\;(\boldsymbol{\omega}\times\dot{\mathbf{r}})\cdot\widehat{\varphi} \;=\; 2\,\ell\omega\,\dot{\theta}\,(\sin\lambda\,\cos\theta \;+\; \cos\lambda\,\sin\theta\,\sin\varphi).$$

If the length ℓ of the pendulum is large, the angular deviation θ of the pendulum can be small enough that $\sin\theta \ll 1$ and $\cos\theta \simeq 1$ and, thus, the azimuthal component of the Coriolis acceleration is approximately

$$-\,2\;(\boldsymbol{\omega}\times\dot{\mathbf{r}})\cdot\widehat{\varphi} \;\simeq\; 2\,\ell\,(\omega\,\sin\lambda)\;\dot{\theta}. \tag{35}$$

Next, the azimuthal component of the pendulum acceleration is

$$\ddot{\mathbf{r}}\cdot\widehat{\varphi} \;=\; \ell\left(\ddot{\varphi}\,\sin\theta \;+\; 2\,\dot{\theta}\,\dot{\varphi}\;\cos\theta\right),$$

which for small angular deviations yields

$$\ddot{\mathbf{r}}\cdot\widehat{\varphi} \;\simeq\; 2\,\ell\;(\dot{\varphi})\;\;\dot{\theta}. \tag{36}$$

By combining these expressions into Eq. (33), we obtain an expression for the precession angular frequency of the Foucault pendulum

$$\dot{\varphi} \;=\; \omega\;\sin\lambda \tag{37}$$

as a function of latitude λ. As expected, the precession motion is clockwise in the Northern Hemisphere and reaches a maximum at the North Pole ($\lambda = 90^o$).

The more traditional approach to describing the precession motion of the Foucault pendulum makes use of Cartesian coordinates (x, y, z). The motion of the Foucault pendulum in the (x, y)-plane is described in terms of Eqs. (30) as

$$\left.\begin{array}{rcl} \ddot{x} \;+\; \omega_0^2\,x & = & 2\,\omega\sin\lambda\;\dot{y} \\ \ddot{y} \;+\; \omega_0^2\,y & = & -\,2\,\omega\sin\lambda\;\dot{x} \end{array}\right\}, \tag{38}$$

where $\omega_0^2 = T/m\ell \simeq g/\ell$ and $\dot{z} \simeq 0$ if ℓ is very large. We now define the complex-valued function

$$q \;=\; y + i\,x \;=\; \ell\,\sin\theta\;e^{i\varphi}, \tag{39}$$

so that Eq. (38) becomes

$$\ddot{q} \;+\; \omega_0^2\,q \;-\; 2i\;\omega\sin\lambda\;\dot{q} \;=\; 0.$$

We now insert the eigenfunction $q(t) \;=\; \rho\,\exp(i\text{-}\;t)$ into this equation and find that the solution for the eigenfrequency - is

$$\text{-} \;\;=\; \omega\;\sin\lambda \;\pm\; \sqrt{\omega^2\;\sin^2\lambda \;+\; \omega_0^2},$$

so that the eigenfunction is

$$q \;=\; \rho\,e^{i\omega\,\sin\lambda\,t}\;\;\sin\left(\sqrt{\omega^2\;\sin^2\lambda \;+\; \omega_0^2}\;t\right).$$

By comparing this solution with Eq. (39), we find

$$\rho\;\sin\left(\sqrt{\omega^2\;\sin^2\lambda \;+\; \omega_0^2}\;t\right) \;=\; \ell\;\sin\theta \;\simeq\; \ell\,\theta(t),$$

and

$$\varphi(t) \;=\; \omega\;\sin\lambda\,t,$$

from which we recover the Foucault pendulum precession frequency (37).