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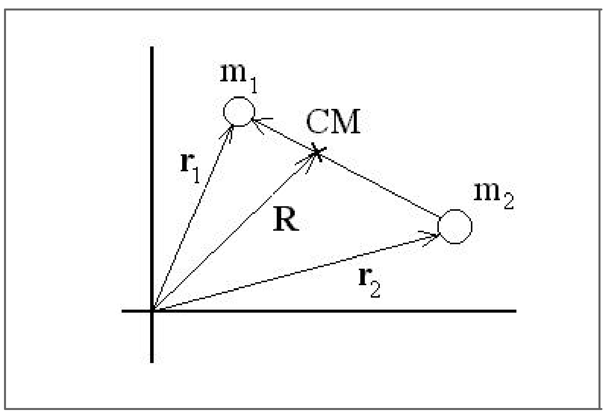
Motion in a Central-Force Field

1 Motion in the Center-of-Mass Frame

The Lagrangian for an isolated two-particle system is

$$L \;=\; \frac{m_1}{2}\,|\dot{\mathbf{r}}_1|^2 \;+\; \frac{m_2}{2}\,|\dot{\mathbf{r}}_2|^2 \;-\; U(\mathbf{r}_1 - \mathbf{r}_2),$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ represent the positions of the particles of mass m_1 and m_2, respectively, and $U(\mathbf{r}_1, \mathbf{r}_2) = U(|\mathbf{r}_1 - \mathbf{r}_2|)$ is the potential energy for an isolated two-particle system (see Figure below).



Let us now define the position $\mathbf{R}$ of the center-of-mass (CM)

$$\mathbf{R} \;=\; \frac{m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2}{m_1 + m_2},$$

and define the inter-particle vector $\mathbf{r} \;=\; \mathbf{r}_1 \;-\; \mathbf{r}_2$, so that the particle positions can be expressed as

$$\mathbf{r}_1 \;=\; \mathbf{R} \;+\; \frac{m_2}{M}\,\mathbf{r} \quad\text{and}\quad \mathbf{r}_2 \;=\; \mathbf{R} \;-\; \frac{m_1}{M}\,\mathbf{r},$$

where $M = m_1 + m_2$ is the total mass of the two-particle system. The Lagrangian of the isolated two-particle system thus becomes

$$L \;=\; \frac{M}{2}\,|\dot{\mathbf{R}}|^2 \;+\; \frac{\mu}{2}\,|\dot{\mathbf{r}}|^2 \;-\; U(\mathbf{r}),$$

where

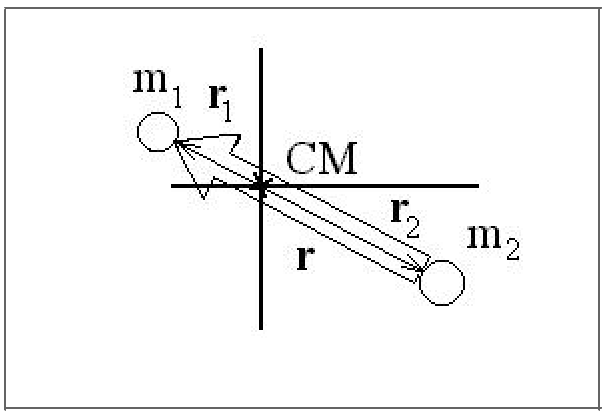
$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{-1}$$

denotes the *reduced* mass of the two-particle system.

For an isolated system, the CM canonical momentum

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{R}}} = M \, \dot{\mathbf{R}}$$

is a constant of the motion. The CM reference frame is defined by the condition $\mathbf{R} = 0$, i.e., we move the origin of our coordinate system to the CM position (the Figure below shows the case where $m_1 > m_2$).



In this case, the Lagrangian for an isolated two-particle system in the CM reference frame is

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{\mu}{2} |\dot{\mathbf{r}}|^2 - U(\mathbf{r}),$$

where

$$\mathbf{r}_1 = \frac{m_2}{M} \mathbf{r} \quad \text{and} \quad \mathbf{r}_2 = -\frac{m_1}{M} \mathbf{r}.$$

Hence, once the Euler-Lagrange equation for $\mathbf{r}$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \right) = \frac{\partial L}{\partial \mathbf{r}} \quad \rightarrow \quad \mu \, \ddot{\mathbf{r}} = -\nabla U(\mathbf{r})$$

is solved for $\mathbf{r} = \mathbf{r}(t)$, the motion of m_1 and m_2 are determined through $\mathbf{r}_1(t) = (m_2/M) \, \mathbf{r}(t)$ and $\mathbf{r}_2(t) = -(m_1/M) \, \mathbf{r}(t)$.

2 Motion in a Central-Force Field

A particle moves under the influence of a central-force field $\mathbf{F}(\mathbf{r}) = F(r) \, \widehat{\mathbf{r}}$ if the force on the particle is independent of the angular position of the particle about the center of force and depends only on its distance r from the center of force. Here, the magnitude $F(r)$ (which is positive for a repulsive force and negative for an attractive force) is defined in terms of the central potential $U(r)$ as $F(r) = -U'(r)$.

2.1 Lagrangian Formalism

The motion of two particles in an isolated system takes place on a two-dimensional plane. When these particles move in a central-force field, the Lagrangian is simply

$$L = \frac{\mu}{2}\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) - U(r), \tag{1}$$

where polar coordinates (r, θ) are most conveniently used. Since the potential U is independent of θ, the canonical momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu\, r^2\,\dot{\theta} \equiv \ell \tag{2}$$

is a constant of motion (here, labeled ℓ). The Euler-Lagrange equation for r, therefore, becomes the radial force equation

$$\mu\left(\ddot{r} - r\,\dot{\theta}^2\right) = \mu\,\ddot{r} - \frac{\ell^2}{\mu\, r^3} = F(r). \tag{3}$$

In this description, the planar orbit is parametrized by time, i.e., once $r(t)$ and $\theta(t)$ are obtained, a path $r(\theta)$ onto the plane is defined.

Since $\dot{\theta}$ does not change sign on its path along the orbit, we may replace $\dot{r}$ and $\ddot{r}$ with $r'(\theta)$ and $r''(\theta)$ as follows. First,

$$\dot{r} = \dot{\theta}\, r' = \frac{\ell\, r'}{\mu\, r^2} = -\frac{\ell}{\mu}\left(\frac{1}{r}\right)'.$$

Next, using Eq. (2)

$$\ddot{\theta} = -\frac{2\,\ell\,\dot{r}}{\mu\, r^3} = -\frac{2\,\ell^2\, r'}{\mu^2\, r^5},$$

we find

$$\ddot{r} = \ddot{\theta}\, r' + \dot{\theta}^2 r'' = -\frac{\ell^2}{\mu^2\, r^5}\left[\,2\,(r')^2 - r\,r''\,\right].$$

Lastly, using the identity

$$\left(\frac{1}{r}\right)'' = \left(-\frac{r'}{r^2}\right)' = \frac{1}{r^3}\left[\,2\,(r')^2 - r\,r''\,\right],$$

we find an expression for $\ddot{r}$:

$$\ddot{r} = -\frac{\ell^2}{\mu^2\, r^2}\left(\frac{1}{r}\right)'',$$

and the radial force equation (3) becomes

$$s'' \;+\; s \;=\; -\;\frac{\mu}{\ell^2\, s^2}\; F(1/s) \;=\; -\;\frac{d\overline{U}(s)}{ds}, \tag{4}$$

where $s(\theta) = 1/r(\theta)$ and $\overline{U}(s) = (\mu/\ell^2)\, U(1/s)$.

Note that the form of the potential can be calculated from the solution $s(\theta) = 1/r(\theta)$ as follows. For example, consider the particle trajectory described in terms of the solution $r(\theta) = r_0\, \sec(\alpha\,\theta)$, where r_0 and α are constants, then

$$s'' \;+\; s \;=\; -\;\left(\alpha^2 \;-\; 1\right)\; s \;=\; -\;\frac{d\overline{U}(s)}{ds},$$

and thus

$$\overline{U}(s) \;=\; \frac{1}{2}\;\left(\alpha^2 \;-\; 1\right)\; s^2 \;\;\rightarrow\;\; U(r) \;=\; \frac{\ell^2}{2\mu\, r^2}\left(\alpha^2 \;-\; 1\right).$$

Note also that the function $\theta(t)$ is determined from the relation

$$\dot{\theta} \;=\; \frac{\ell}{\mu r^2(\theta)} \;\;\rightarrow\;\; t(\theta) \;=\; \frac{\mu}{\ell}\int_0^{\theta} r^2(\theta)\, d\theta.$$

Returning to our example, we find

$$t(\theta) \;=\; \frac{\mu r_0^2}{\alpha\ell}\int_0^{\alpha\theta}\sec^2\phi\, d\phi \;=\; \frac{\mu r_0^2}{\alpha\ell}\,\tan(\alpha\theta) \;\;\rightarrow\;\; r(t) \;=\; r_0\,\sqrt{1 \;+\; \left(\frac{\alpha\ell\, t}{\mu r_0^2}\right)^2}$$

and the total energy

$$E \;=\; \frac{\alpha^2\,\ell^2}{2\mu r_0^2},$$

is determined from the initial conditions $r(0) = r_0$ and $\dot{r}(0) = 0$.

2.2 Hamiltonian Formalism

The Hamiltonian for the central-force problem is

$$H \;=\; \frac{p_r^2}{2\mu} \;+\; \frac{\ell^2}{2\,\mu\, r^2} \;+\; U(r),$$

where $p_r = \mu\,\dot{r}$ is the radial canonical momentum. Since energy is also conserved, we solve

$$E \;=\; \frac{\mu\,\dot{r}^2}{2} \;+\; \frac{\ell^2}{2\,\mu\, r^2} \;+\; U(r) \;=\; \frac{\mu\,\dot{r}^2}{2} \;+\; V(r),$$

as

$$\dot{r} \;=\; \pm\,\sqrt{\frac{2}{\mu}\,[\,E \;-\; V(r)\,]}, \tag{5}$$

where $V(r)$ is known as the *effective* potential and the sign $\pm$ depends on initial conditions. This equation can then be used with Eq. (2) to yield

$$d\theta \;=\; \frac{\ell}{\mu\, r^2}\, dt \;=\; \frac{\ell}{\mu\, r^2}\, \frac{dr}{\dot{r}} \;=\; \frac{-\, ds}{\sqrt{\epsilon - 2\, \overline{U}(s) \;-\; s^2}}, \tag{6}$$

where $\epsilon = 2\mu\, E/\ell^2$, or

$$s'(\theta) \;=\; \pm\, \sqrt{\epsilon - 2\, \overline{U}(s) \;-\; s^2}. \tag{7}$$

We readily check that this equation is a proper solution of the radial force equation (4) since

$$s'' \;=\; \frac{s'\, [d\overline{U}/ds + s]}{\sqrt{\epsilon - 2\, \overline{U}(s) \;-\; s^2}} \;=\; -\, \frac{d\overline{U}}{ds} \;-\; s$$

is indeed identical to Eq. (4). Hence, for a given central-force potential $U(r)$, we can solve for $r(\theta) = 1/s(\theta)$ by integrating

$$\theta(s) \;=\; -\, \int_{s_0}^{s} \frac{d\sigma}{\sqrt{\epsilon - 2\, \overline{U}(\sigma) \;-\; \sigma^2}}, \tag{8}$$

where s_0 defines $\theta(s_0) = 0$, and performing the inversion $\theta(s) \to s(\theta)$.

2.3 Turning Points

Eq. (7) yields the following energy equation

$$E \;=\; \frac{\mu}{2}\, \dot{r}^2 \;+\; \frac{\ell^2}{2\mu\, r^2} \;+\; U(r) \;=\; \frac{\ell^2}{2\mu} \left[\, (s')^2 \;+\; s^2 \;+\; 2\, \overline{U}(s)\, \right],$$

where $s' = -\, \mu\dot{r}/\ell = -\, p_r/p_\theta$. *Turning* points are those special values of r_n (or s_n) ($n = 1, 2, ...$) for which

$$E \;=\; U(r_n) \;+\; \frac{\ell^2}{2\mu\, r_n^2} \;=\; \frac{\ell^2}{\mu} \left[\, \overline{U}(s_n) \;+\; \frac{s_n^2}{2}\, \right],$$

i.e., $\dot{r}$ (or s') vanishes at these points. If two non-vanishing turning points $r_2 < r_1 < \infty$ (or $0 < s_1 < s_2$) exist, the motion is said to be *bounded* in the interval $r_2 < r < r_1$ (or $s_1 < s < s_2$), otherwise the motion is *unbounded*.

3 Kepler Problem

We now solve the Kepler problem where $U(r) = -\, k/r$, where k is a constant, so that $\overline{U}(s) = -\, s_0 s$, where $s_0 = \mu k/\ell^2$. The turning points for the Kepler problem are solutions of the quadratic equation

$$s^2 \;-\; 2\, s_0\, s \;-\; \epsilon \;=\; 0,$$

which can be written as $s = s_0 \pm \sqrt{s_0^2 + \epsilon}$

$$s_1 \;=\; s_0\,(1 - \mathsf{e}) \quad \text{and} \quad s_2 \;=\; s_0\,(1 + \mathsf{e}),$$

where

$$\mathsf{e} \;=\; \sqrt{1 + \epsilon/s_0^2} \;=\; \sqrt{1 \;+\; 2\,E\ell^2/\mu k^2}.$$

We note that motion is bounded when $E < 0$ ($0 < \mathsf{e} < 1$) and unbounded when $E \geq 0$ ($\mathsf{e} > 1$).

3.1 Bounded Keplerian Orbits

We will now look at the bounded case ($\mathsf{e} < 1$). We define $\theta(s_2) = 0$, so that for the Kepler problem, Eq. (8) becomes

$$\theta(s) \;=\; -\int_{s_0\,(1+\mathsf{e})}^{s} \frac{d\sigma}{\sqrt{s_0^2\mathsf{e}^2 - (\sigma - s_0)^2}}, \tag{9}$$

which can easily be integrated by using the identity

$$\frac{-\,dx}{\sqrt{a^2 - x^2}} \;=\; d\;\arccos\left(\frac{x}{a}\right),$$

so that Eq. (9) yields

$$\theta(s) \;=\; \arccos\left(\frac{s - s_0}{s_0\,\mathsf{e}}\right).$$

This equation can easily be inverted to yield

$$s(\theta) \;=\; s_0\,(1 \;+\; \mathsf{e}\,\cos\theta). \tag{10}$$

We can readily check that this solution also satisfies the radial force equation (4).

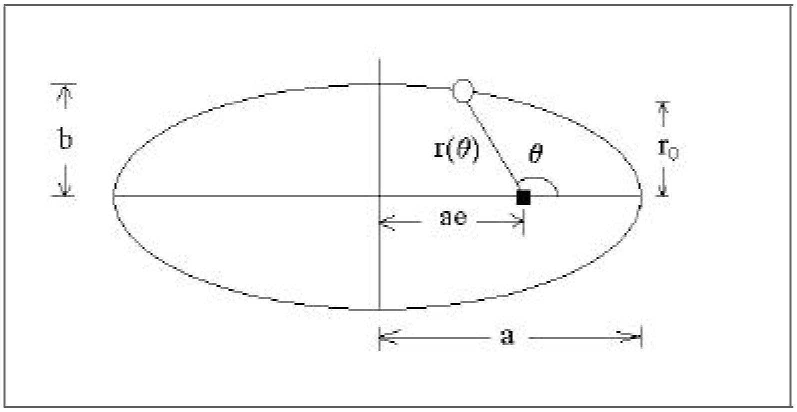
3.1.1 Kepler's First Law

The solution for $r(\theta)$ is now trivially obtained from $s(\theta)$ as

$$r(\theta) \;=\; \frac{r_0}{1 \;+\; \mathsf{e}\,\cos\theta}, \tag{11}$$

where $r_0 = 1/s_0$ denotes the position of the minimum of the *effective* potential

$$V'(r_0) \;=\; \frac{1}{r_0^2}\left(k \;-\; \frac{\ell^2}{\mu\,r_0}\right) \;=\; 0.$$



Eq. (11) generates an ellipse of major radius

$$a = \frac{r_0}{1 - \mathsf{e}^2} = \frac{k}{2\,|E|}$$

and minor radius

$$b = a\sqrt{1 - \mathsf{e}^2} = \sqrt{\frac{\ell^2}{2\mu\,|E|}}$$

and, therefore, yields Kepler's First Law.

3.1.2 Kepler's Second Law

Using Eq. (2), we find

$$dt = \frac{\mu}{\ell}\, r^2\, d\theta = \frac{2\mu}{\ell}\, dA(\theta),$$

where $dA(\theta) = \int r\ dr\, d\theta = \frac{1}{2}\,[r(\theta)]^2\, d\theta$ denotes an infinitesimal area swept by $d\theta$. When integrated, this relation yields Kepler's Second law

$$\Delta t = \frac{2\mu}{\ell}\, \Delta A, \tag{12}$$

i.e., equal areas are swept in equal times since μ and ℓ are constants.

3.1.3 Kepler's Third Law

The orbital period T of a bound system is defined as

$$T = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \frac{\mu}{\ell} \int_0^{2\pi} r^2\, d\theta = \frac{2\mu}{\ell}\, A = \frac{2\pi\,\mu}{\ell}\, a\, b$$

where $A = \pi\, ab$ denotes the area of an ellipse with major radius a and minor radius b. Using the expressions for a and b found above, we find

$$T \;=\; \frac{2\pi\,\mu}{\ell} \,\cdot\, \frac{k}{2|E|} \,\cdot\, \sqrt{\frac{\ell^2}{2\mu\,|E|}} \;=\; 2\pi\,\sqrt{\frac{\mu\,k^2}{(2\,|E|)^3}}.$$

If we now substitute the expression for $a = k/2|E|$ and square both sides of this equation, we obtain Kepler's Third Law

$$T^2 \;=\; \frac{(2\pi)^2\mu}{k}\, a^3. \tag{13}$$

Note that in Newtonian gravitational theory, $k/\mu = G\,(m_1 + m_2)$; although Kepler's Third Law states that T^2/a^3 is a constant for all planets in the solar system, we find that this is only an approximation that holds for $m_1 \gg m_2$.

3.2 Unbounded Keplerian Orbits

We now look at the case where the total energy is positive or zero (i.e., $\mathsf{e} \geq 1$). Eq. (11) yields $r\,(1 + \mathsf{e}\,\cos\theta) = r_0$ or

$$\left(\sqrt{\mathsf{e}^2 - 1}\; x \;-\; \frac{\mathsf{e}\, r_0}{\sqrt{\mathsf{e}^2 - 1}} \right)^2 \;-\; y^2 \;=\; \frac{r_0^2}{\mathsf{e}^2 - 1}.$$

For $\mathsf{e} = 1$, the particle orbit is a parabola $x = (r_0^2 - y^2)/2r_0$, with distance of closest approach at $x(0) = r_0/2$, while for $\mathsf{e} > 1$, the particle orbit is a hyperbola.

3.3 Laplace-Runge-Lenz Vector

Let us now investigate an additional constant of the motion for the Kepler problem. First, we consider the time derivative of the vector $\mathbf{p} \times \mathbf{L}$, where the linear momentum $\mathbf{p}$ and angular momentum $\mathbf{L}$ are

$$\mathbf{p} \;=\; \mu \left(\dot{r}\, \widehat{r} \;+\; r\dot{\theta}\, \widehat{\theta} \right) \quad \text{and} \quad \mathbf{L} \;=\; \ell\, \widehat{z} \;=\; \mu r^2 \dot{\theta}\, \widehat{z}.$$

The time derivative of the linear momentum is $\dot{\mathbf{p}} = -\,\nabla U(r) = -\,U'(r)\,\widehat{r}$ while the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is itself a constant of the motion so that

$$\begin{aligned} \frac{d}{dt}\,(\mathbf{p} \times \mathbf{L}) \;&=\; \frac{d\mathbf{p}}{dt} \times \mathbf{L} \;=\; -\,\mu\dot{\mathbf{r}}\cdot\nabla U\;\mathbf{r} \;+\; \mu\mathbf{r}\cdot\nabla U\;\dot{\mathbf{r}} \\ &=\; -\,\frac{d}{dt}\,(\mu\, U\,\mathbf{r}) \;+\; \mu\,(\mathbf{r}\cdot\nabla U \;+\; U)\,\dot{\mathbf{r}}, \end{aligned}$$

hence the vector $\mathbf{A} = \mathbf{p} \times \mathbf{L} + \mu\, U(r)\ \mathbf{r}$ is a constant of the motion if the potential $U(r)$ satisfies the condition $\mathbf{r} \cdot \nabla U(r) + U(r) = 0$. For the Kepler problem, with central potential $U(r) = -\,k/r$, the Laplace-Runge-Lenz (LRL) vector

$$\mathbf{A} \;=\; \mathbf{p} \times \mathbf{L} \;-\; k\mu\,\widehat{r} \;=\; \left(\frac{\ell^2}{r} \;-\; k\mu \right) \widehat{r} \;-\; \ell\,\mu\dot{r}\;\widehat{\theta}$$

is, therefore, a constant of the motion since $\mathbf{r} \cdot \nabla U = -\,U$.

Since the vector $\mathbf{A}$ is constant in both magnitude and direction, we choose its direction to be along the x-axis and its amplitude is determined at the distance of closest approach $r_{min} = r_0/(1 + \mathsf{e})$ and we can easily show that $\mathbf{A} \cdot \widehat{r} = A\,\cos\theta$ leads to the Kepler solution

$$r(\theta) \;=\; \frac{r_0}{1 + \mathsf{e}\,\cos\theta},$$

where $r_0 = \ell^2/k\mu$ and $\mathsf{e} = A/k\mu$.

Note that if the Keplerian orbital motion is perturbed by the introduction of an additional potential term $\delta U(r)$, we can show that the LRL vector is no longer conserved (i.e., $d\mathbf{A}/dt \neq 0$) and that the direction of the Keplerian elliptical orbit precesses with a precession frequency

$$\omega_{\rm p}(\theta) \;=\; \widehat{z} \cdot \frac{\mathbf{A}}{A^2} \times \frac{d\mathbf{A}}{dt},$$

where the unperturbed Kepler solution $r(\theta)$ is to be used.

4 Isotropic Simple Harmonic Oscillator

We now investigate the case when the central potential is of the form

$$U(r) \;=\; \frac{k}{2}\,r^2 \quad\rightarrow\quad \overline{U}(s) \;=\; \frac{\mu k}{2\ell^2\;s^2}. \tag{14}$$

The turning points for this problem are expressed as

$$r_1 \;=\; r_0 \left(\frac{1-\mathsf{e}}{1+\mathsf{e}}\right)^{\frac{1}{4}} \;=\; \frac{1}{s_2} \quad\text{and}\quad r_2 \;=\; r_0 \left(\frac{1+\mathsf{e}}{1-\mathsf{e}}\right)^{\frac{1}{4}} \;=\; \frac{1}{s_1},$$

where $r_0 = (\ell^2/\mu k)^{1/4} = 1/s_0$ is the radial position at which the effective potential has a minimum $V_0 = k\,r_0^2$ and

$$\mathsf{e} \;=\; \sqrt{1 \;-\; \left(\frac{kr_0^2}{E}\right)^2}.$$

Here, we see that orbits exist and are always bounded for $E > V_0$ (and thus $0 \quad \mathsf{e} \quad 1$). Next, using the change of coordinate $q = s^2$ in Eq. (8), we obtain

$$\theta \;=\; \frac{-\,1}{2} \int_{q_2}^{q} \frac{dq}{\sqrt{(\mu/\ell^2)\;[2E\,q - k] \;-\; q^2}}, \tag{15}$$

where $q_2 = (1+\mathsf{e})\,\mu E/\ell^2$. We now substitute $q(\varphi) = (1+\mathsf{e}\cos\varphi)\,\mu E/\ell^2$ in Eq. (15) to obtain

$$\theta = \frac{1}{2}\,\arccos\left[\,\frac{1}{\mathsf{e}}\left(\frac{\ell^2 q}{\mu E} - 1\right)\right]$$

or

$$r(\theta) = \frac{r_0\,\sqrt{1-\mathsf{e}^2}}{\sqrt{1+\mathsf{e}\cos 2\theta}}. \tag{16}$$

This equation describes the ellipse

$$\frac{x^2}{(1-\mathsf{e})} + \frac{y^2}{(1+\mathsf{e})} = r_0^2$$

of semi-major axis $a = r_0\sqrt{1+\mathsf{e}}$ and semi-minor axis $b = r_0\sqrt{1-\mathsf{e}}$. Lastly, we note that one revolution along the orbit $r(\theta)$ corresponds to an angular period of π, i.e., $r(\theta+\pi) = r(\theta)$, and not 2π as found in the Kepler problem. The area of the ellipse $A = \pi\,ab = \pi\,(\ell^2/\mu E)$ while the period is

$$T(E,\ell) = \int_0^{\pi} \frac{d\theta}{\dot{\theta}} = \frac{\mu\,A}{\ell} = \frac{\pi\,\ell}{E}.$$

If we introduce the angular frequency $\omega = \pi/T$, then we find the important relation between energy and angular momentum $E = \ell\,\omega$.