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Lagrangian Mechanics

1 Principle of Least Action

The configuration of a mechanical system evolving in an n-dimensional space, with coordinates $\mathbf{x} = (x^1, x^2, ..., x^n)$, may be described in terms of *generalized* coordinates $\mathbf{q} = (q^1, q^2, ..., q^k)$ in a k-dimensional *configuration* space, with $k < n$.

The Principle of Least Action (also known as Hamilton's principle) is expressed in terms of a function $L(\mathbf{q}, \dot{\mathbf{q}};\, t)$ known as the *Lagrangian*, which appears in the *action* integral

$$A[\mathbf{q}] \;=\; \int_{t_i}^{t_f} L(\mathbf{q}, \dot{\mathbf{q}};\, t)\, dt, \tag{1}$$

where the action integral is a functional of the vector function $\mathbf{q}(t)$, which provides a path from the initial point $\mathbf{q}_i = \mathbf{q}(t_i)$ to the final point $\mathbf{q}_f = \mathbf{q}(t_f)$. The variational principle

$$0 \;=\; \delta A[\mathbf{q}] \;=\; \frac{d}{d\epsilon} A[\mathbf{q} + \epsilon\, \delta\mathbf{q}]\Big|_{\epsilon=0} \;=\; \int_{t_i}^{t_f} \delta q^j \left[\frac{\partial L}{\partial q^j} \;-\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j} \right) \right] dt,$$

where the variation $\delta\mathbf{q}$ is assumed to vanish at the integration boundaries ($\delta\mathbf{q}_i = 0 = \delta\mathbf{q}_f$), yields the *Euler-Lagrange* equation for the generalized coordinate q^j ($j = 1, ..., k$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j} \right) \;=\; \frac{\partial L}{\partial q^j}, \tag{2}$$

The Lagrangian also satisfies the second Euler equation

$$\frac{d}{dt}\left(L \;-\; \dot{q}^j\, \frac{\partial L}{\partial \dot{q}^j} \right) \;=\; \frac{\partial L}{\partial t}, \tag{3}$$

and thus for time-independent Lagrangian systems ($\partial L/\partial t = 0$) we find that $L - \dot{q}^j\, \partial L/\partial \dot{q}^j$ is a conserved quantity whose interpretation will be discussed shortly.

The form of the Lagrangian function $L(\mathbf{r}, \dot{\mathbf{r}};\, t)$ is dictated by our requirement that Newton's Second Law $m\, \ddot{\mathbf{r}} = -\nabla U(\mathbf{r}, t)$ describing the motion of a particle of mass m in a nonuniform (possibly time-dependent) potential $U(\mathbf{r}, t)$ be written in the Euler-Lagrange form (2). One easily obtains the form

$$L(\mathbf{r}, \dot{\mathbf{r}};\, t) \;=\; \frac{m}{2}\, |\dot{\mathbf{r}}|^2 \;-\; U(\mathbf{r}, t), \tag{4}$$

which is simply the kinetic energy of the particle **minus** its potential energy. For a time-independent Lagrangian ($\partial L/\partial t = 0$), we also find that the energy function

$$\dot{\mathbf{r}} \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}} \;-\; L \;=\; \frac{m}{2}\,|\dot{\mathbf{r}}|^2 \;+\; U(\mathbf{r}) \;=\; E,$$

is a constant of the motion. Hence, for a simple mechanical system, the Lagrangian function is obtained by computing the kinetic energy of the system and its potential energy and then construct Eq. (4).

2 Examples

The construction of a Lagrangian function for a system of N particles proceeds in three steps as follows.

Step I. Define k generalized coordinates $\{q^1(t), ..., q^k(t)\}$ that represent the instantaneous *configuration* of the system of N particles.

Step II. Construct the position vector $\mathbf{r}_a(\mathbf{q}; t)$ and its associated velocity

$$\mathbf{v}_a(\mathbf{q}, \dot{\mathbf{q}}; t) \;=\; \frac{\partial \mathbf{r}_a}{\partial t} \;+\; \sum_{j=1}^{k} \dot{q}^j\, \frac{\partial \mathbf{r}_a}{\partial q^j}$$

for each particle ($a = 1, ..., N$).

Step III. Construct the kinetic energy

$$K(\mathbf{q}, \dot{\mathbf{q}}; t) \;=\; \frac{1}{2} \sum_a m_a\, |\mathbf{v}_a(\mathbf{q}, \dot{\mathbf{q}}; t)|^2$$

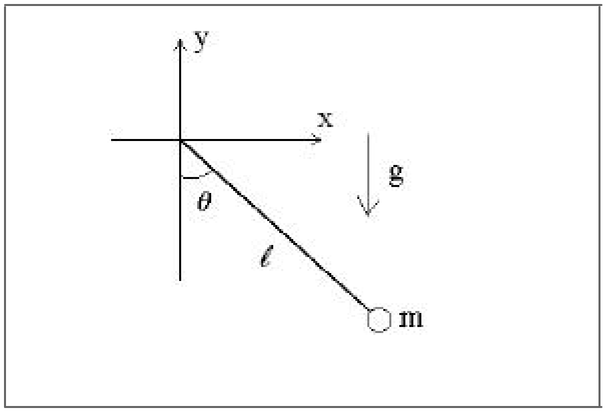
and the potential energy $U(\mathbf{q}; t)$ for the system and combine them to obtain the Lagrangian

$$L(\mathbf{q}, \dot{\mathbf{q}}; t) \;=\; K(\mathbf{q}, \dot{\mathbf{q}}; t) \;-\; U(\mathbf{q}; t),$$

from which the Euler-Lagrange equations (2) are derived.

2.1 Example I: Pendulum

Consider a pendulum composed of an object of mass m and a massless string of constant length ℓ in a constant gravitational field with acceleration g.



Although the motion of the pendulum is two-dimensional, a single generalized coordinate is needed to describe the configuration of the pendulum: the angle θ measured from the negative y-axis (see Figure above). Here, the position of the object is given as

$$x(\theta) \;=\; \ell\,\sin\theta \quad\text{and}\quad y(\theta) \;=\; -\,\ell\,\cos\theta,$$

with associated velocity components

$$\dot{x}(\theta,\dot{\theta}) \;=\; \ell\,\dot{\theta}\,\cos\theta \quad\text{and}\quad \dot{y}(\theta,\dot{\theta}) \;=\; \ell\,\dot{\theta}\,\sin\theta.$$

Hence, the kinetic energy of the pendulum is

$$K \;=\; \frac{m}{2}\,\ell^2\dot{\theta}^2,$$

and choosing the zero potential energy point when $\theta = 0$ (see Figure above), the gravitational potential energy is

$$U \;=\; mg\ell\,(1 - \cos\theta).$$

The Lagrangian $L = K - U$ is, therefore, written as

$$L(\theta,\dot{\theta}) \;=\; \frac{m}{2}\,\ell^2\dot{\theta}^2 \;-\; mg\ell\,(1 - \cos\theta),$$

and the Euler-Lagrange equation for θ is

$$\frac{\partial L}{\partial\dot{\theta}} = m\ell^2\,\dot{\theta} \;\rightarrow\; \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) \;=\; m\ell^2\,\ddot{\theta}$$

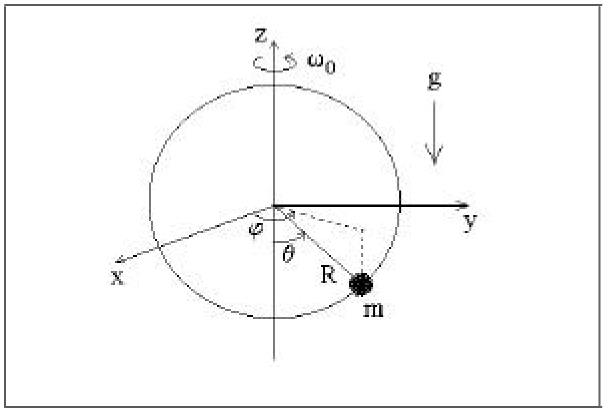
$$\frac{\partial L}{\partial\theta} \;=\; -\,mg\ell\,\sin\theta$$

or

$$\ddot{\theta} \;+\; \frac{g}{\ell}\,\sin\theta \;=\; 0$$

2.2 Example II: Bead on a Rotating Hoop

Consider a bead of mass m sliding freely on a hoop of radius R rotating with angular velocity ω_0 in a constant gravitational field with acceleration g.



Here, since the bead of the rotating hoop moves on the surface of a sphere of radius R, we use the generalized coordinates given by the two angles θ (measured from the negative z-axis) and φ (measured from the positive x-axis), where $\dot{\varphi} = \omega_0$. The position of the bead is given as

$$
\begin{aligned}
x(\theta, t) &= R \sin\theta \; \cos(\varphi_0 + \omega_0 t), \\
y(\theta, t) &= R \sin\theta \; \sin(\varphi_0 + \omega_0 t), \\
z(\theta, t) &= -\, R \; \cos\theta,
\end{aligned}
$$

where $\varphi(t) = \varphi_0 + \omega_0\, t$ and its associated velocity components are

$$
\begin{aligned}
\dot{x}(\theta, \dot{\theta}; t) &= R\left(\dot{\theta}\; \cos\theta\; \cos\varphi \;-\; \omega_0\; \sin\theta\; \sin\varphi\right), \\
\dot{y}(\theta, \dot{\theta}; t) &= R\left(\dot{\theta}\; \cos\theta\; \sin\varphi \;+\; \omega_0\; \sin\theta\; \cos\varphi\right), \\
\dot{z}(\theta, \dot{\theta}; t) &= R\, \dot{\theta}\; \sin\theta,
\end{aligned}
$$

so that the kinetic energy of the bead is

$$
K(\theta, \dot{\theta}) \;=\; \frac{m}{2}\, |\mathbf{v}|^2 \;=\; \frac{m\, R^2}{2}\left(\dot{\theta}^2 \;+\; \omega_0^2\; \sin^2\theta\right).
$$

The gravitational potential energy is

$$
U(\theta) \;=\; mgR\,(1 \;-\; \cos\theta),
$$

where we chose the zero potential energy point at $\theta = 0$ (see Figure above). The Lagrangian $L = K - U$ is, therefore, written as

$$
L(\theta, \dot{\theta}) \;=\; \frac{m\, R^2}{2}\left(\dot{\theta}^2 \;+\; \omega_0^2\; \sin^2\theta\right) \;-\; mgR\,(1 - \cos\theta),
$$

and the Euler-Lagrange equation for θ is

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\,\dot{\theta} \;\rightarrow\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mR^2\,\ddot{\theta}$$

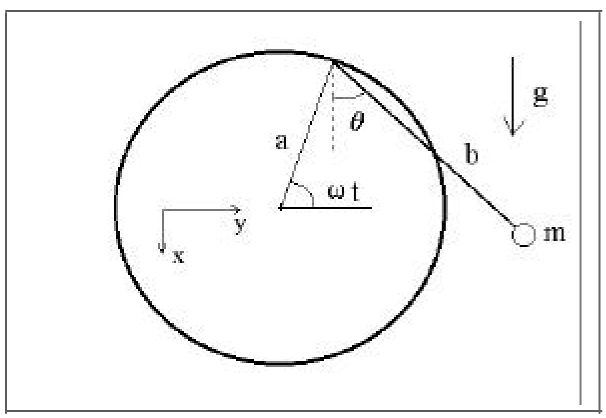
$$\frac{\partial L}{\partial \theta} = -\,mgR\,\sin\theta + mR^2\omega_0^2\,\cos\theta\,\sin\theta$$

or

$$\ddot{\theta} + \sin\theta\left(\frac{g}{R} - \omega_0^2\,\cos\theta\right) = 0$$

2.3 Example III: Rotating Pendulum

Consider a pendulum of mass m and length b attached to the edge of a disk of radius a rotating at angular velocity ω in a constant gravitational field with acceleration g.



Placing the origin at the center of the disk, the coordinates of the pendulum mass are

$$x = -\,a\,\sin\omega t + b\,\cos\theta$$
$$y = a\,\cos\omega t + b\,\sin\theta$$

so that the velocity components are

$$\dot{x} = -\,a\omega\,\cos\omega t - b\dot{\theta}\,\sin\theta$$
$$\dot{y} = -\,a\omega\,\sin\omega t + b\dot{\theta}\,\cos\theta$$

and the squared velocity is

$$v^2 = a^2\omega^2 + b^2\dot{\theta}^2 + 2\,ab\,\omega\,\dot{\theta}\,\sin(\theta - \omega t).$$

Setting the zero potential energy at $x = 0$, the gravitational potential energy is

$$U = -\,mg\,x = mga\,\sin\omega t - mgb\,\cos\theta.$$

The Lagrangian $L = K - U$ is, therefore, written as

$$
\begin{aligned}
L(\theta, \dot{\theta}; t) \;\; &= \;\; \frac{m}{2} \left[\, a^2\omega^2 \; + \; b^2\dot{\theta}^2 \; + \; 2\, ab\, \omega\, \dot{\theta} \; \sin(\theta - \omega t) \, \right] \\
& \quad - \; mga \, \sin\omega t \; + \; mgb \, \cos\theta,
\end{aligned} \tag{5}
$$

and the Euler-Lagrange equation for θ is

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} \; &= \; mb^2 \, \dot{\theta} + m\, ab\omega \; \sin(\theta - \omega t) \;\; \rightarrow \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) \; &= \; mb^2 \, \ddot{\theta} \; + \; m\, ab\omega \, (\dot{\theta} - \omega) \; \cos(\theta - \omega t)
\end{aligned}
$$

and

$$
\frac{\partial L}{\partial \theta} \; = \; m\, ab\, \omega\, \dot{\theta} \, \cos(\theta - \omega t) \; - \; mg\, b \, \sin\theta
$$

or

$$
\ddot{\theta} \; + \; \frac{g}{b} \, \sin\theta \; - \; \frac{a}{b}\, \omega^2 \; \cos(\theta - \omega t) \; = \; 0
$$

We recover the standard equation of motion for the pendulum when a or ω vanish.

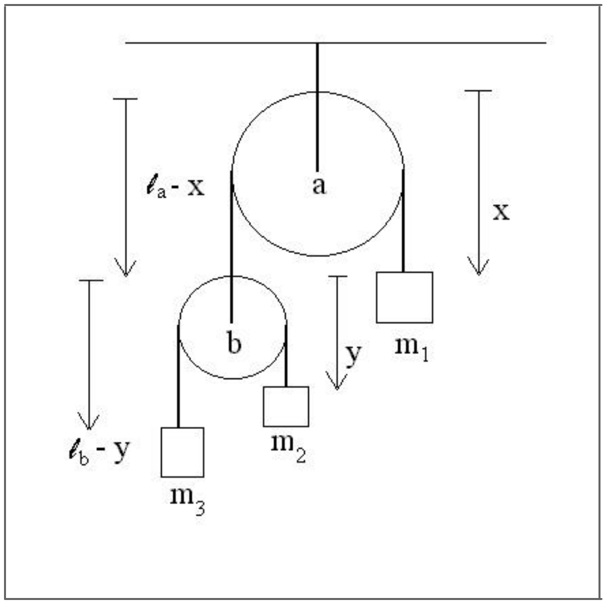
Note that the terms $[(m/2)\, a^2\omega^2]$ and $[-\, mga \, \sin\omega t]$ in the Lagrangian (5) play no role in determining the dynamics of the system. In fact, as can easily be shown, a Lagrangian L is always defined up to an exact time derivative, i.e., the Lagrangians L and $L' = L + df/dt$, where $f(\mathbf{q}, t)$ is an arbitrary function, lead to the same Euler-Lagrange equations. In the present case,

$$
f(t) \; = \; [(m/2)\, a^2\omega^2]\, t + (mga/\omega) \; \cos\omega t
$$

and thus this term can be omitted from the Lagrangian (5) without changing the equations of motion.

2.4 Example IV: Compound Atwood Machine

Consider a compound Atwood machine composed three masses (labeled m_1, m_2, and m_3) attached by two massless ropes through two massless pulleys in a constant gravitational field with acceleration g. The two generalized coordinates for this system (see Figure) are the distance x of mass m_1 from the top of the first pulley and the distance y of mass m_2 from the top of the second pulley; here, the lengths ℓ_a and ℓ_b are constants.



The coordinates and velocities of the three masses m_1, m_2, and m_3 are

$$\begin{aligned} x_1 &= x \;\rightarrow\; v_1 = \dot{x}, \\ x_2 &= \ell_a - x + y \;\rightarrow\; v_2 = \dot{y} - \dot{x}, \\ x_3 &= \ell_a - x + \ell_b - y \;\rightarrow\; v_3 = -\dot{x} - \dot{y}, \end{aligned}$$

respectively, so that the total kinetic energy is

$$K = \frac{m_1}{2}\,\dot{x}^2 + \frac{m_2}{2}\,(\dot{y} - \dot{x})^2 + \frac{m_3}{2}\,(\dot{x} + \dot{y})^2 .$$

Placing the zero potential energy at the top of the first pulley, the total gravitational potential energy, on the other hand, can be written as

$$U = -\,g\,x\,(m_1 - m_2 - m_3) - g\,y\,(m_2 - m_3),$$

where constant terms were omitted. The Lagrangian $L = K - U$ is, therefore, written as

$$\begin{aligned} L(x, \dot{x}, y, \dot{y}) = &\frac{m_1}{2}\,\dot{x}^2 + \frac{m_2}{2}\,(\dot{x} - \dot{y})^2 + \frac{m_3}{2}\,(\dot{x} + \dot{y})^2 \\ &+ g\,x\,(m_1 - m_2 - m_3) + g\,y\,(m_2 - m_3). \end{aligned}$$

The Euler-Lagrange equation for x is

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= (m_1 + m_2 + m_3)\,\dot{x} + (m_3 - m_2)\,\dot{y} \;\rightarrow \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= (m_1 + m_2 + m_3)\,\ddot{x} + (m_3 - m_2)\,\ddot{y} \\ \frac{\partial L}{\partial x} &= g\,(m_1 - m_2 - m_3) \end{aligned}$$

while the Euler-Lagrange equation for y is

$$\begin{aligned}
\frac{\partial L}{\partial \dot{y}} &= (m_3 - m_2)\,\dot{x} \;+\; (m_2 + m_3)\,\dot{y} \;\;\rightarrow \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) &= (m_3 - m_2)\,\ddot{x} \;+\; (m_2 + m_3)\,\ddot{y} \\
\frac{\partial L}{\partial y} &= g\;(m_2 - m_3)
\end{aligned}$$

or

$$\begin{aligned}
(m_1 + m_2 + m_3)\,\ddot{x} + (m_3 - m_2)\,\ddot{y} &= g\;(m_1 - m_2 - m_3) \\
(m_3 - m_2)\,\ddot{x} \;+\; (m_2 + m_3)\,\ddot{y} &= g\;(m_2 - m_3)
\end{aligned}$$

3 Symmetries and Conservation Laws

The Noether theorem states that for each symmetry of the Lagrangian there corresponds a conservation law (and *vice versa*). When the Lagrangian L is invariant under a time translation, a space translation, or a spatial rotation, the conservation law involves energy, linear momentum, or angular momentum, respectively.

When the Lagrangian is invariant under time translations, $t \rightarrow t + \delta t$, the Noether theorem states that energy is conserved, $dE/dt = 0$, where

$$E \;=\; \frac{d\mathbf{q}}{dt}\cdot\frac{\partial L}{\partial \dot{\mathbf{q}}} \;-\; L$$

defines the energy invariant. When the Lagrangian is invariant under spatial translations or rotations, $\alpha \rightarrow \alpha + \delta\alpha$, the Noether theorem states that the component

$$p_\alpha \;=\; \frac{\partial L}{\partial \dot{\alpha}} \;=\; \frac{\partial \mathbf{q}}{\partial \alpha}\cdot\frac{\partial L}{\partial \dot{\mathbf{q}}}$$

of the canonical momentum $\mathbf{p} = \partial L/\partial \dot{\mathbf{q}}$ is conserved, $dp_\alpha/dt = 0$. Note that if α is a *linear* spatial coordinate, p_α denotes a component of the linear momentum while if α is an *angular* spatial coordinate, $p_\alpha = L_\zeta$ denotes a component of the angular momentum (with ζ denoting the axis of symmetry about which α is measured).