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Unifying Input-Output and State-Space Perspectives of Predictive Control

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Abstract

Predictive control refers to the concept where the current control decision is based on a prediction of the system controlled response a number of times into the future. Originally derived from input-output models, recent years have seen efforts to interpret and develop predictive control from the state-space domain. This paper presents a formulation that unifies the two perspectives. Although a general state-space model is used as the starting point of the derivation, the predictive controller gains are ultimately synthesized from input-output data. There is no need for explicit model identification, and no observer is required in the implementation. Central to this development is a special interaction matrix which can be explained in the context of a "generalized" Cayley-Hamilton theorem. Experimental results are provided to illustrate this unified perspective of predictive control, including a comparison between data-based and model-based predictive controller designs.

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1. Introduction

Predictive control is the concept in which the current control action is derived by minimizing the difference between a prediction of the system (controlled) output and the desired output at some time step in the future. Originated from chemical process engineering,[1] several input-output model based predictive control methods have emerged, including Model Algorithmic Control (MAC),[2] Dynamic Matrix Control (DMC),[3] Extended Prediction Self-Adaptive Control (EPSAC),[4] Extended Horizon Adaptive Control,[5] MUltiStep Multivariable Adaptive Regulator (MUSMAR),[6] and the well-known Generalized Predictive Control (GPC).[7,8] Sharing the same common philosophy, the details of these controllers are different from each other due to different choices of cost functions, constraints, and dynamic models. Recently, the predictive control concept has made its way into the aerospace control community.[9-11]

Predictive control has gained popularity in recent years because the concept is intuitively appealing, the theory is relatively easy to understand, and actually works in practice. From the input-output perspective, one starts out with an one-step ahead auto-regressive representation, turns it into a multi-step ahead prediction model, specifies a cost function, and minimizes it to obtain a predictive control law. Since the dynamic model is in input-output form, and so is the cost function, it follows naturally that the controller assumes a dynamic output feedback form. There is no state-space model to consider, no observer to design, no Riccati equation to solve. On the other hand, much of modern control theory is built on the state-space model where the state information

provides direct insight into the system dynamics that is implicit in the input-output view. There has been considerable effort to interpret predictive control from the state-space perspective.[12-15] To this end, some basic issues remain. For example, if the starting point is a state-space representation in a general coordinate system, how can one justify the existence and structures of various input-output predictive models? One attractive feature of the input-output approach is that the model parameters can be identified directly from input-output data. Identification of a state-space model from input-output data, however, is not a trivial task. Is there a way to avoid explicit state-space model identification? Unlike chemical process problems where it is preferable to use step or impulse responses to design a controller, dynamic response to a rich excitation is often available or can be obtained easily in the control of flexible structures. What is an efficient strategy to synthesize a predictive controller from this type of data without having to resort to model identification?

Central to a clear understanding of these issues is seeing how a general state-space model is related to various input-output predictive models. Although one may start with a state-space model and wish to remain with it, unless the model is exactly known one ultimately still has to arrive at input-output models because information about the system comes in the form of input-output data. Of course, the relationship from an input-output model to equivalent state-space models and vice versa can be explained through certain specialized canonical forms and the well-known Cayley-Hamilton theorem, respectively. The downside of this connection is that as canonical forms are concerned it requires a non-trivial jump in going from the single-input single-output (SISO) case to

the multi-input multi-output (MIMO) case. This extension may be simple in concepts, but the details are complex, hence it is usually considered as difficult,[16] and extensive literature is devoted to this topics. Consequently, it is not immediately clear how to justify the existence of certain input-output maps and not others, how to characterize the freedom in going from a general state-space model to various input-output predictive models, and why for a multi-output system, the order of the difference equation can be less than the dimension of the state vector in a state-space representation. In this paper we offer a way to close this conceptual gap by introducing a new mechanism to relate the two kinds of models. This is accomplished through a special interaction matrix that can be explained as a generalization to the standard Cayley-Hamilton theorem. Once this step is accomplished, the divergence that arises when one starts the derivation with an input-output model or a state-space model becomes less significant. In other words, predictive control derived within this framework would then have both the state-space and input-output models merge into one common package.

We first start out with a state-space model, and describe a mechanism by which various kinds of input-output models can be justified and produced. We show how a predictive control law can be derived with these results. We then describe how to identify only the relevant parameters needed for this predictive control law, and thus avoiding the need for explicit state-space model identification or observer design. Following the theoretical development, experimental results will be presented to illustrate how these controllers actually work in practice on a very lightly damped and flexible system, including a comparison between data-based and model-based design approaches.

2. Mathematical Preliminaries

Consider an n-th order, r-input, m-output discrete-time model of a system in state-space form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

In the following we describe a key step in bridging the above state space model to various input-output models through the introduction of a special interaction matrix. This bridge will be useful for later use in system identification and controller design. By repeated substitution we have for some $p \geq 0$,

$$\begin{aligned} x(k+p) &= A^p x(k) + \mathcal{C}_p u_p(k) \\ y_p(k) &= \mathcal{O}_p x(k) + \mathcal{T}_p u_p(k) \end{aligned} \tag{2}$$

where $u_p(k)$ and $y_p(k)$ are column vectors of input and output data going into p steps into the future starting with $u(k)$ and $y(k)$, respectively,

$$u_p(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+p-1) \end{bmatrix}, \quad y_p(k) = \begin{bmatrix} y(k) \\ y(k+1) \\ \vdots \\ y(k+p-1) \end{bmatrix} \tag{3}$$

For a sufficiently large value of p, $\mathcal{C}_p$ is an $n \times pr$ controllability matrix, $\mathcal{O}_p$ a $pm \times n$ observability matrix, $\mathcal{T}_p$ a $pm \times pr$ Toeplitz matrix of the system Markov parameters,

$$\mathcal{C}_p = \left[A^{p-1}B, \ \cdots, AB, B\right], \quad \mathcal{O}_p = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}, \quad \mathcal{T}_p = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & \ddots & \ddots & \vdots \\ CAB & CB & D & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ CA^{p-2}B & \dots & CAB & CB & D \end{bmatrix} \tag{4}$$

As long as $pm \geq n$ then it is guaranteed for an observable system that a matrix M exists such that

$$A^p + M\mathcal{O}_p = 0 \tag{5}$$

so that for $k \geq 0$, Eq. (2) can be expressed as

$$\begin{aligned} x(k+p) &= A^p x(k) + \mathcal{C}_p u_p(k) \\ &= A^p x(k) + \mathcal{C}_p u_p(k) + M\left[\mathcal{O}_p x(k) + \mathcal{T}_p u_p(k)\right] - M y_p(k) \\ &= \left(A^p + M\mathcal{O}_p\right)x(k) + \left(\mathcal{C}_p + M\mathcal{T}_p\right)u_p(k) - M y_p(k) \\ &= \left(\mathcal{C}_p + M\mathcal{T}_p\right)u_p(k) - M y_p(k) \end{aligned} \tag{6}$$

The interaction matrix M allows us to eliminate explicit dependence of the state variable in the right hand side of Eq. (2). Of course, if the system is known exactly then M can be easily computed from Eq. (5). More interestingly, the existence of M will be used to justify various input-output model structures, and to determine how the predictive

controller gains can be computed from input-output data. This step can be done without actually having to compute M explicitly nor having to know the original system model itself.

3. Multi-Step Ahead Input-Output Model

In the following, we describe how the above interaction matrix allows us to produce directly from the state-space model the well-known multi-step ahead output predictor model that is commonly used in predictive control theory. Normally, this step is done by solving a set of Diophantine equations starting with a one-step ahead input-output model. The connection to a state-space model can be traced through specialized cannonical forms, but in so doing, the algebra will become rather complicated quickly to the point where insights into the conversion are lost. With the use of an interaction matrix M, however, the conversion process becomes extremely transparent. First note that

$$x(k+q) = A^q x(k) + \mathcal{C}_q u_q(k) \tag{7}$$

Also, we re-write Eq. (6) by shifting the time indices back so that for $k \geq p$, the expression for the current state $x(k)$ is

$$x(k) = \left(\mathcal{C}_p + M\mathcal{T}_p\right)u_p(k-p) - My_p(k-p) \tag{8}$$

Next, substituting Eq. (8) into Eq. (7) yields

$$x(k+q) = A^q\left(\boldsymbol{e}_p + M\boldsymbol{\tau}_p\right)u_p(k-p) - A^q M y_p(k-p) + \boldsymbol{e}_q u_q(k) \tag{9}$$

Substituting Eq. (9) into $y(k+q) = Cx(k+q) + Du(k+q)$, we immediately arrive at a multi-step ahead input-output model,

$$y(k+q) = CA^q\left(\boldsymbol{e}_p + M\boldsymbol{\tau}_p\right)u_p(k-p) - CA^q M y_p(k-p) + \left[C\boldsymbol{e}_q,\ D\right]u_{q+1}(k) \tag{10}$$

When examining the input and output terms involving in the above expression,

$$u_p(k-p) = \begin{bmatrix} u(k-p) \\ u(k-p+1) \\ \vdots \\ u(k-1) \end{bmatrix},\quad y_p(k-p) = \begin{bmatrix} y(k-p) \\ y(k-p+1) \\ \vdots \\ y(k-1) \end{bmatrix},\quad u_{q+1}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+q) \end{bmatrix} \tag{11}$$

we observe that the expression for $y(k+q)$ involves past p past input values from $u(k-1)$ back to $u(k-p)$, p past output values from $y(k-1)$ to $y(k-p)$, the present input $u(k)$ and q future input values from $u(k+1)$ up to $u(k+q)$. Notice the absence of the output terms from $y(k)$ to $y(k+q-1)$. For this reason, it is a multi-step ahead output predictor model. The expression for $y(k+q)$ assumes the standard form

$$y(k+q) = \sum_{i=1}^{p} \alpha_i y(k-i) + \sum_{i=1}^{p} \beta_i u(k-i) + \sum_{i=0}^{q} \gamma_i u(k+i) \tag{12}$$

whose coefficients are related to the original space-space model *A, B, C, D* by

$$\begin{aligned}
\left[\alpha_p,\ \ldots,\ \alpha_2,\ \alpha_1\right] &= -CA^q M \\
\left[\beta_p,\ \ldots,\ \beta_2,\ \beta_1\right] &= CA^q\left(\boldsymbol{e}_p + M\boldsymbol{\tau}_p\right) \\
\left[\gamma_0,\ \gamma_1,\ \ldots,\ \gamma_{q-1}\right] &= C\boldsymbol{e}_q\ ,\quad \gamma_q = D
\end{aligned} \tag{13}$$

We note here that the coefficients γ_i in the multi-step model are precisely the system Markov parameters. For this reason, these models belong to the class of ARMarkov models which arise in connection with the problems of aerospace structural system identification and disturbance rejection.[17]

4. One-Step Ahead Input-Output Model

A special case of the multi-step ahead model is the well-known one-step ahead model known as Auto-Regressive moving average model with eXogenous input (ARX). This model can be easily derived by setting q to zero in Eq. (12),

$$y(k) = \sum_{i=1}^{p} \alpha_i y(k-i) + \sum_{i=1}^{p} \beta_i u(k-i) + \gamma_0 u(k) \tag{14}$$

The coefficients of the above ARX model are related to the state-space model by the same matrix M,

$$\begin{aligned} \left[\alpha_p,\ \ldots,\ \alpha_2,\ \alpha_1\right] &= -CM \\ \left[\beta_p,\ \ldots,\ \beta_2,\ \beta_1\right] &= C\left(\boldsymbol{e}_p + M\boldsymbol{\tau}\right)\ ,\quad \gamma_0 = D \end{aligned} \tag{15}$$

What is remarkable about these relationships is that they are extremely simple without any complexity in going from the SISO case to the MIMO case. We will now examine what these relationships reveal about the connection between input-output and state-space models.

5. Interpretation of the interaction matrix M

First, for a chosen value of p whether a particular input-output map exists depends on whether M exists. The existence condition is very simple. As long as p is chosen sufficiently large such that $pm \geq n$, then M is guaranteed to exist for an observable system. When M exists, whether it is unique will dictate whether a certain input-output map is unique. If $pm = n$ then M is unique. If $pm > n$ then M is not unique. One possible choice for M is $M = -A^p \boldsymbol{O}_p^+$ where $\boldsymbol{O}_p^+$ denotes the pseudo-inverse of $\boldsymbol{O}_p$ corresponding to the case where the norm of M is minimized. In general, any left-inverse of $\boldsymbol{O}_p$ can be used.

Second, the Cayley-Hamilton theorem is the mechanism by which various input-output models can be derived from a state space model. Since the interaction matrix M also provides this connection, it is useful to see how M is related to the standard

approach. To this end, partition the $n \times pm$ matrix M into p sub-matrices, each is an $n \times m$ matrix, $M = \left[M_1, M_2, \ldots, M_p\right]$. We then rewrite Eq. (5) as

$$A^p + \Lambda_p A^{p-1} + \cdots + \Lambda_2 A + \Lambda_1 = 0 \tag{16}$$

where each coefficient $\Lambda_i = M_i C$, $i = 1, 2, \ldots, p$, is an $n \times n$ matrix. Equation (16) resembles the standard Cayley-Hamilton theorem, but with several important differences: (a) The coefficients Λ_i are matrices, not scalars. (b) The coefficients Λ_i are defined not only with respect to A but also an auxiliary matrix C. These coefficients are unique for an observable pair A and C if $pm = n$, and not unique if $pm > n$. (c) For multiple-output systems, p can be smaller than the true order n of the system whereas p must be at least n in Cayley-Hamilton case. For example, for a 4-th order observable system with two (independent) outputs, $n = 4$, $m = 2$, p can be taken to be as small as 2, which is less than 4, the true order of the system. The matrix coefficients Λ_i are unique in that case. Of course, any larger value of p can be used, in which case the coefficients Λ_i are not unique.

The contrast to the standard approach can be further seen by performing the following operation. Pre-multiplying Eq. (16) by C and post-multiplying it by B produces a relationship among the system Markov parameters,

$$CA^p B + \Gamma_p CA^{p-1} B + \cdots + \Gamma_2 CAB + \Gamma_1 CB = 0 \tag{17}$$

where $\Gamma_i = CM_i$, $i = 1, 2, \ldots, p$, and p is any (integer) value such that $pm \geq n$. Note that Γ_i now has M_i pre-multiplied by C as opposed to Λ_i having C pre-multiplied by M_i previously. Using the Cayley-Hamilton theorem, the corresponding version of Eq. (17) is

$$CA^n B + \lambda_n CA^{n-1} B + \cdots + \lambda_2 CAB + \lambda_1 CB = 0 \tag{18}$$

where the coefficients λ_i are scalars regardless of the number of outputs in the system. In contrast, the coefficients Γ_i are scalars only for the single-output case. In that particular case, if p is chosen to be n then $\Gamma_i = \lambda_i$ exactly. In the multi-output case, Γ_i are matrices. The higher-order Markov parameters can then be related to fewer than n lower-order Markov parameters, a result that is not at all obvious from the standard approach. The interaction matrix can be further generalized by noting that a more general version of Eq. (5) is $A^s + M\boldsymbol{O}_p = 0$ where s can be a negative or positive integer or zero. Backward-time, forward-time, mixed backward and forward-time, one-step, multi-step, and other specialized models can be derived under one common framework, each with its own interaction matrix.

In view of the above discussion, the above development can be thought of as a "generalized" version of the standard Cayley-Hamilton theorem. The generalized version characterizes the available freedom when a state-space model is converted into various input-output models. Questions regarding the existence of certain input-output maps for certain application-specific uses can be answered within this framework in a

straightforward manner. Trying to address such questions using the conventional approach would be very difficult, if not impossible. Since this is not the sole focus of the paper, we would not dwell into this matter further. We now show how the above results are applicable to the problem of predictive control.

6. Predictive Control

Predictive control refers to the strategy where the decision for the current control action is based on a minimization of a quadratic cost that involves a prediction of the system response some number of time steps into the future. To use the above developed models for predictive control, we start by assigning further meanings to the parameters p and q. Let q denote the prediction horizon and p denote the duration (the number of time steps) beyond which the prediction horizon is extended. In other words, if the current time step is k, then the output prediction horizon extends from $y(k+q)$ until $y(k+q+p-1)$. During this period the relevant control extends from the control to be used $u(k)$ until $u(k+q+p-1)$. One can form various cost functions from which the current control action to be used is the one that minimizes it. For example, one such cost function is

$$J = y_p^T(k+q)Qy_p(k+q) + u_{p+q}^T(k)Ru_{p+q}(k) \tag{19}$$

where for clarity, we write down the relevant terms explicitly,

$$y_p(k+q) = \begin{bmatrix} y(k+q) \\ y(k+q+1) \\ \vdots \\ y(k+q+p-1) \end{bmatrix}, \qquad u_{p+q}(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+q+p-1) \end{bmatrix} \tag{20}$$

The two parameters q and p in the above cost function are related to the minimum (q) and maximum ($q+p$) costing horizons in generalized predictive control.[8] One often sees another parameter known as the control horizon. Instead of extending the control input from $u(k)$ to $u(k+q+p-1)$ as done here, this parameter limits it to some $u(k+n_u-1)$ where $n_u \le q+p$ even though the output extends all the way to $y(k+q+p-1)$. Using $n_u < q+p$ is equivalent to setting the control from $u(k+n_u)$ to $u(k+q+p-1)$ to zero identically. This constraint can be trivially accommodated within our framework but its significance is questionable. As shown later in the section, one theoretical exception is setting n_u to a minimum q while imposing this constraint results in a predictive control equivalent of a full-state feedback deadbeat controller. In this particular situation, the control input vanishes identically after q steps. Otherwise it is difficult to justify why the control input should do so in finite time for any other feedback controllers.

Before performing the minimization we need an expression for $y_p(k+q)$,

$$\begin{aligned} y_p(k+q) &= \mathcal{O}_p x(k+q) + \mathcal{T}_p u_p(k+q) \\ &= \mathcal{O}_p A^q \left(\mathcal{C}_p + M\mathcal{T}_p \right) u_p(k-p) - \mathcal{O}_p A^q M y_p(k-p) + \mathcal{W} u_{p+q}(k) \end{aligned} \tag{21}$$

where $\mathcal{H}$ as defined in $\mathcal{W}$ is a $pm \times qr$ Hankel matrix of the system Markov parameters,

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{O}}_p \boldsymbol{\mathcal{C}}_q \ , \quad \mathcal{W} = \left[\boldsymbol{\mathcal{H}},\ \boldsymbol{\tau}_p\right] , \quad u_{p+q}(k) = \begin{bmatrix} u_q(k) \\ u_p(k+q) \end{bmatrix} \tag{22}$$

Note that there are no overlapping terms between $u_p(k-p)$ and $u_{p+q}(k)$. This check is important because any overlapping terms must be combined prior to performing the minimization. Substituting Eq. (21) into Eq. (19) then performing the minimization with respect to $u_{p+q}(k)$ yields,

$$u_{p+q}(k) = \mathcal{A}_1 u_p(k-p) + \mathcal{A}_2 y_p(k-p) \tag{23}$$

where

$$\mathcal{A}_1 = -\mathcal{B}\boldsymbol{\mathcal{O}}_p A^q \left(\boldsymbol{\mathcal{C}}_p + M\boldsymbol{\tau}_p\right), \qquad \mathcal{A}_2 = \mathcal{B}\boldsymbol{\mathcal{O}} A^q M , \qquad \mathcal{B} = \left(R + \mathcal{W}^T Q \mathcal{W}\right)^+ \mathcal{W}^T Q \tag{24}$$

The predictive controller assumes the general dynamic output feedback form,

$$u(k) = \sum_{i=1}^{p} G_i u(k-i) + \sum_{i=1}^{p} H_i y(k-i) \tag{25}$$

where the controller gains $\left[G_p,\ G_{p-1},\ \cdots,\ G_1\right]$ and $\left[H_p,\ H_{p-1},\ \cdots,\ H_1\right]$ are the first r rows of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.

We have noted before that there exists a lower bound for p, $pm \geq n$. We now examine if there is a similar lower bound for q. Starting with the current state $x(k)$, the deadbeat control sequence that will drive the state to zero in q steps must satisfy $A^q x(k) + \mathcal{C}_q u_q(k) = 0$. To insure existence of such a control, $\mathcal{C}_q$ must be full row rank, i.e., for a controllable system we must have $qr \geq n$. This is the requirement needed for full state suppression as in the vibration suppression problem. Solving for $u_q(k)$ yields

$$\begin{aligned} u_q(k) &= -\mathcal{C}_q^+ A^q x(k) \\ &= -\mathcal{C}_q^+ A^q \left(\mathcal{C}_p + M\mathcal{T}_p\right) u_p(k-p) + \mathcal{C}_q^+ A^q M y_p(k-p) \end{aligned} \tag{26}$$

To see how this same result can be derived from Eq. (21), we set $y_p(k+q) = 0$ and $u_p(k+q) = 0$, and solve for $u_q(k)$

$$\begin{aligned} u_q(k) &= \mathcal{C}_q^+ \mathcal{O}_p^+ \mathcal{O}_p \left[-A^q \left(\mathcal{C}_p + M\mathcal{T}_p\right) u_p(k-p) + A^q M y_p(k-p)\right] \\ &= -\mathcal{C}_q^+ A^q \left(\mathcal{C}_p + M\mathcal{T}_p\right) u_p(k-p) + \mathcal{C}_q^+ A^q M y_p(k-p) \end{aligned} \tag{27}$$

which is exactly the same as Eq. (26) above. Thus for full state suppression then there is a lower bound for q, $qr \geq n$. If q is chosen smaller than this minimum value, denoted by $q_{\min}$, say $q = 0$, then the control tries to suppress the measured output immediately, an extreme situation known as one-step ahead control. Both of these cases will be discussed and illustrated with experimental results.

7. Identification of Predictive Controller Gains from Input-Output Data

If a model of the system is known exactly, then Eqs. (24) provide the formulas for the predictive controller gains. A more useful case is where a model of the system is not known but only input-output data is available. One option would be to identify a state-space model of the system first, compute the appropriate M, and then use Eqs. (24) to obtain the controller gains. This option turns out to be unnecessary because the parameter combinations that are involved in the controller gains can be identified directly. The ability to bypass traditional model identification is one major advantage of this predictive controller design. Let us now examine what specific parameter combinations are needed. First, to build $\mathcal{B}$ we need the combination $\mathcal{W} = \left[\boldsymbol{H},\ \boldsymbol{\tau}_p\right]$. To build $\mathcal{A}_1$ and $\mathcal{A}_2$ we also need the combinations $\mathcal{P}_1 = \boldsymbol{O}_p A^q\left(\boldsymbol{e}_p + M\boldsymbol{\tau}_p\right)$ and $\mathcal{P}_2 = \boldsymbol{O}_p A^q M$, respectively. To see how $\mathcal{W}$, $\mathcal{P}_1$, and $\mathcal{P}_2$ can be identified from input-output data, re-write Eq. (21) as

$$y_p(k+q) = \left[\mathcal{P}_1 \quad \mathcal{P}_2 \quad \mathcal{W}\right]\begin{bmatrix} u_p(k-p) \\ -y_p(k-p) \\ u_{p+q}(k) \end{bmatrix} \tag{28}$$

Given a set of input-output data of sufficient length, we can write for $k \geq p$,

$$Y = \left[y_p(p+q) \quad y_p(p+q+1) \quad \cdots \quad y_p(p+q+\ell-1)\right] \tag{29}$$

$$V = \begin{bmatrix} u_p(0) & u_p(1) & \cdots & u_p(\ell-1) \\ -y_p(0) & -y_p(1) & \cdots & -y_p(\ell-1) \\ u_{p+q}(p) & u_{p+q}(p+1) & \cdots & u_{p+q}(p+\ell-1) \end{bmatrix} \tag{30}$$

The unknown parameter combinations can then be computed from

$$\begin{bmatrix} \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{W} \end{bmatrix} = YV^{+} = YV^{T}\left(VV^{T}\right)^{+} \tag{31}$$

The controller gains are obtained from $\mathcal{A}_1 = -\mathcal{B}\mathcal{P}_1$, $\mathcal{A}_2 = \mathcal{B}\mathcal{P}_2$, $\mathcal{B} = \left(R + \mathcal{W}^T Q \mathcal{W}\right)^{+} \mathcal{W}^T Q$. To perform the pseudo-inverse of V or VV^T, good numerical practice calls for the use of the singular value decomposition where singular values that are zero or several orders of magnitudes smaller than the rest should be discarded. In so doing, one has the opportunity to produce a controller with gains smaller than that obtained if all singular values are kept. Note that this opportunity is not available if one is to compute the controller gains from a given state-space model instead. We also note here that whether the controller synthesis is model-based or data-based, the singular value decomposition should still be used in the computation of $\left(R + \mathcal{W}^T Q \mathcal{W}\right)^{+}$ especially when $R = 0$. Discarding insignificant singular values is an effective way to remove the source of numerical ill-conditioning if it arises. Finally, Eq. (31) is a batch-type solution, but recursive forms of the solution can be easily used for adaptive on-line implementation.

8. Experimental Results

A series of experiments is carried out to study the developed predictive controller designs on a compact yet very lightly damped and flexible system. The experimental setup consists of a series of parallel steel rods mounted on a thin center wire as shown in Figs. 1 and 2. A linear actuator controls the vertical movement of the tip of the first rod. A laser-based optical sensing system is used to detect the torsional motion of the third rod, and the last (tenth) rod is clamped to a rigid support. Once perturbed, it takes more than 30 seconds for the vibrations to subside. Data acquisition and active control are implemented through a PC-based D-Space system with a chosen sampling interval of 0.005 second. To obtain data for system identification and controller design, the system is excited by a random input for 80 seconds and the resultant output data is recorded.

The following steps are carried out in this study. First, we perform system identification to obtain a state-space model for the system. The technique used is OKID (Observer/Kalman filter Identification) [18,19] which produces a state-space model from open-loop input-output experimental data. This model is used to simulate closed-loop results for later comparison with actual experimental closed-loop results. Second, the same set of input-output data is used to identify the predictive controller gains directly as outlined in the previous section, thus bypassing the intermediate model identification step. These data-based controller gains are referred to as *identified* gains. Third, since a model of a system is available by identification, it can also be used to compute the predictive controller gains. These model-based controller gains are referred to as

computed gains, meaning they are computed from an experimentally identified state-space model.

System Identification

We now report the system identification results. It is established in OKID theory that optimal identification can be expected when a sufficiently large upper bound on the order of the input-output model is assumed for it to contain implicitly an optimal Kalman filter. For the system under consideration, setting $p = 100$ is quite sufficient for this purpose in that further increase in p does not improve the quality of the identification model further. As large as a value of $p = 200$ is used to make this assessment. The Bode plot of the identified 18th-order state-space model is shown in Fig. 3 clearly indicating that there are 9 flexible modes between 5 Hz and 40 Hz. Figure 4 shows a comparison between measured output and predicted output with the identified state-space model for the first 5 seconds of data. The high level of matching between the two time histories confirms that the identified model is quite accurate as far as system identification is concerned. In this study, an accurate identification model is desired because it will be used to design model-based predictive controllers for later comparison with data-based designs.

Data-Based Predictive Control with Identified Gains

As mentioned before, the same set of data is used to compute the controller gains without having to obtain an intermediate model for the system. With p set at 100, we first illustrate the case with $q = 0$ corresponding to output (not full state) suppression.

Such a controller is equivalent to one-step ahead control. What can happen for a non-minimum phase system or identification model is that the control input will grow unbounded while trying to maintain the output at zero. Even for a continuous minimum phase system, discretization alone may cause unstable zeros to appear.[20] To avoid actuator saturation, a "small" control weighting is used, $R = 5 \times 10^{-4}$. The system is excited for about 1 second after which the controller is turned on. The controller quickly suppresses the output rod motion, but the input rod continues to vibrate, indicating that the remaining rods continue to vibrate. Figure 5 shows the simulated and actual open-loop (dashed) and closed-loop (solid) experimental results. Although the quality of the identified model is high from the perspective of system identification, the observed discrepancy between closed-loop simulated and experimental results should not be a surprise because setting the prediction horizon $q = 0$ produces a highly "aggressive" controller which attenuates modeling error. As discussed in the theoretical section, to suppress the vibration of the entire structure, the prediction horizon q must be chosen to be at least 18 in this case (for a 9-mode system). In our formulation choosing $q = 18$ with no control weighting does not result in a predictive control equivalent of a full-state feedback deadbeat controller because we are not setting $u(k + q_{\min})$ to $u(k + q_{\min} + p - 1)$ to zero. Nevertheless, working with a minimum q places high demand on control energy. In our set-up, this quickly saturates the actuator. Two options are available to handle the situation. One can use control input weighting as in the previous case, or simply increase the prediction horizon, which is the case reported here with $q = 25$. Figure 6 shows the simulated and actual open-loop and closed-loop experimental results. Notice that the control effort is well within reach, the vibration takes slightly longer to decay, the control

input converges to zero after the vibration is suppressed, and the agreement between simulated and experimental results improves significantly. When the prediction horizon is increased to $q = 200$, it takes longer for the controller to suppress the vibration but the control effort is significantly less, and the agreement between simulated and experimental results is even better, indicating reduced sensitivity to modeling error, Fig. 7.

Model-Based Predictive Control with Computed Gains

We now compare the above results with the case where the predictive controller gains are computed from the identified state-space model. Recall that in the identified gain case with $q = 0$, we already observe significant discrepancy between simulated and actual closed-loop results. Therefore it is not expected that a model-based predictive controller with $q = 0$ would work in practice. This is indeed the case without slowing it down considerably with input weighting. Figure 8 shows the case with $q = 25$ with no input weighting. The controller suppresses the most of the vibration but observe the difference between the simulated and actual closed-loop experimental results, and also to their identified gains counterparts shown previously in Fig. 6. One may argue that using $q = 25$ is still rather aggressive, and thus expect the situation to improve if a larger value of q is used. Indeed, this is the case observed when q is increased to $q = 200$. A much improved closed-loop result is obtained and closer agreement between simulation and experimental results is seen, Fig. 9. Upon closer examination, however, this closed-loop result is still a bit worse than its counterpart with identified gains as previously shown in Fig. 7. These results suggest that although the identified model is considered to be of good quality as one would hope for under normal circumstances, it is still difficult to

predict closed-loop performance accurately when used to design a model-based predictive controller. One is better off identifying the predictive controller gains directly from input-output data. The experimental results reported here are consistent with those reported in Ref. 21 where it was shown that it is better to identify a multi-step ahead observer from input-output data directly than to repeatedly propagate the one-step ahead model. Finally, we note that by truncating the singular values that are significantly smaller than the rest, the identified gains have much smaller magnitudes than the computed gains. This is a definite advantage of the direct identification approach.

9. Conclusions

In this paper we have presented a predictive control formulation that bridges the input-output and state-space approaches of predictive control. A key element in this bridge is the relationship between state-space and input-output models. Through a special interaction matrix which can be viewed as a generalization of the standard Cayley-Hamilton theorem, the relationship between the two kinds of models becomes extremely simple. The present predictive control formulation is neither input-output model based nor state-space model based, but rather it is both due to the transparent connection provided here. From the input-output perspective, the formulation clearly explains and justifies the structures of various input-output models. From the state-space perspective, it shows how the coefficients of various predictive models and the predictive controller gains are related to those of the original state-space model.

The experimental results reported in this paper also show among other things the benefit of designing a controller directly from input-output data as opposed to working through an intermediate identification model. The general philosophy here is to identify the needed parameters for a particular controller design instead of identifying a model of the system first from which the needed parameters for the controller are computed. This general philosophy will be further extended to the problem of state-space system identification with accurate order estimation, the problem of designing an optimal state estimator directly from input-output data, and the problem of periodic disturbance cancellation without explicit disturbance or system identification. For each of these problems, an appropriate interaction matrix arises that helps provide a considerably concise solution. These results will be reported in the immediate future.

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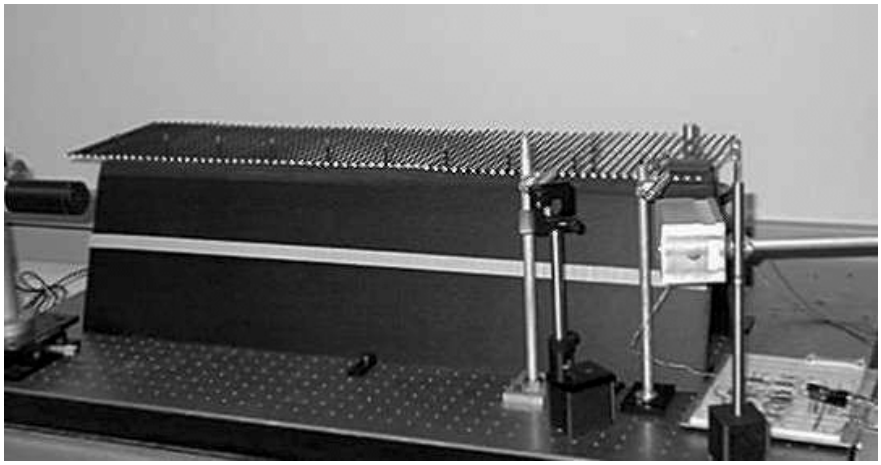

Figure 1. Flexible dynamic system.

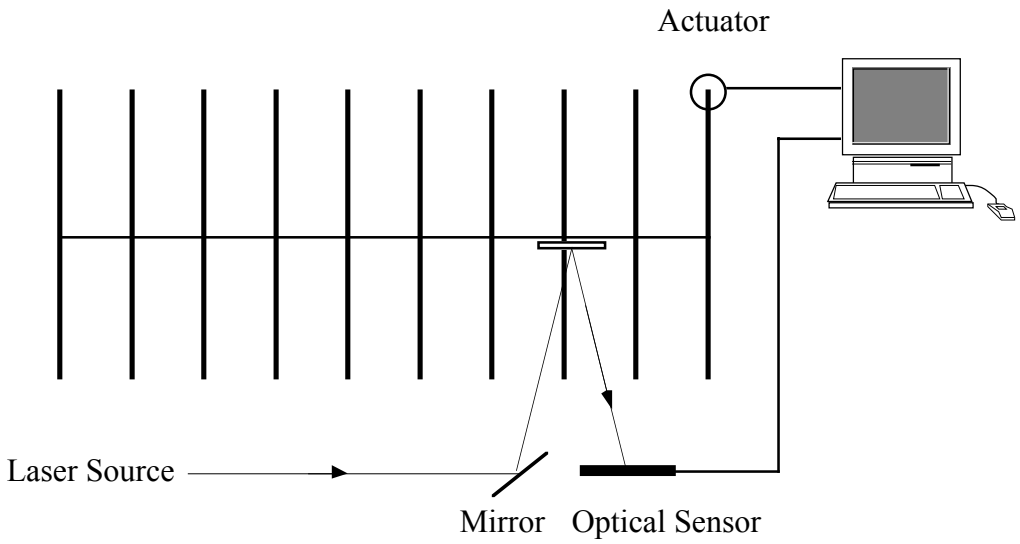

Figure 2. Schematic layout of experimental setup.

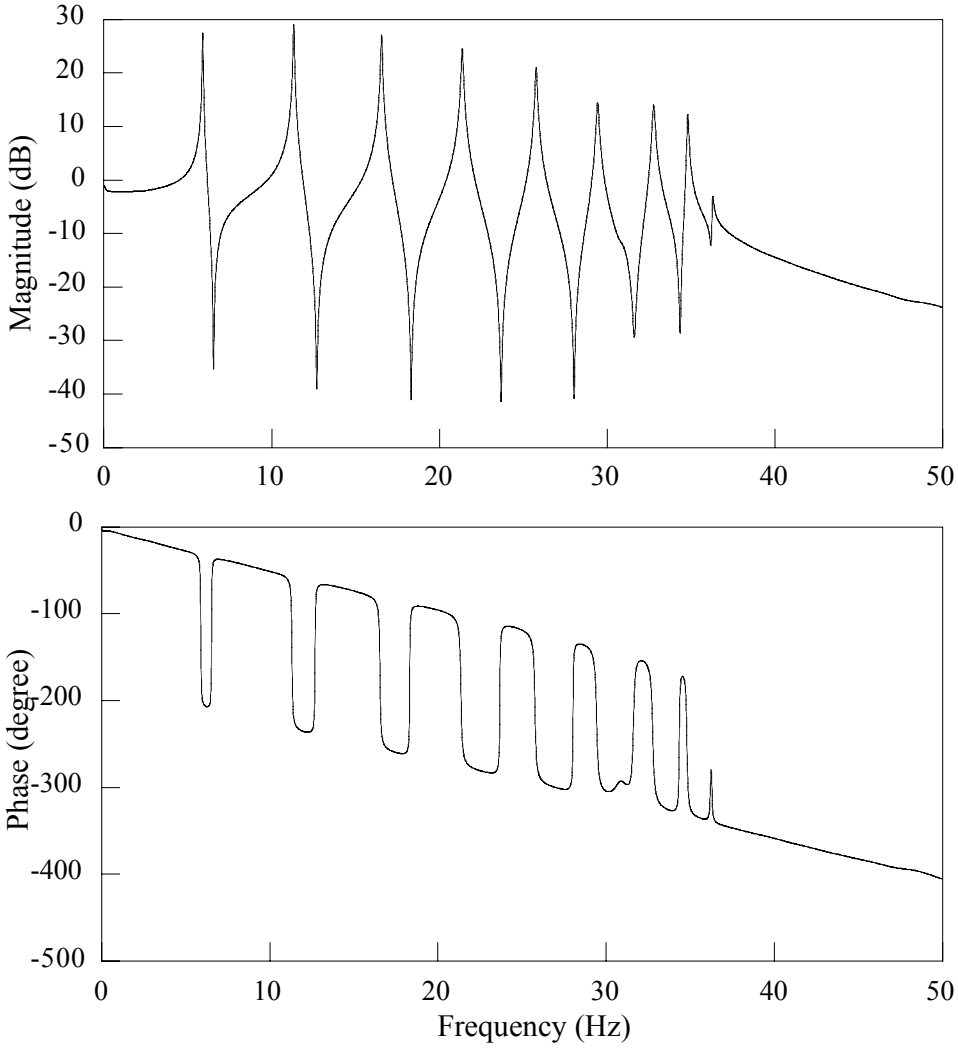

Figure 3. Frequency responses of identified model.

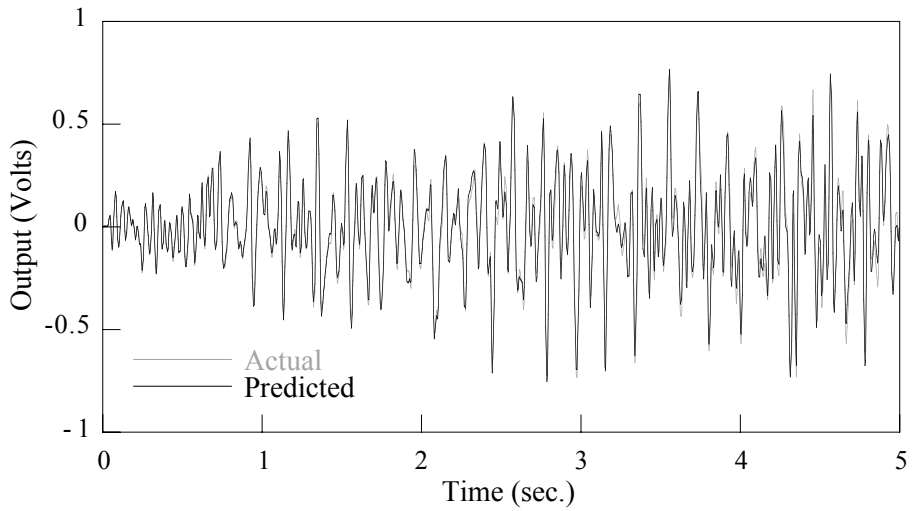

Figure 4. Predicted versus actual responses.

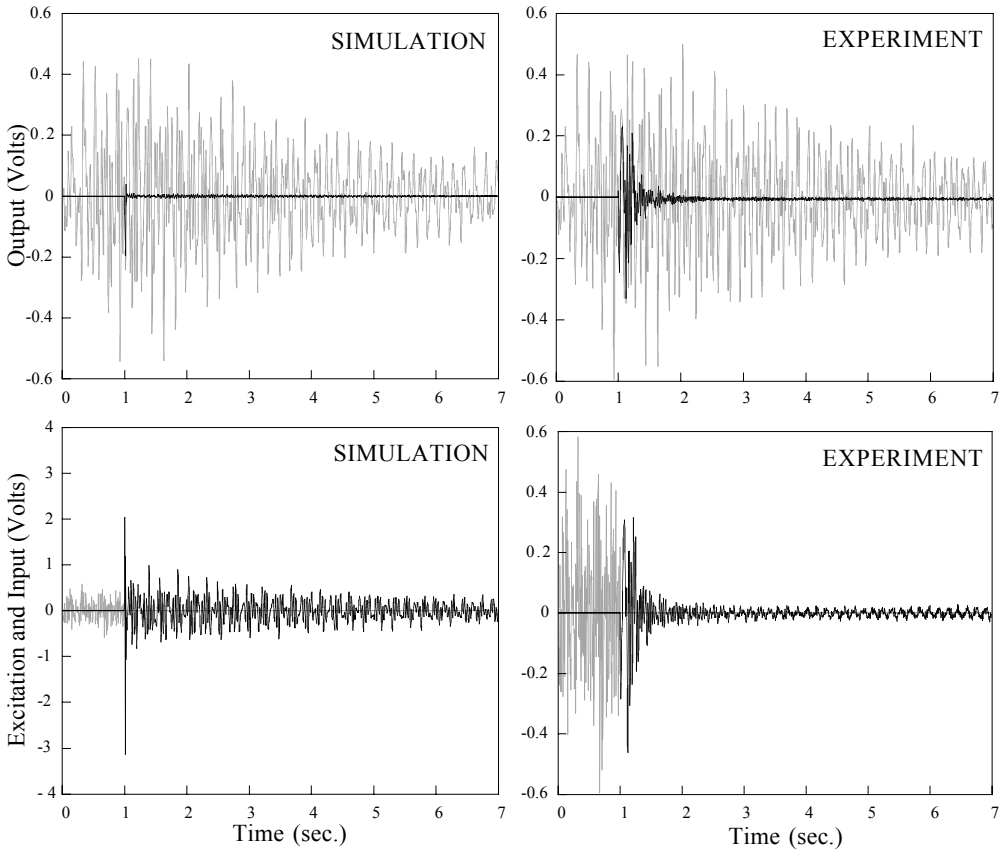

Figure 5. Output suppression with identified gains ($q = 0$).

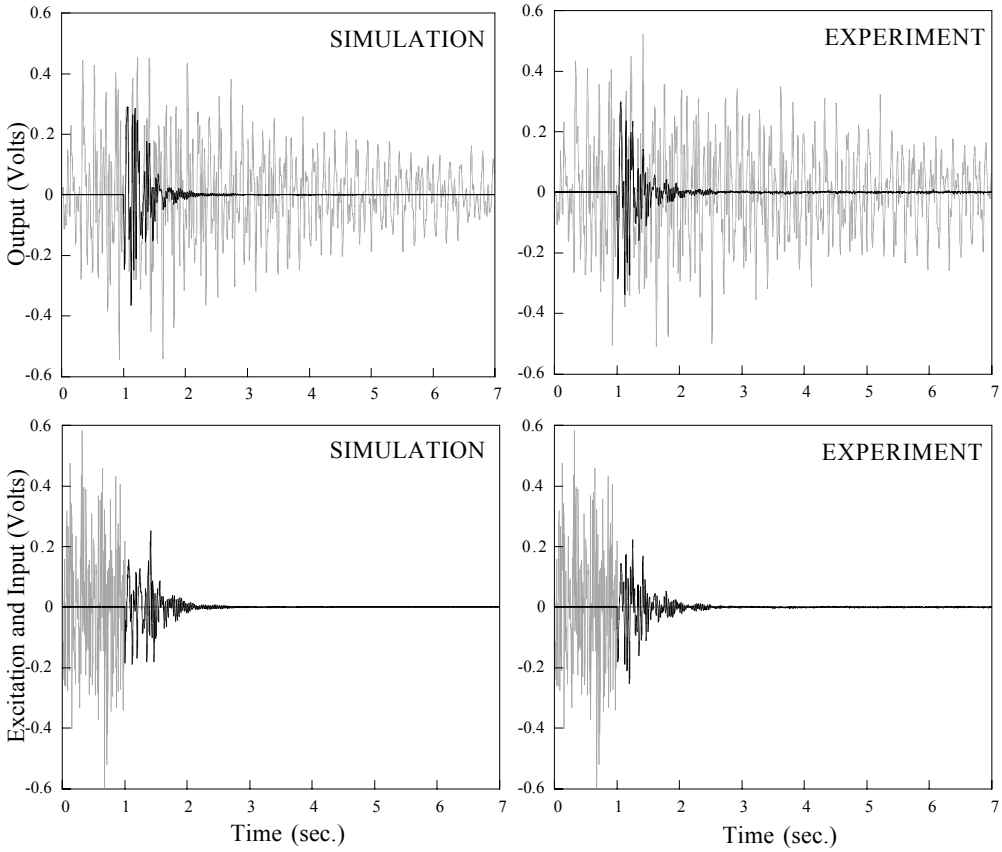

Figure 6. Vibration suppression with identified gains ($q = 25$).

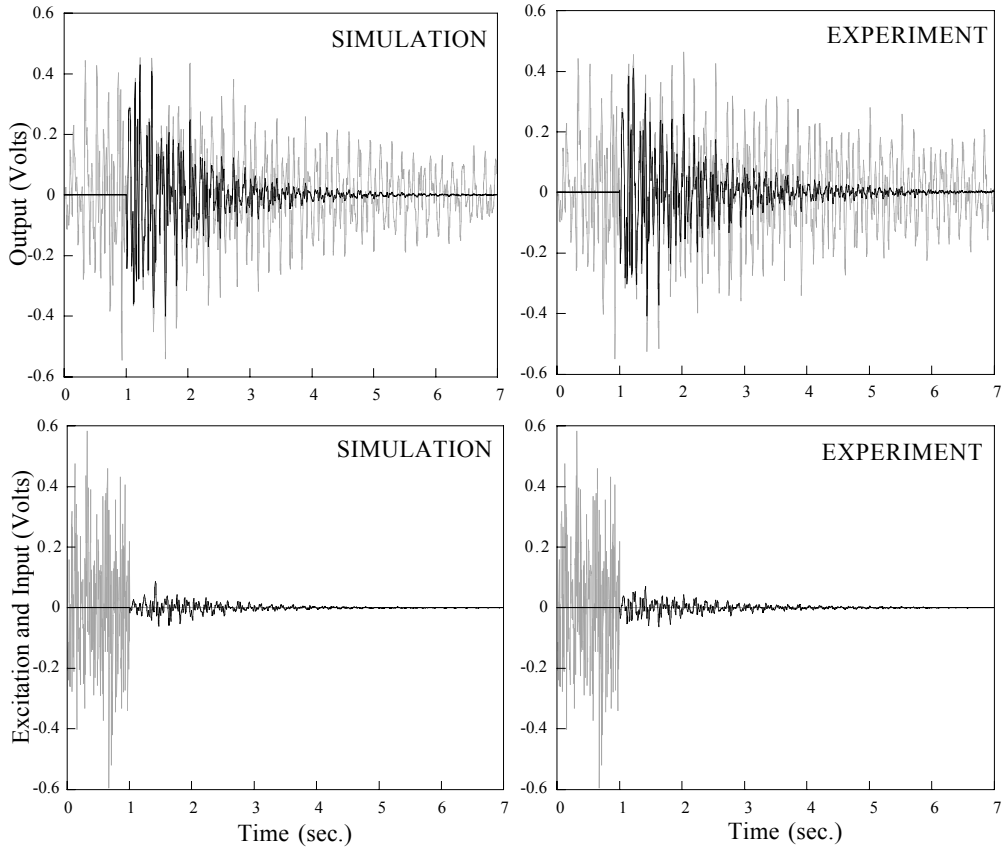

Figure 7. Vibration suppression with identified gains ($q = 200$).

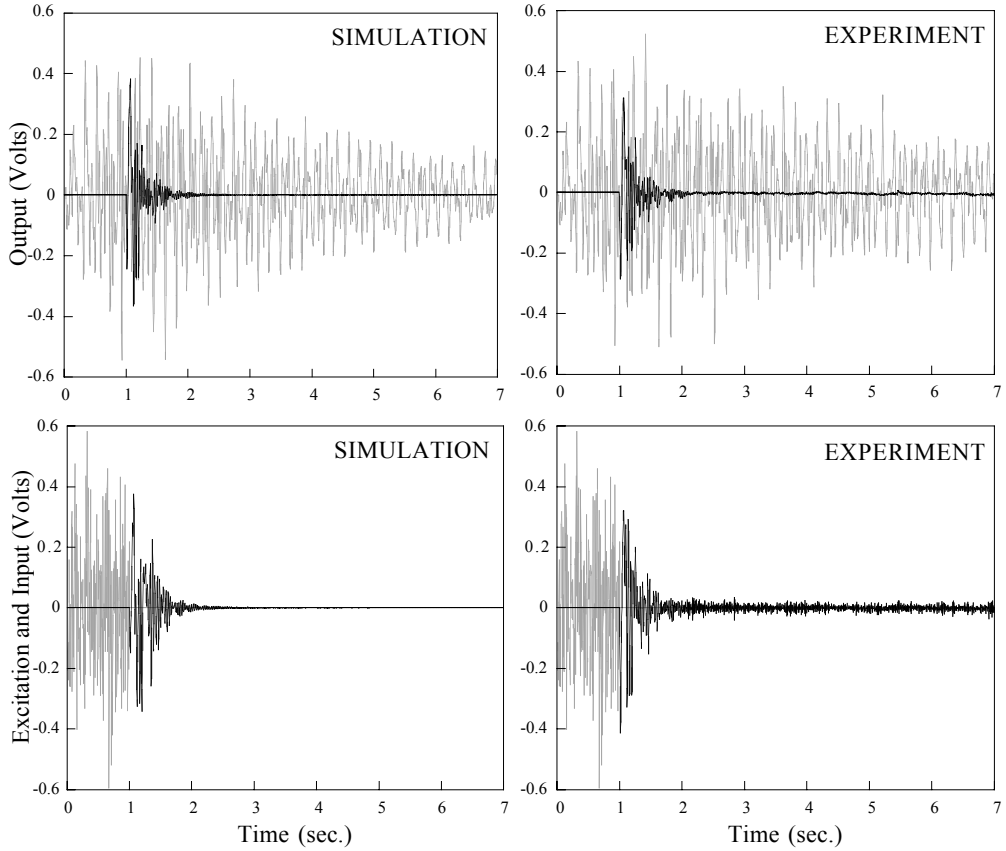

Figure 8. Vibration suppression with computed gains ($q = 25$).

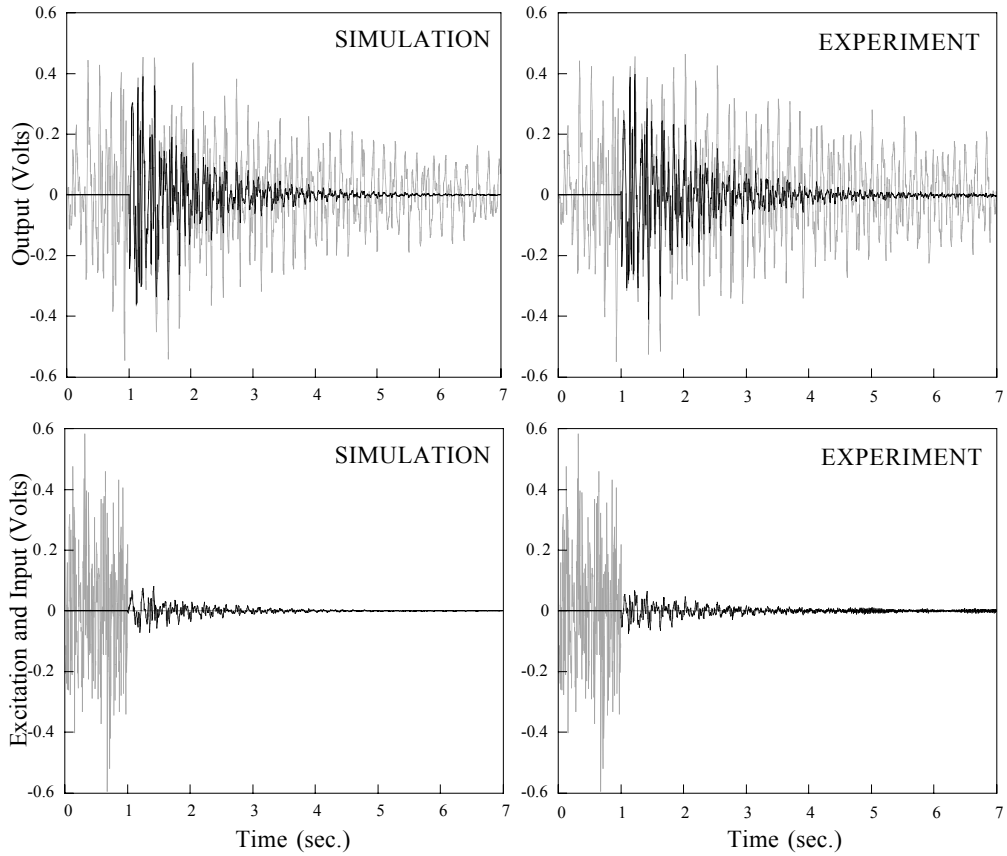

Figure 9. Vibration suppression with computed gains ($q = 200$).