Nonlinear Dynamics and Limit Cycle Analysis in Biomedical Engineering

Mike Kokko
December 1, 2017


Part I: Theory and Methods
• Linear Dynamics
• Nonlinear Dynamics
  • Phase Portraits
  • Limit Cycle Analysis (Poincare Maps)

Part II: Examples from Biomedical Literature

Physical System $\rightarrow$ Differential Equation $\rightarrow$ Full Trajectory
Nonlinear vs. Linearized Pendulum
Fixed Point Classification

\[ \dot{x} \approx Ax \]

- **Stable Nodes**
- **Unstable Nodes**
- **Stable Spirals / Foci**
- **Unstable Spirals / Foci**
- **Saddle Points**

**Stable Nodes**
- degenerate sink
- spiral sink
- uniform motion

**Unstable Nodes**
- degenerate source
- spiral source

**Saddle Points**
- line of stable fixed points
- saddle
- line of unstable fixed points

**Determinant**
- \( \det A = \frac{1}{4} (\text{Tr } A)^2 \)

https://i.stack.imgur.com/duPPi.png
Limit Cycles

- **Isolated**, closed orbits in phase plane (state space)
- Only possible in nonlinear systems
- **Proving** (or **ruling out**) existence in a region can be tricky
  - Gradient field?
  - Lyapunov function?
  - Dulac’s criterion?
  - Poincaré-Bendixson theorem?
- **Stable**, semi-stable, or unstable
van der Pol Oscillator Limit Cycle ($\mu = 1.0$)

\[
\ddot{x} + \mu (x^2 - 1) \dot{x} + x = 0
\]

\[
\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\mu (x_1^2 - 1) x_2 - x_1 \end{bmatrix}
\]
van der Pol Oscillator Limit Cycle \((\mu = 0.2)\)

\[
\ddot{x} + \mu (x^2 - 1) \dot{x} + x = 0
\]

\[
\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\mu (x_1^2 - 1) x_2 - x_1 \end{bmatrix}
\]

http://www.dos4ever.com/centenial/centenial_en.html
Poincaré Maps

- First return map relative to a **surface of section** \( S \) (\( P: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} \))
- In \( \mathbb{R}^2 \) fixed points and closed orbits fall on the line of slope 1
- "Easily" extended to higher dimensions
- Continuous time system becomes discrete \( \{ \dot{x} = f(x) \} \rightarrow \{ x_{k+1} = P(x_k) \} \)
- Stability related to eigenvalues of linearized \( P \) (slope < 1 in \( \mathbb{R}^2 \))

---

Henri Poincaré
1854 - 1912
Poincaré Maps

• First return map relative to a **surface of section** $S$ ($P: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$)
• In $\mathbb{R}^2$ fixed points and closed orbits fall on the line of slope 1
• “Easily” extended to higher dimensions
• Continuous time system becomes discrete $\{\dot{x} = f(x)\} \rightarrow \{x_{k+1} = P(x_k)\}$
• Stability related to eigenvalues of linearized $P$ (slope < 1 in $\mathbb{R}^2$)

Henri Poincaré
1854 - 1912

http://underactuated.mit.edu/underactuated.html?chapter=4
http://www.mlahanas.de/Physics/Bios/images/HenriPoincare.jpg
Gait Cycle Analysis

Phase Plane Analysis, Limit Cycles, Poincaré Maps

Eadweard Muybridge, 1887. Wikimedia Commons.
Gait Cycle Analysis

Fig. 1. Five-link biped robot.

Fig. 2. Poincaré map for a continuous system.


Gait Cycle Analysis

Fig. 3. (a) The detailed phase portraits of a typical female subject (the left column). (b) Phase portraits of healthy female subjects in sagittal plane (the middle column). (c) Phase portraits of healthy male subjects in sagittal plane (the right column). Phase plane portraits combine position and velocity data on a single plot. Steady state joint, velocities can be correlated directly with positions by eliminating the time variable.

Fig. 4. Sagittal maps of a normal male subject. First return maps are graphical tools that facilitate in distinguishing between transient and steady state locomotion. Steady state locomotion can be observed from clustering of points inside the shown circles.

\( \phi_1, \phi_4, \phi_7 \) — sagittal plane excursion of ankle, knee, and hip joints, respectively.

\( \phi_2, \phi_5, \phi_8 \) — coronal plane excursion of ankle, knee, and hip joints, respectively.

\( \phi_3, \phi_6, \phi_9 \) — transverse plane excursion of ankle, knee, and hip joints, respectively.

Gait Cycle Analysis

Figure 5: First return maps of comofemoral joint rotations. (a) for subject 1, and (b) for subject 2, under conditions of normal gait and gait during acute synovitis. The angular position of the comofemoral joint at equilibrium is denoted by $q_e$.  

Figure 7: The stability index for greyhound locomotion.

Circadian Rhythms

Limit Cycles

1am: Teenagers start melatonin production

4am: Morning people reach their lowest body temperature

7am: Cortisol levels spike, to prepare for the stresses of the day

2pm: Peak afternoon sleepiness – a good time for a nap

7pm: Highest alertness and fastest reaction times

10pm: Digestion slows and bowel movements are suppressed

3am: The deepest part of your sleep

6am: Night owls take another two hours to minimum temperature

8am: 6 per cent lower muscle performance than in the evening

6pm: Lowest levels of thyroid stimulating hormone

9pm: Adults begin producing melatonin, making them sleepy

Circadian Rhythms

\[
\left(\frac{12}{\pi}\right)^2 \ddot{x} + \mu (4x^2 - 1) \left(\frac{12}{\pi}\right) \dot{x} + \left(\frac{24}{\tau_x}\right)^2 x = 0
\]

\[
\mu = 0.13 \quad \text{and} \quad \tau_x = 24.2
\]

\[
\dot{x} = \left(\frac{\pi}{12}\right) \left[ x_c + \mu \left( x - \frac{4}{3} x^3 \right) \right] \quad \text{Core Body Temp.}
\]

\[
\dot{x}_c = -\left(\frac{\pi}{12}\right) \left(\frac{24}{\tau_x}\right)^2 x
\]

\[\dot{x} = \left(\frac{\pi}{12}\right) \left[ x_c + \mu \left( \frac{1}{3} x + \frac{4}{3} x^3 - \frac{256}{105} x^7 \right) + B \right]
\]

\[
\dot{x}_c = \left(\frac{\pi}{12}\right) \left( qBx_c - \left[ \left( \frac{24}{0.99729\tau_x} \right)^2 + kB \right] x \right)
\]


Figure 1. Phase response curves (PRC) to a three-cycle (5 h per cycle) bright light (-10,000 lux) stimulus. Initial phase is defined as the center of the light stimulus relative to the fitted minimum of the core body temperature measured during constant routine conditions (CBT\text{min}), which is defined as 0 h. Simulations from Kronauer’s (1990) model (dashed line) and our current model (solid line) are compared with experimental data (filled circles) from Khalsa et al. (1997). Note that one data point from that PRC was omitted because the subject (#1579) had undergone eye surgery prior to his experimental trial (Khalsa, personal communication, 1998).
Fig. 1. Experimental results from a 22-year-old man (subject 1111) living in an environment free of time cues on a 20-hour forced desynchrony protocol (left panel), a classical free-running protocol (center panel), and a 28-hour forced desynchrony protocol (right panel). The rest-activity cycle is plotted in a double raster format, with successive days plotted both next to and beneath each other and clock hour indicated on the abscissa. Baseline sleep episodes were scheduled at their habitual times (based on an average of their schedule during the week before laboratory admission). Thereafter, sleep/dark episodes (solid bars, light intensity <0.03 lux) were scheduled for 6.67 hours (33% of imposed day) in the 20-hour protocol, self-selected by subject (averaging 28% of cycle) in the free-running protocol, and scheduled for 9.33 hours (33% of imposed day) in the 28-hour protocol. During wake episodes, the light intensity was ~15 lux (20- and 28-hour protocols) or ~150 lux (free-running protocol). Constant routines (open bars) for phase assessments of the endogenous circadian temperature nadir (O) and the fitted melatonin maximum (△) were conducted before and after forced desynchrony in all subjects except 1209, who began forced desynchrony immediately after the three baseline days. Period estimations were performed with the use of temperature data (continuously collected via rectal thermistor throughout all studies) and plasma melatonin and cortisol data (assayed from samples collected every 20 to 60 min during segments of the study in the 20- and 28-hour protocols). The estimated phase of the circadian temperature rhythm (dashed line) was determined by nonorthogonal spectral analysis (31, 32). The temperature period estimates are nearly equivalent under both forced desynchrony protocols (20-hour protocol, 24.29 hours; 28-hour protocol, 24.28 hours), independent of the imposed rest-activity cycle. However, the estimated temperature period (25.07 hours) observed during free-running conditions (with self-selected rest-activity cycle averaging 27.07 hours) was much longer.

Neural Pattern Generators

Limit Cycles, Phase Plane Analysis
Half section of lobster cut in median plane to illustrate general anatomy. From soft-shell female, 6 1/2 inches long, slightly favored in head to show nervous system. Esophageal and gastric ganglion (the latter below reference line to anterior gastric muscle) and anterior visceral and median nerves are shown. Muscle marked levator abdominis (thoraco-abdominis) originates far forward in the thorax and joins enveloping muscles of the flexor system of abdomen. Note that abdominal sternal spines are much longer than in sexually mature animals.

Neural Pattern Generators

![Diagram of neural pattern generators](http://www.bio.brandeis.edu/marderlab/figures/gastricMill_col.jpg)

![Image of stomach](http://www.bio.brandeis.edu/marderlab/figures/HomarusStomach.jpg)
Neural Pattern Generators

Neural Pattern Generators

- Studying the role of $I_h$ in regulating pyloric and gastric mill cycles
- Phase portrait reveals rhythmic similarities in the absence of $I_h$
- Spontaneous recovery when blocking relaxed -> limit cycle

Cardiac Pacing

Limit Cycles, Poincaré Maps

https://commons.wikimedia.org/wiki/File:Cardiac_conduction_system.jpg
https://commons.wikimedia.org/wiki/File:2023_ECG_Tracing_with_Heart_ContractionN.jpg
Cardiac Pacing


Pushing the Limit (Cycle)

- Graphical methods can provide insight into the structure of nonlinear dynamical systems, even when differential equations cannot be solved analytically.
- Limit cycles possible in nonlinear systems.
- Periodic (and aperiodic) oscillations in biological systems can be analyzed - and sometimes controlled - using nonlinear techniques.