Technical Theory Appendix
This appendix gives the details of the math theory underlying the claims made in the main text. It is written for someone with an advanced undergraduate understanding of economics as well as an understanding of differential calculus.

The claims explored in this appendix are that: first, a skill-balanced inflow of immigrants has no impact on wages; second, that the prediction that it does derives entirely from imposing the assumption that the stock of capital is fixed; third, that immigration only impacts wages when it affects the relative supply of different skill types of workers. I also briefly comment on other ways in which the economy may adjust to immigration that are not covered in the main text.

A skill-balanced inflow of immigrants has no impact on wages.
There are two key assumptions underlying this proposition. The first is constant returns to scale, which implies that if you double the inputs used in production, output doubles. This assumption is standard in immigration research (e.g., Card, 2001; Borjas, 2003; Ottaviano and Peri, 2012). The second is that the supply of capital is elastic, which means that investors respond to elevated returns on capital that may temporarily result from immigration – due to a scarcity of capital per worker – by investing in new capital until the returns are bid down to what they were before immigrants came. Again, it is standard in economics to treat this as holding “in the long run”; the main text provides the argument why the long run is essentially immediate with regard to immigration. Together, these assumptions imply that as immigrants come into the U.S., the economy replicates itself proportionately. As a result, wages are not affected.

Let us now see this formally. Consider the setup described in the main text. Output is produced by two types of labor – skilled (S) and unskilled (U) – and capital (K). For the purpose of simple illustration, it is also convenient to assume a competitive economy characterized by a single good. Thus, output of the economy is given by the production function \( Y = h(K, S, U) \), where \( h \) is constant returns to scale.

In a competitive economy, input prices are equal to their marginal products. So the return on capital \( r \), skilled wage \( (w_s) \), and unskilled wage \( (w_u) \) are given, respectively, by:

\[
\begin{align*}
  r &= h_k \\
  w_s &= h_{s} \\
  w_u &= h_{u}
\end{align*}
\]

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1 Incidentally, growth theory suggests that there may be increasing returns to scale, that is, output increases more than proportionately with inputs. If so, constant returns is a conservative assumption for assessing immigration’s labor market impact. Indeed, di Giovanni et al. (2015) calculate that the benefits of immigration-generated increases in scale are substantial in most countries.

2 \( h(K, S, U) \) is also assumed twice continuously differentiable, and characterized by diminishing returns to each input.
where $h_i$ is the derivative of the production function with respect to factor $i \in \{K, S, U\}$. Take the natural log of both sides of each of the expression $r = h_k$ above and totally differentiate:

$$d \ln r = \frac{h_{kk}}{h_k} dK + \frac{h_{ks}}{h_k} dS + \frac{h_{ku}}{h_k} dU$$

With some further identical transformations, we can rewrite this as:

$$d \ln r = \frac{h_{kk} Y K dK}{h_k Y K} + \frac{h_{ks} Y S dS}{h_k Y S} + \frac{h_{ku} U dU}{h_k Y U}$$

$$= \frac{h_{kk} Y}{h_k Y} rK dK + \frac{h_{ks} Y W_s S dS}{h_k Y S} + \frac{h_{ku} Y W_u U dU}{h_k Y U}$$

$$= c_{kk} s_k d \ln K + c_{ks} s_s d \ln S + c_{ku} s_u d \ln U$$

The first step divides and multiplies each term by a factor/output ratio (e.g., $K/Y$), the second step divides and multiplies by a corresponding factor price (and substitutes in the marginal product conditions above, e.g., $w_s = h_s$), and the last step is just notation. In particular, for any two factors $i, j \in \{K, S, U\}$ define $c_{ij} = \frac{h_{ij} Y}{h_i h_j}$ as the “elasticity of complementarity” (Hamermesh, 1993) between those factors. (It describes how the relative wages or prices of the two factors respond to relative supplies.) $s_i$ is the share of output paid to factor $i \in \{K, S, U\}$ (for example, $s_k = \frac{rK}{Y}$).

One can similarly derive identities for the two kinds of labor:

$$d \ln w_s = c_{sk} s_k d \ln K + c_{ss} s_s d \ln S + c_{su} s_u d \ln U$$

$$d \ln w_u = c_{uk} s_k d \ln K + c_{us} s_s d \ln S + c_{uu} s_u d \ln U$$

These demand identities can be further simplified for the case of a “skill balanced” inflow, defined as an inflow of immigrants that increases the size of the skilled and unskilled workforces by the same proportion. In mathematical terms, $d \ln S = d \ln U = d \ln L$, where $L = S + U$ is the size of the workforce in total, is a skill-balanced inflow. $d \ln L \approx \% \Delta L$ is the proportional amount by which immigration grows the workforce. Substituting this in produces simplified demand identities:

$$d \ln r = c_{kk} s_k d \ln K + (c_{ks} s_s + c_{ku} s_u) d \ln L$$

$$d \ln w_s = c_{sk} s_k d \ln K + (c_{ss} s_s + c_{su} s_u) d \ln L$$

$$d \ln w_u = c_{uk} s_k d \ln K + (c_{us} s_s + c_{uu} s_u) d \ln L$$

This is where our assumptions begin to bite. Consider each in turn:

**1. Elastic Capital Supply.** Elastic capital supply means that investors create new capital in response to higher returns until the point where the return on capital is
held constant, so \( d \ln r = 0 \). Substituting this into the first equation above we have that:

\[
0 = c_{kk}s_k d \ln K + (c_{ks}s_s + c_{ku}s_u) d \ln L
\]

\[
\leftrightarrow d \ln K = -\frac{c_{ks}s_s + c_{ku}s_u}{c_{kk}s_k} d \ln L
\]

2. **Constant returns to scale.** Constant returns is defined to mean that if you multiply each input by the same factor \( a > 0 \) (for example, 2), output rises by this same factor. Mathematically:

\[
h(aK, aS, aU) = ah(K, S, U) = aY
\]

Now notice what happens when we take the derivative of both sides with respect to \( K \):

\[
ah_a(aK, aS, aU) = ah_a(K, S, U)
\]

\[
\leftrightarrow h_a(aK, aS, aU) = h_a(K, S, U) = r
\]

That is, factor prices are scale invariant. Doubling the amount capital, skilled labor, and unskilled labor will not affect the return to capital. Notice that the same scale invariance applies to wages:

\[
h_s(aK, aS, aU) = h_s(K, S, U) = w_s
\]

\[
h_u(aK, aS, aU) = h_u(K, S, U) = w_u
\]

Now take the derivative of both sides of \( h_k(aK, aS, aU) = h_k(K, S, U) \) with respect to \( a \):

\[
h_{kk}K + h_{ks}S + h_{ku}U = 0
\]

Because right-hand side does not contain an “\( a \)” its derivative is zero. Using the definitions of elasticity of complementarity and factor shares this can be rewritten as:

\[
c_{kk}s_k + c_{ks}s_s + c_{ku}s_u = 0
\]

Similarly, from the wage invariance expressions, \( c_{sk}s_K + c_{ss}s_s + c_{su}s_u = 0 \) and \( c_{uk}s_K + c_{us}s_s + c_{uu}s_u = 0 \).

3. **Both together.** Notice that \( c_{kk}s_k + c_{ks}s_s + c_{ku}s_u = 0 \) can be rewritten as

\[
\frac{c_{ks}s_s + c_{ku}s_u}{c_{kk}s_k} = -1
\]

Substituting this into the expression we obtained for \( d \ln K \) above we have that:
This equation says that capital grows proportionately with the size of the labor force. Importantly, this result did not depend on any functional form assumptions about production; it is just a general property of constant returns to scale with elastic capital supply.

Finally, wages. Substitute \( d \ln K = d \ln L \) into the expressions for \( d \ln w_s \) and \( d \ln w_u \) above produces:

\[
d \ln w_s = (c_{sk}s_k + c_{ss}s_s + c_{su}s_u) d \ln L = 0
\]
\[
d \ln w_u = (c_{uk}s_k + c_{us}s_s + c_{uu}s_u) d \ln L = 0
\]

The last step uses that \( c_{sk}s_K + c_{ss}s_s + c_{su}s_u = 0 \) and \( c_{uk}s_K + c_{us}s_s + c_{uu}s_u = 0 \).

And this proves the proposition: both skilled and unskilled wages do not change in response to a skill-balanced inflow. In words, because all factors (S, U and now K) grow by the same proportion, and since wages are invariant to scale, wages do not fall in response to a skill-balanced inflow of immigrants.

### Wage Impacts With a Fixed Capital Stock

Given that wages do not fall in response to a skill imbalanced inflow under the quite general conditions shown here, from where comes the common prediction that adding more workers means that wages “have to” go down? It comes only from the implausible assumption that the stock of capital stays fixed after immigration. To see this, suppose for the moment that \( d \ln K = 0 \) -- the capital stock cannot grow.

Using the identities above, you can show that this implies:

\[
d \ln w_s = -c_{sk}s_k d \ln L
\]
\[
d \ln w_u = -c_{uk}s_k d \ln L
\]

In the most commonly assumed production functions \( c_{uk}, c_{sk} > 0 \) (in words, capital is “complementary” with labor), thus since \( s_k \) (capital’s share of output) is also positive, wages fall. As mentioned in the main text, the intuition is that capital becomes “diluted”: each worker has less capital to work with, is therefore less productive and paid lower wages (in a competitive market economy).

A common simplification is to further impose that \( c_{sk} = c_{uk} = 1 \), as Borjas (2003) did.\(^3\) Imposing this, we have that \( d \ln w_s = d \ln w_u = -s_k d \ln L \). In words, wages fall

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\( ^3 \) Specifically, Borjas (2003) assumed the production function has a Cobb-Douglass functional form respect to capital, which looks like this: \( h(K, S, U) = K^\alpha g(S, U)^{1-\alpha} \), where \( \alpha \) is some positive number representing capital’s share of output, and \( g \) is a function. This assumption is quite commonly made; however, as common as it is, it
by the proportion by which immigration grows the workforce, \( d \ln L \), times \( s_k \). The report claiming "Immigration cuts salaries of Americans $2,470 a year" is based on imposing these assumptions, plus that capital's share \( s_k \approx 0.35 \), and that immigrants represent 16.6% of workers. Thus, the calculation is approximately:

\[
0.166 \times 0.35 \times $42,500 = $2,470
\]

The source for the $42,500 used in the report is not cited, but it is in the ballpark of average US income.

Again, this only comes to pass if the capital stock is fixed. One way to see how implausible this is to assume that the capital stock had remained fixed over the past 50 years of immigration (ignoring the fact that it has actually more than doubled – in per worker terms! – over that period. See Figure 3.) and ask what this would imply for the return on capital. These same assumptions imply that

\[
d \ln r = (1 - s_k) d \ln L \approx (1 - 0.35) \times 0.166 = 0.11
\]

In words, you must assume that the return on capital investment increased by a considerable 11%, and investors did not increase their investments in response.

**Additional Results**

In the more general case when immigrant inflows are not skill balanced, one can derive (again using the identities above) that, after capital adjusts \( d \ln r = 0 \),

\[
d \ln \left( \frac{W_u}{W_s} \right) = \left[ (c_{sk} - c_{uk}) s_k \frac{c_{sk}s_s}{c_{sk}s_s + c_{ku}s_u} + (c_{ss} - c_{us}) s_s \right] d \ln \left( \frac{U}{S} \right)
\]

Notice that relative wages depend only on the growth in the skilled/unskilled ratio, just as the formula in the text described, and in no way on changes in the absolute number of workers \( L \).

The term in brackets is the slope of the relative demand curve. (See Figure 4.) It has two pieces. The second term, \( (c_{ss} - c_{us}) s_s \) is negative; indeed, it is the reason the relative demand curve slopes down.\(^4\) The first term comes from the adjustment of capital, and its sign is related to the sign of \( (c_{sk} - c_{uk}) \). Recall that the “$2,470” simulations were based on imposing that \( c_{sk} = c_{uk} = 1 \), as it is common to do in the immigration literature. Notice that this simplification conveniently makes this term go away. In contrast, there is considerable evidence that capital is actually “skill complementary,” that is, that \( c_{sk} > c_{uk} \) (capital is more complementary with skilled

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\(^4\) Since \( c_{ss} < 0 \) (by virtue of diminishing returns) and \( s_s > 0 \). Also, that \( c_{us} > 0 \) is not a required feature of \( h \), but for the skill types considered here it is appropriate.
than unskilled workers). This implies that the first term is unambiguously positive. Thus, the adjustment of capital flattens the slope of the relative demand curve, something that is generally not considered in standard treatments of the labor market impact of immigration. It has also been shown to be quantitatively important (Lewis, 2013), particularly in absorbing the large wave of Southern and Eastern European immigration in the early twentieth century (Lafortune et al., 2016).

This model imposes a fixed production technology, $h$, while the real world is more dynamic. Acemoglu (2007) shows that skill-mix induced changes in production technology (due to, for example, immigration) will also flatten the relative demand curve under very general conditions. Adjustments in production technology seem to be important in practice. For example, when the U.S. kicked out Mexican farm laborers in the 1960s (so-called braceros) California tomato farms began using mechanized tomato harvesters rather than raise wages or replace the braceros with U.S. workers (Clemens et al., 2017).

**Absolute Wages.** One can also derive separate expressions for skilled and unskilled wages. These show that an increase in $U/S$ is associated with a decline in $w_u$ and an increase in $w_s$. Note that this allows for the possibility that the average native worker has higher wages as a result of immigration (say, if natives are disproportionately skilled and immigrants are disproportionately unskilled), as has been found to be true empirically using a richer model (Ottaviano and Peri, 2012).

Additional References


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5 As a side note, this is thought to be one reason why inequality has been rising in the U.S. (e.g., Goldin and Katz, 2008; Krusell et al., 2000). As new capital equipment gets adopted (computers, industrial robots) the productivity of skilled workers increases relative to unskilled workers due to capital-skill complementarity.

6 By virtue of the identity $c_{ks}s_s + c_{ku}s_u = -c_{kk}s_k > 0$ (since $c_{kk} < 0$ by virtue of diminishing returns), $c_{sk}$ and $c_{uk}$ must either be both positive or be opposite in sign. These facts imply that $c_{sk} > c_{uk}$ is sufficient for the first term in brackets to be positive.

7 Generalizing the model to include multiple products also tends to flatten the relative demand curve, as changing the mix of products (towards, say, more unskilled-intensive goods) is another margin of adjustment to immigration. In practice, this effect appears to be small (e.g., Card and Lewis, 2007).


