SOLUTIONS TO MIDTERM EXAM **MAY 11, 2005**

1.

1 \ 2	L	C	R
U	6, 0	0,5	2,2
M	1, 8	4, 0	8, 6
D	~~3, 3~~	~~2, 6~~	~~5,5~~

a) Strategy D is dominated by a mixed strategy over U and M (e.g. $\sigma_1 = (.45,.55,0)$).

b) The game has two congruous sets:

- S, which is the cross product over all strategies, namely $S_1 \times S_2$
- {U,M}x{L,C}. By checking best responses in pure strategies (this is how we defined a congruous set), we can see that there are no best responses for {U,M} outside {L,C} and vice versa.

c) Since D is dominated, we know that U and M are supports.

	1 \ 2	L	C	R
p	U	6, 0	0,5	2,2
1-p	M	1, 8	4, 0	8, 6
	D	3, 3	2, 6	5,5

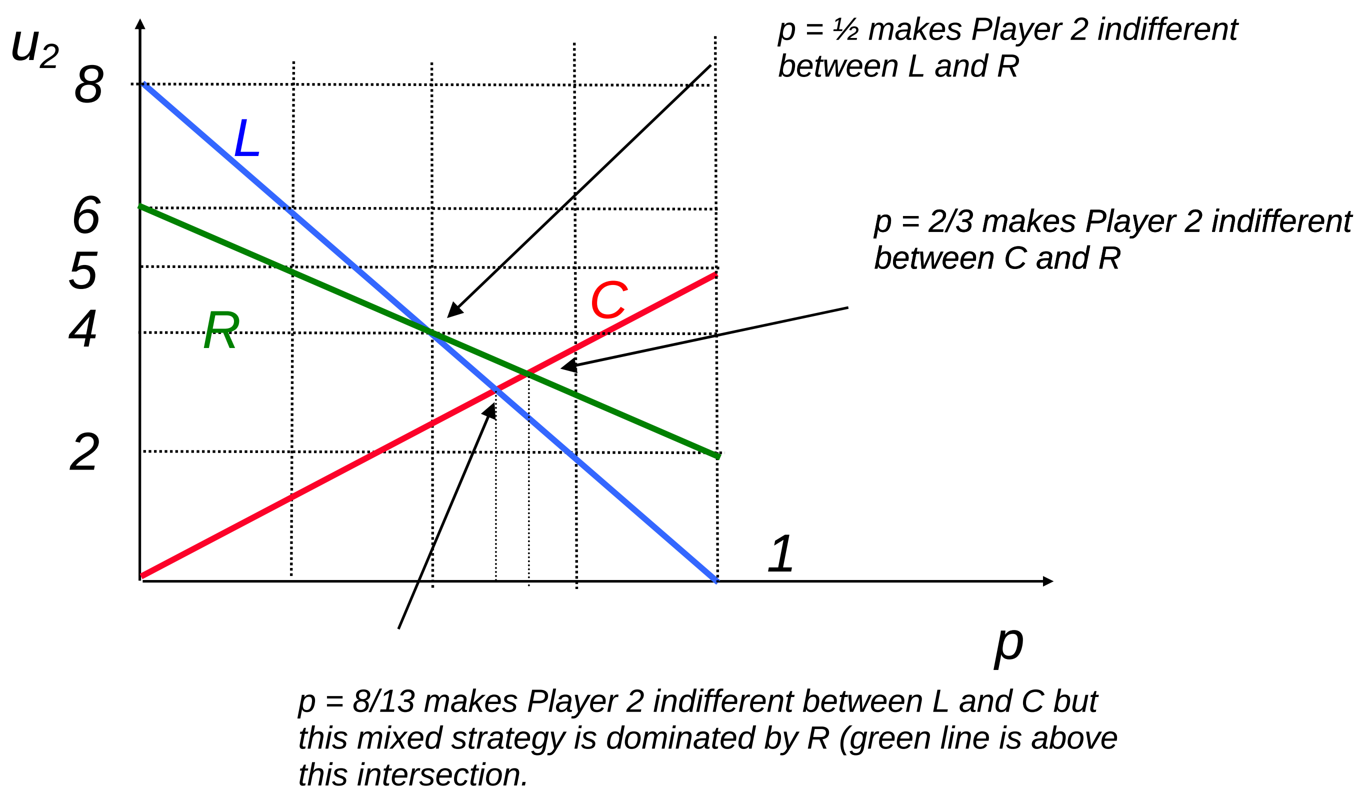
Computing the payoffs that Player 2 receives when Player 1 mixes over U and M:

$u_2(\sigma_{1,} L) = 8(1-p)$ (1)

$u_2(\sigma_{1,} C) = 5p$ (2)

$u_2(\sigma_{1,} R) = 2p + 6(1-p)$. (3)

We graph the payoffs as functions over *p*:



Mathematically, we set the following equations equal to see the cut-offs illustrated above:

$(1) = (2) : 8 - 8p = 5p \Leftrightarrow p = \frac{8}{13}$ (Player 2 indifferent between L and C).
$(1) = (3) : 8 - 8p = 2p + 6 - 6p \Leftrightarrow p = \frac{1}{2}$ (Player 2 indifferent between L and R).
$(2) = (3) : 5p = 2p + 6 - 6p \Leftrightarrow p = \frac{2}{3}$ (Player 2 indifferent between C and R).

The graph tells us that mixing between L and C is never a best response but dominated by R.[1]

We therefore can reduce the rest of our analysis to two supports: L and R, and C and R.

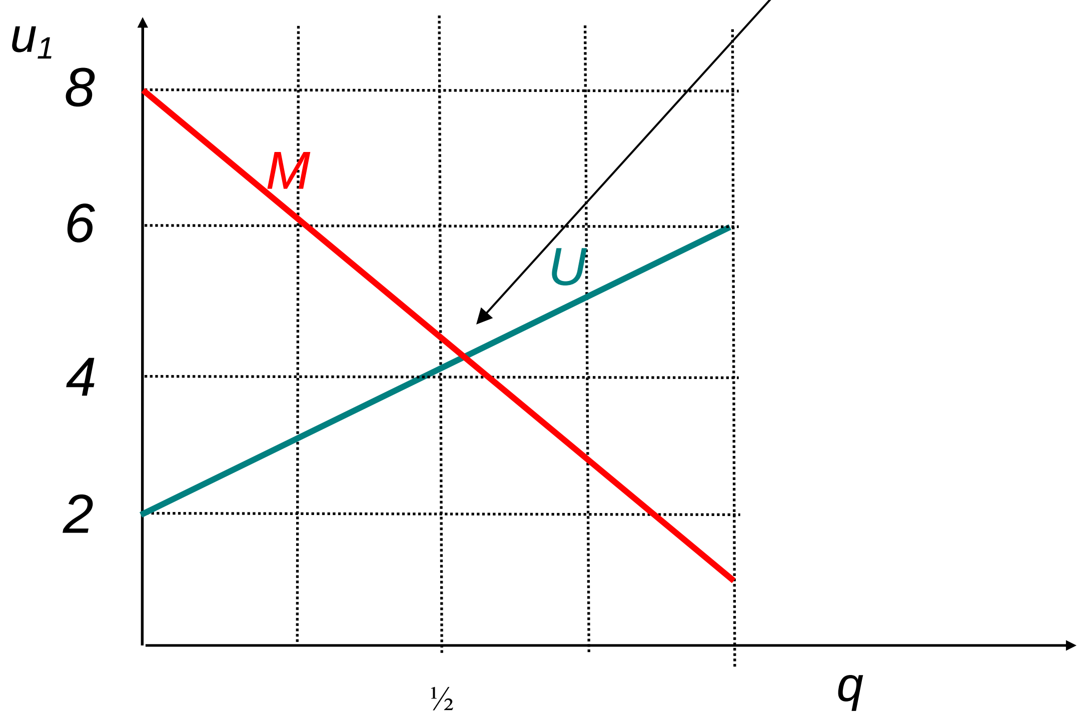
- L and R: Whenever Player 1 mixes with p = ½, Player 1 is indifferent. We need to double-check: Would a mixture between L and R make Player 1 want to mix with p = ½, or would he prefer instead to reply with a pure strategy?

[1] This led to the unexpected result that L and C, although elements of the strategy set that forms a best response complete in pure strategies, is never a best reply when Player 1 mixes with ½ <p<2/3 (see graph above). Reason for this unexpected result is that congruous sets are defined in pure strategies and its definition does not permit mixed strategies.

Putting *q* and *1-q* over L and R gives:

		q		*1-q*
	1 \ 2	L	C	R
p	U	6, 0	0,5	2,2
1-p	M	1, 8	4, 0	8, 6
	D	3, 3	2, 6	5,5

$$u_1(U,\sigma_2) = 6q + 2(1-q)$$
$$u_1(M,\sigma_2) = q + 8(1-q)$$
$$Equating: 6q + 2 - 2q = q + 8 - 8q \Leftrightarrow 4q + 2 = 8 - 7q \Leftrightarrow 11q = 6 \Leftrightarrow q = \frac{6}{11}$$



That is, we have a MSNE ((½,½,0),(6/11,0,5/11)).

- Now we use C and R as support:

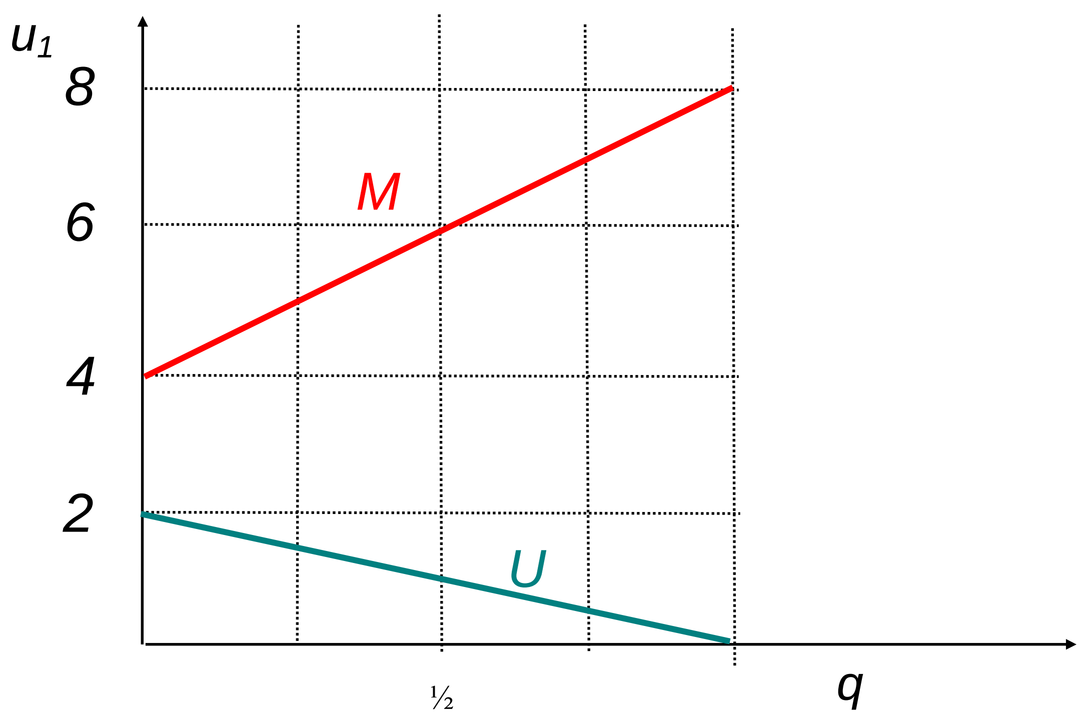
Putting *q* and *1-q* over C and R gives:

1 \ 2	L	C (q)	R ($1-q$)
U	6, 0	0,5	2,2
M	1, 8	4, 0	8, 6
D	3, 3	2, 6	5,5

One can already observe that M does always better than U for any mixture over C and R:

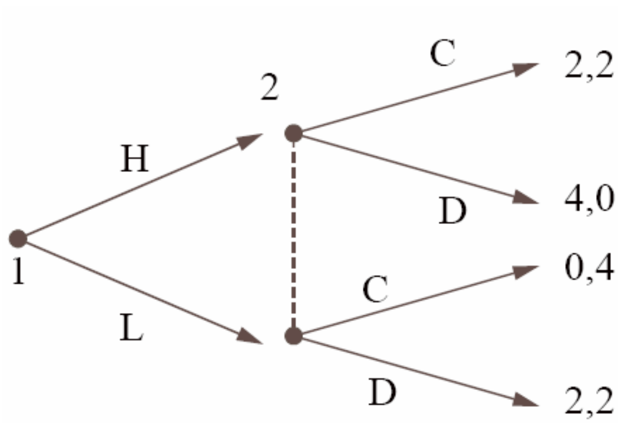
$$u_1(U,\sigma_2) = 2(1-q)$$
$$u_1(M,\sigma_2) = 4q + 8(1-q)$$
$$Equating: 2-2q = 4q+8-8q \Leftrightarrow 2-2q = -4q+8 \Leftrightarrow 2q = 6.$$

This leads to a contradiction since q<1. Thus, we have no MSNE in this case.



The only MSNE of this game is ((½,½,0),(6/11,0,5/11)).

2. (5 points) Consider the following game in extensive form:



a) Convert this game into normal form and offer a complete description of the game.

1 \ 2	C	D
H	2,2	4,0
L	0,4	2,2

Description:

- List of Players $\{1,2\}$.

- Strategy spaces: $S_1 = \{H, L\}$, $S_2 = \{C, D\}$.

- Payoff functions (see matrix).

If someone wanted to write it down, it should read:

$$u_1(H,C) = 2,\ u_1(L,C) = 0,\ u_1(H,D) = 4,\ u_1(L,D) = 2,$$
$$u_2(H,C) = 2,\ u_2(L,C) = 4,\ u_1(H,D) = 0,\ u_1(L,D) = 2.$$

b) Is this game strictly competitive? Answer by writing down the conditions and checking each and every strategy profile. If it is strictly competitive, find the players' security strategies.

A game is strictly competitive if for every two strategy profiles $s,\ s' \in S, u_1(s) > u_1(s')$ *iff* $u_2(s) < u_2(s')$.

It is easy to see that this holds: 4>2 comes with 0<2 and vice versa.

The security strategy $\underline{s}_i \in S_i$ solves $\max_{s \in S_i} w_i(s_i)$ with $w_i(s_i)$ being Player i's worst payoff that he can get when he plays strategy s_i.

Security strategy for Player 1 is H, security strategy for Player 2 is C.

c) PSNE=(H,C)

This is a dominant-strategy equilibrium: solving for any mixed strategy does not give a different result. The only NE is PSNE=(H,C).

3) a) Offer a complete description of the game and illustrate it it in extensive form.

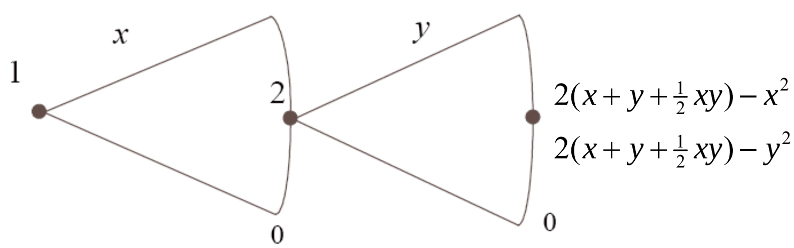
- List of Players $\{1,2\}$.

- Strategy spaces: $S_1 = [0,10],\ S_2 = [0,10]$.

- Payoff functions:

$$\pi_1 = 2(x + y + \tfrac{1}{2}xy) - x^2$$
$$\pi_2 = 2(x + y + \tfrac{1}{2}xy) - y^2$$

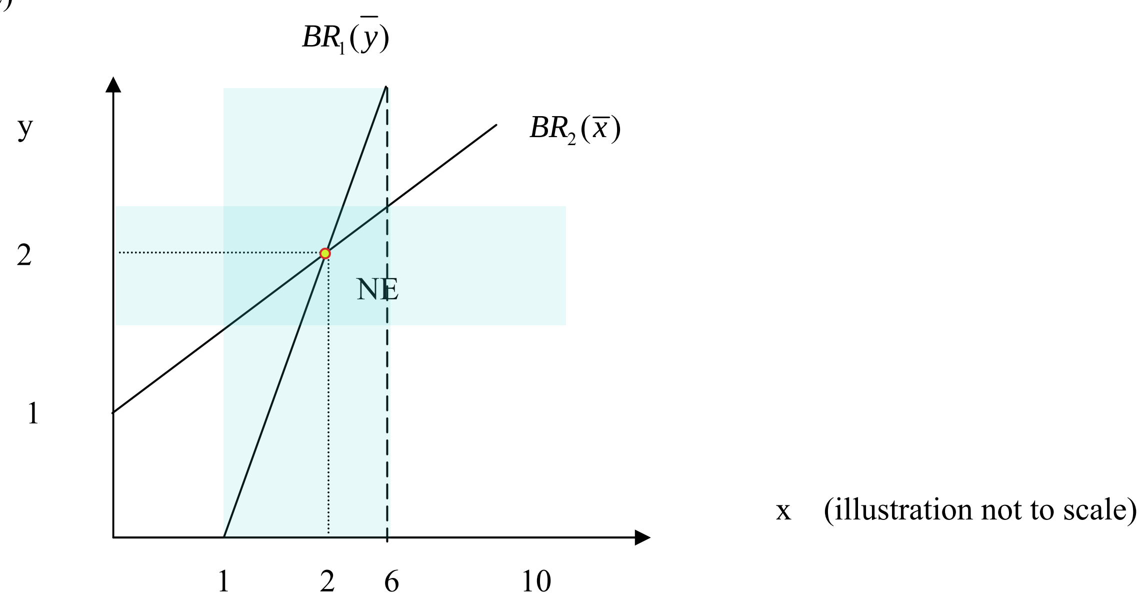
- Illustration in tree form:



b) We find the BR functions by deriving individual payoffs w.r. to the strategic variable (x for Player 1, y for 2) and setting it to zero.

$$\frac{\partial \pi_1}{\partial x} = 2 + y - 2x = 0 \Leftrightarrow BR_1(\bar{y}) = x = 1 + \tfrac{1}{2}\bar{y}.$$
$$\frac{\partial \pi_2}{\partial y} = 2 + x - 2y = 0 \Leftrightarrow BR_2(\bar{x}) = y = 1 + \tfrac{1}{2}\bar{x}.$$

c)



c) The NE is found by equating the two BR curves (in their intersection, since we are looking for mutually best responses):

$$BR_1 \doteq BR_2$$
$$x = 1 + \tfrac{1}{2}(1 + \tfrac{1}{2}x)$$
$$\tfrac{3}{4}x = \tfrac{3}{2}$$
$$x = 2.$$

The set of rationalizable strategies is found by eliminating strictly dominated strategies.

We may start with Player 2: He can eliminate any response of Player 1 less than 1 and more than 6. Iterating, Player 1, supposing rational behavior of Player 2, may, in the next round eliminate all strategies of Player 2 that are best responses to any strategy less than 1 and more than 6 etc. (the two rectangles above illustrate the first two iterations..).

Thus, R={(2,2)}. Comparing with (c) shows that the result is the same since this game is solvable by iterated dominance and has only one NE.

4. The set of rationalizable strategies is R={(UF)X(BC)}.

1 \ 2	AC	AD	BC	BD
UE	5,4	4,4	4,5	12,2
UF	3,7	8,7	5,8	10,6
DE	2,10	7,6	4,6	9,5
DF	4,4	5,9	4,10	10,9

5) Evaluate the following payoffs for the game pictured here:

a) $u_1(\sigma_1, O)\ for\ \sigma_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = 0$

b) $u_2(\sigma_1, I)\ for\ \sigma_1 = (\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8}) = 9/8$

c) $u_2(\sigma_1, \sigma_2)\ for\ \sigma_1 = (\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8}), \sigma_2 = (\frac{2}{3}, \frac{1}{3}) = 5/6$

d) $u_1(\sigma_1, \sigma_2)\ for\ \sigma_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), \sigma_2 = (\frac{1}{2}, \frac{1}{2}) = 9/8.$

6) Find all mixed strategies of the following two games:

a)

1 \ 2	*q* X	*1-q* Y
A	8,8	0,0
B	1,1	1,1
C	1,1	1,1

We start by putting probabilities over X and Y. Obviosly, **any** *q* makes Player 1 indifferent between B and C. Thus, B and C is a candidate for a support. The only question we need to solve is: When would A beat this candidate?

That is: we need to check for which *q* $u_1(A, \sigma_2) > u_1(\sigma_1, \sigma_2)$, with $\sigma_1 = (0, p, 1-p)$.

$u_1(\sigma_1, \sigma_2) = 1$ (trivial).
$u_1(A, \sigma_2) = 8q.$
Equating $8q = 1$ yields $q = \frac{1}{8}$.

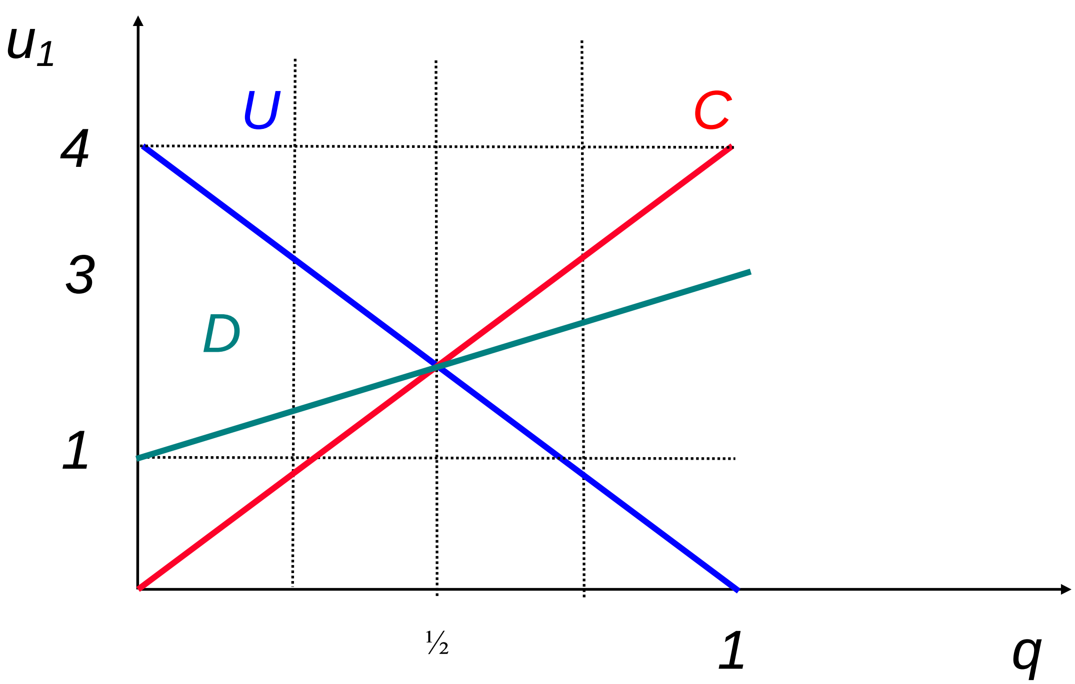
This game has MSNEs for all values of *p* and *q* satisfying ((0,*p*,1-*p*),(*q*,1-*q*)) with *q* <1/8.

b)

1 \ 2		L	M	R
			q	1-q
p	U	8,1	0,2	4,3
r	C	3,1	4,4	0,0
	D	5,0	3,3	1,4

$(1)\, u_1(U,\sigma_2) = 4-4q$
$(2)\, u_1(C,\sigma_2) = 4q$
$(3)\, u_1(D,\sigma_2) = 3q+1-q = 2q+1$

$(1) \doteq (2) \Rightarrow 4-4q = 4q \Leftrightarrow q = \tfrac{1}{2}$
$(1) \doteq (3) \Rightarrow 4-4q = 2q+1 \Leftrightarrow q = \tfrac{1}{2}$
$(2) \doteq (3) \Rightarrow 4q = 2q+1 \Leftrightarrow q = \tfrac{1}{2}$



Note that this q solves the entire system of equation, that is, the q that makes player 1 indifferent between two strategies, makes him indifferent between all three strategies. Our candidate is $\sigma_2 = (0, \tfrac{1}{2}, \tfrac{1}{2})$.

Checking for player 1's distribution that makes Player 2 indifferent:

$$\left.\begin{array}{l} u_2(\sigma_1, M) = 2p + 4r + 3(1-p-r) \\ u_2(\sigma_1, R) = 3p + 4(1-p-r) \end{array}\right\} r = \tfrac{1}{5}$$

Since $r = \frac{1}{5}$, the other 2 probabilities need to fulfill $\sigma_1 = (x, \frac{1}{5}, y), with\, x, y \geq 0, x + y = \frac{4}{5}$.

The game has a MNSE $((x, \frac{1}{5}, y), (0, \frac{1}{2}, \frac{1}{2}))$ for all $x, y \geq 0, x + y = \frac{4}{5}$. ■