**FINAL EXAM – ECON 35 – SPRING 2005**

(1) Find all mixed-strategy Nash equilibria of the following game.
Note: If you don't check whether a candidate for a mixed-strategy Nash equilbrium indeed forms an equilibrium or whether it is beaten by another pure strategy, you should explain verbally why you actually would not need to check.

| 1 \ 2 | D | E | F |
|---|---|---|---|
| A | 1,3 | 5,10 | 2, 1 |
| B | 3,4 | 2, 3 | 3,4 |
| C | 2, 0 | 3, 1 | 2, 1 |

First we see that C is dominated by a mixture between A and B. One candidate that dominates C is e.g. $\sigma_1 = (\frac{2}{5}, \frac{3}{5}, 0)$. We see that

$u_1(\sigma_1, D) = 1 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{11}{5} = 2\frac{1}{5} > 2 \ \ (\textit{fulfilled}).$

$u_1(\sigma_1, E) = 5 \cdot \frac{2}{5} + 2 \cdot \frac{3}{5} = \frac{15}{5} = 3\frac{1}{5} > 3 \ \ (\textit{fulfilled}).$

$u_1(\sigma_1, F) = 2 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{13}{5} = 2\frac{3}{5} > 2 \ \ (\textit{fulfilled}).$

Thus, C is dominated.

Now we put a probability distribution over A and B, which form the support for a mixed strategy played by Player 1. We use *p* and 1-*p*.

Player 2's payoffs as function of *p* are:

$u_2(\sigma_1, D) = 3p + 4(1-p)$

$u_2(\sigma_1, E) = 10p + 3(1-p)$

$u_2(\sigma_1, F) = p + 4(1-p)$

Before proceeding, we graph the payoffs of player 2 as a function of *p*:

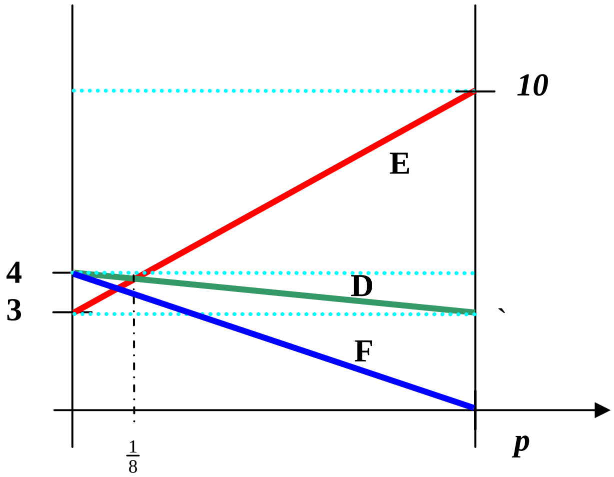



*Fig. 1*

From this we can already see: To find a support for Player 2's possible mixed strategies, we should never use E and F, since whenever Player 2 is indifferent between E and F, D would be the best reply. The only two possible supports are D and F and D and E.

D and F is easy to treat: We see that this can be only a mixed if *p*=0 and Player 1 plays B as pure strategy, while Player 2 mixes between D and F. We disregard this candidate further because of the property that Player 1 plays a pure strategy in this case (she is not indifferent between two strategies), and thus it cannot lead to a proper mixed strategy.

The graph shows that we have one suitable candidate for support: D and E. Mathematically, the value of *p* that makes Player 2 indifferent between D and E is found by setting the two equations

$$u_2(\sigma_1, D) = 3p + 4(1-p)$$

$$u_2(\sigma_1, E) = 10p + 3(1-p)$$

equal: $3p + 4(1-p) = 10p + 3(1-p) \Leftrightarrow 4 - p = 7p + 3 \Leftrightarrow 1 = 8p \Leftrightarrow p = \frac{1}{8}$.

We now put probabilities *q* and 1-*q* over D and E to check which mixed strategy Player 2 could use in equilibrium:

$$u_1(A, \sigma_2) = q + 5(1-q) = 5 - 4q$$

$$u_1(B, \sigma_2) = 3q + 2(1-q) = q + 2$$

Equating the right-hand sides yields $5 - 4q = 2 + q \Leftrightarrow 3 = 5q \Leftrightarrow q = \frac{5}{3}$.

The candidate for a MSNE in which both player play a mixed strategy is:
$((\frac{1}{8}, \frac{7}{8}, 0), (\frac{3}{5}, \frac{2}{8}, 0))$. We need to show that this is really an equilibrium.

That is, we need to check if the following condition is fulfilled:

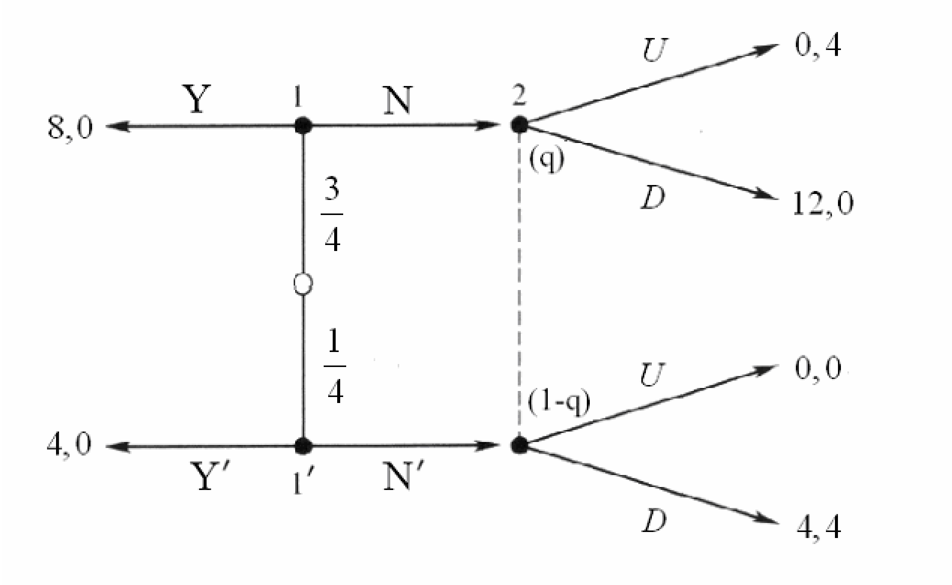
$u_i(\sigma_i, \sigma_j) \geq u_i(s_i, \sigma_j),\ \forall i, j.$

a) Do we need to check for player 1 (that is i=1, j=2?)
- A and B form the support of the mixed strategy, thus neither A nor B can beat it.
- C is dominated, thus we don't need to check.

b) Do we need to check for player 2? (that is i=2, j=1?)
- Again, we don't need to check whether D and E beat it since they form the support.
- We could find whether $u_2(\sigma_1, \sigma_2) \geq u_2(F, \sigma_2)$. An easier way is to observe from *Fig. 1* that whenever Player 1 plays mixed, F is strictly dominated. Since in equilibrium, Player 1 plays mixed with $p$ = 1/8, *F* is strictly dominated.

Thus, the MSNE of this game is $((\frac{1}{8}, \frac{7}{8}, 0), (\frac{3}{5}, \frac{2}{5}, 0))$.

(2) Consider the following sequential move game of incomplete information:




(a) Find all pooling and all separating PBE in this game.

1. We check for separating equilibria.

1a) Suppose Player 1 plays YN': This sets $q$ = 0. Thus, Player 2 chooses D.

Is (YN', D) a PBE? Only if Player 1 does not want to deviate. Since 12>8, the high type of Player 1 would like to play N. No equilibrium.

1b) Suppose Player 1 plays strategy Y'N. Then, $q$=1 and Player 2 replies with U. The candidate (Y'N, U) is not an equilibrium either since 0<8 and the high type of Player 1 would like to play Y.

2. We check for pooling equilibria.

2a) Suppose Player 1 plays strategy NN'. Then $q$ = 3/4. Note that Player 2 chooses U whenever $q$ > 1/2 and D when $q$ < 1/2 ( *from* $4q = 4(1-q)$ ).
Thus, 2 plays U.
(NN'U) is not a PBE since each type of Player 1 would deviate to the left side (8>0, 4>0).

2b) Suppose Player 1 plays YY'. This can only be an equilibrium if Player 1 would not deviate to the right, following Player 2's strategy.
*Case 1:* $q$ < ½. Then Player 2 plays D. In this case, YY' can never lead to a PBE because Player 1 would prefer to play N rather than Y (12>8).
*Case 2:* Now suppose that Player 2 holds out-of equilibrium beliefs that $q$ > ½. This would include him to play U. (YY',U) is a pooling PBE since no type of Player 1 would like to deviate from YY'.

The only PBE in this game is (YY',U) for $q$ > ½.

b) We derive the Bayesian normal form:

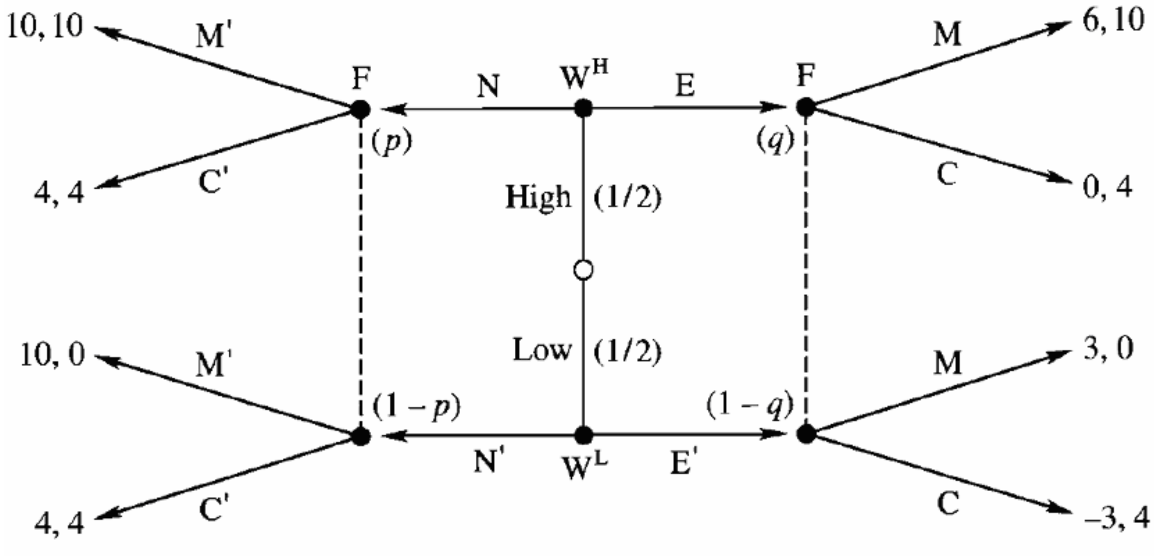
| 1 \ 2 | U | D |
|---|---|---|
| YY′ | 7, 0 | 7, 0 |
| YN′ | 6, 0 | 7, 1 |
| NY′ | 1, 3 | 10, 0 |
| NN′ | 0, 3 | 10, 1 |

Finding the strategy profile with mutually best responses we see that the only BNE is (YY',U), which is the same as the pooling equilibrium in the sequential move version of the game.
c) As shown below, rationalizability does not solve the game in this case. The set of rationalizable strategies is {YY', NY', NN'} for Player 1 and {U,D} for Player 2.

YN' is dominated by a mixture between YY' and NY'.

| 1 \ 2 | U | D |
|---|---|---|
| YY′ | 7, 0 | 7, 0 |
| YN′ | ~~6, 0~~ | ~~7, 1~~ |
| NY′ | 1, 3 | 10, 0 |
| NN′ | 0, 3 | 10, 1 |

3)




1. We check first for separating equilibria.

1a) Suppose Player 1 plays NE': This sets $p$ = 1, $q$ = 0. Player 2 chooses M'C.

Is (NE',M'C) a PBE? Only if Player 1 does not want to deviate. This is not the case: the low type would deviate to N' since 10>-3. No equilibrium.

1b) Suppose Player 1 plays strategy N'E. Then, $p$ = 0, $q$ = 1. Player 2 replies with C'M.
Is (N'E,C'M) a PBE? Only if Player 1 does not want to deviate. This is the case: No deviation occurs since 4<6 and 4>3. (N'E,C'M) is a separating PBE.

2. We check for pooling equilibria.

2a) Suppose Player 1 plays strategy NN'. Then $p$ = ¾.

Before proceeding note that Player 2 chooses M' whenever $p$ > 1/2 and chooses C' when $p$ < 1/2 ( *from* $p = 1 - q$ ). Similarly, on the right side Player 2 chooses M whenever $q$ > 1/2 and C when $q$ < 1/2 ( *from* $q = 1 - q$ ).

From $p$ = ¾ follows: Player 1 chooses M' on the left side.
_Case 1:_ $q$ < ½. Then Player 2 chooses C. In this case, NN' forms a pooling PBE since 10>0 and 10>-3.
_Case 2:_ $q$ > ½. Then Player 2 chooses M. In this case as well, NN' forms a pooling PBE since 10>6 and 10>3.

2b) Suppose Player 1 plays strategy EE'. Then $q$ = ¾. Player 2 chooses M on the right side. We again check for the two cases of out-of-equilibrium beliefs:

<u>*Case 1:*</u> $p$ < ½. Then Player 2 chooses C'. In this case, EE' cannot form a pooling PBE since the L-type deviates to N' since 4>3.

<u>*Case 2:*</u> $p$ > ½. Then Player 2 chooses M'. In this case as well, EE' cannot form a pooling PBE since the H-type deviates to N' since 10>6.

Thus, the game has the following two Perfect Bayesian Equilibria:

- The separating equilibrium (N'E,C'M),
- and the pooling equilibrium in which Player 1 never invests in education. This second equilibrium is characterized by any out-of equilibrium beliefs of Player 2. Player 2 chooses to offer the managerial job, independent of the player's type.

(4) Consider the following game in normal form:

| 1 \ 2 | L | C | R |
|---|---|---|---|
| U | 6, 0 | 0,5 | 2,2 |
| M | 1, 8 | 4, 0 | 8, 6 |
| D | 3, 3 | 2, 6 | 5,5 |

(a) Find $BR_1(\mu_2)$ for $\mu_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

We compute:

$$u_1(U, \mu_2) = 6 \cdot \tfrac{1}{3} + 2 \cdot \tfrac{1}{3} = \tfrac{8}{3}$$
$$u_1(M, \mu_2) = \tfrac{1}{3} + 4 \cdot \tfrac{1}{3} + \tfrac{8}{3} = \tfrac{13}{3}$$
$$u_1(D, \mu_2) = 3 \cdot \tfrac{1}{3} + 2 \cdot \tfrac{1}{3} + 5 \cdot \tfrac{1}{3} = \tfrac{10}{3}$$

Comparing the payoffs shows that $BR_1(\mu_2)$ = {M}.

(b) Find $BR_2(\mu_1)$ for $\mu_1 = (\frac{1}{2}, \frac{1}{2}, 0)$.

$$u_2(L, \mu_1) = 0 \cdot \tfrac{1}{2} + 8 \cdot \tfrac{1}{2} = 4$$
$$u_2(C, \mu_1) = 5 \cdot \tfrac{1}{2} = 2.5$$
$$u_2(R, \mu_1) = 2 \cdot \tfrac{1}{2} + 6 \cdot \tfrac{1}{2} = 4$$

$BR_2(\mu_1)$ = {L,R}.

c) D is dominated by a mixture over U and D:

| 1 \ 2 | L | C | R |
|---|---|---|---|
| U | 6, 0 | 0,5 | 2,2 |
| M | 1, 8 | 4, 0 | 8, 6 |
| D | ~~3, 3~~ | ~~2, 6~~ | ~~5,5~~ |

It can be shown that e.g. putting a probability distribution of 0.45 on U and 0.55 on M does the trick:

$u_1(\sigma_1, L) = 0.45 \cdot 6 + 0.55 = 3.25 > 3(\textit{fulfilled}).$
$u_1(\sigma_1, C) = 0.55 \cdot 4 = 2.2 > 2(\textit{fulfilled}).$
$u_1(\sigma_1, R) = 0.45 \cdot 2 + 0.55 \cdot 8 = 5.3 > 5(\textit{fulfilled}).$

No further dominated strategies.

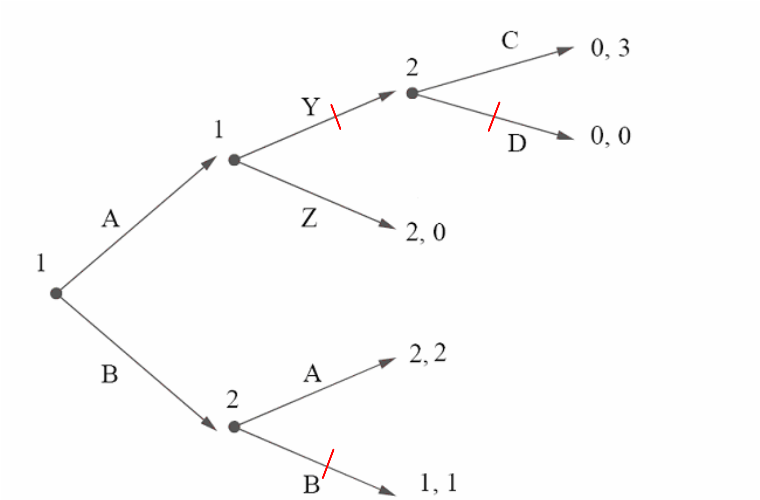
(5) Consider the following two sequential move games in extensive form. Write down the strategy space for each player, find all pure-strategy Nash equilibria and all subgame perfect Nash equilibria.

a) The strategy spaces read: $S_1 = \{AY, AZ, BY, BZ\}, S_2 = \{AC, AD, BC, BD\}$.
To find the PSNE, we convert this game into normal form (thanks, Tatsuya):

| 1 \ 2 | AC | AD | BC | BD |
|---|---|---|---|---|
| AY | 0 3 | 0 0 | 0 3 | 0 0 |
| AZ | 2 0 | 2 0 | 2 0 | 2 0 |
| BY | 2 2 | 2 2 | 1 1 | 1 1 |
| BZ | 2 2 | 2 2 | 1 1 | 1 1 |

It is easy to see that the game has the following PSNE: {(AZ,AC), (AZ,AD), (AZ,BC), (AZ,BD), (BY,AC), (BY,AD), (BZ,AC), (BZ, AD)}.

To find the PSNE, we consider the game in tree form:



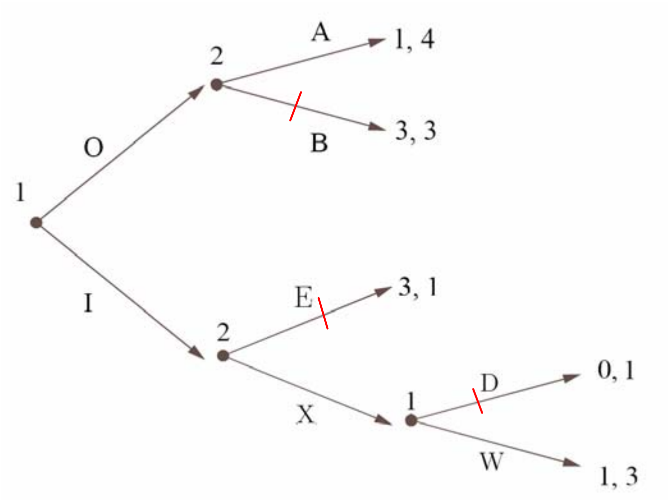

Among all PSNE, only {(AZ, AC), (BZ, AC)} are subgame perfect.

b) $S_1 = \{UD, UW, ID, IW\}$, $S_2 = \{AE, AX, BE, BX\}$. The game in normal form reads:

| 1 \ 2 | AE | AX | BE | BX |
|---|---|---|---|---|
| OD | 1 4 | 1 4 | 3 3 | 3 3 |
| OW | 1 4 | 1 4 | 3 3 | 3 3 |
| ID | 3 1 | 0 1 | 3 1 | 0 1 |
| IW | 3 1 | 1 3 | 3 1 | 1 3 |

from which we see that the set of PSNE={(OD, AX), (OW, AX), (ID, AE), (ID, BE), (IW, AX)}.

To find the SPNE, we again consider the tree form:




SPNE={(OW, AX), (IW, AX)}.

(6) Find the set of rationalizable strategies for the following game:

| 1 \ 2 | L | C | R |
|---|---|---|---|
| U | 5, 1 | 0, 4 | 1, 0 |
| M | 3, 1 | 0, 0 | 3, 5 |
| D | 3, 3 | 4, 4 | 2, 5 |

By applying iterated removal of strictly dominated strategies we see that the only rationalizable strategy profile is {M,R}.

(7) a) $S_A = \{Ea, En, Da, Dn\}$, $S_2 = \{aa', an', na', nn'\}$.

b) The game in normal form reads

| A \ B | aa' | an' | na' | nn' |
|---|---|---|---|---|
| Ea | 3, 3 | 3, 3 | 1, 6 | 1, 6 |
| En | 6, 1 | 6, 1 | 5,5 | 5,5 |
| Da | 0, 4 | 0,3.5 | 0, 4 | 0, 3.5 |
| Dn | 0, 4 | 0,3.5 | 0, 4 | 0,3.5 |

The set of PSNE is {(En,na'),(En,nn')}.

(8) a) The game in Bayesian normal form reads:

| A \ B | Y | N |
|---|---|---|
| YY′ | 5, 0 | 5, 0 |
| YN′ | 7.5,2.5 | 7.5,0 |
| NY′ | 0 , –5 | 0, 0 |
| NN′ | -2.5,-7.5 | -2.5,0 |

The BNE is the strategy profile with mutually best responses: BNE={(YN',Y)}.

c)

| A \ B | Y | N |
|---|---|---|
| YY′ | ~~5, 0~~ | ~~5, 0~~ |
| YN′ | 7.5,2.5 | ~~7.5,0~~ |
| NY′ | ~~0, –5~~ | ~~0, 0~~ |
| NN′ | ~~-2.5,-7.5~~ | ~~-2.5,0~~ |

We observe that YN' dominates all other strategies. Applying iterated removal of strictly dominated strategies shows that Player B never plays N if YN' is played. R={(YN',Y)}=BNE.