Solutions to Practice Midterm #2

(1) Show that the demand curve $x(p) = ap^{-e}$ is isoelastic, which means it shows a constant price elasticity of demand for any *x*.

Solution

$$\varepsilon_{x,p} = \frac{dx}{dp}\frac{p}{x} = -e \cdot \underbrace{a \cdot p^{-e-1} \cdot \frac{p}{ap^{-e}}}_{=1}$$

$$\varepsilon_{x,p} = -e$$

For any constant value of *e* the demand curve is isoelastic.

(2) Remember Kelly Umpleby from the last problem set? Here is the story of her sister Kim. She is a Cobb-Douglas consumer as well. Her preferences follow the utility function $u(x_1, x_2) = x_1 x_2$. Good 1 (gasoline) has a price of p_1. Good 2 is consumption, its price being normalized to one. The representative consumer's budget is m.

a) Find optimum consumption (individual demands). Explain verbally why the utility function $\tilde{u}(x_1, x_2) = x_1^{1/2} x_2^{1/2}$ leads to the same individual demands. Show that this is the case.

Solution: Using Lagrange:

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(m - p_1 x_1 - x_2)$$

with

$$\frac{\partial L}{\partial x_1} = x_2 - \lambda p_1 = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = x_1 - \lambda = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0 \quad (3)$$

By using (1) and (2) we reach $x_2 = p_1 x_1$, substituting into (3) we reach the individual demands

$$x_1^* = \frac{m}{2p_1}, x_2^* = \frac{m}{2}.$$

The function $\tilde{u}(x_1, x_2) = x_1^{1/2} x_2^{1/2}$ is a strictly monotonic transformation of the original function $u(x_1, x_2) = x_1 x_2$. We have $\tilde{u}(x_1, x_2) = \sqrt{u(x_1, x_2)}$. Since the consumer's preferences are not influenced by any strictly monotonic transformation, the function $\tilde{u}$ describes the same preferences as the original function.

That this is true can be shown via the MRS of the two functions:

1. $MRS(u(x_1, x_2)) = -\frac{x_2}{x_1}.$

2. $MRS(\tilde{u}(x_1, x_2)) = -\frac{\frac{\partial \tilde{u}}{\partial x_1}}{\frac{\partial \tilde{u}}{\partial x_2}} = -\frac{\frac{1}{2\sqrt{x_1}}\sqrt{x_2}}{\sqrt{x_1}\frac{1}{2}\sqrt{x_2}} = -\frac{2\sqrt{x_2}\sqrt{x_2}}{2\sqrt{x_1}\sqrt{x_1}} = -\frac{x_2}{x_1}.$

Since $MRS(u) = MRS(\tilde{u})$, the individual demands must be the same.

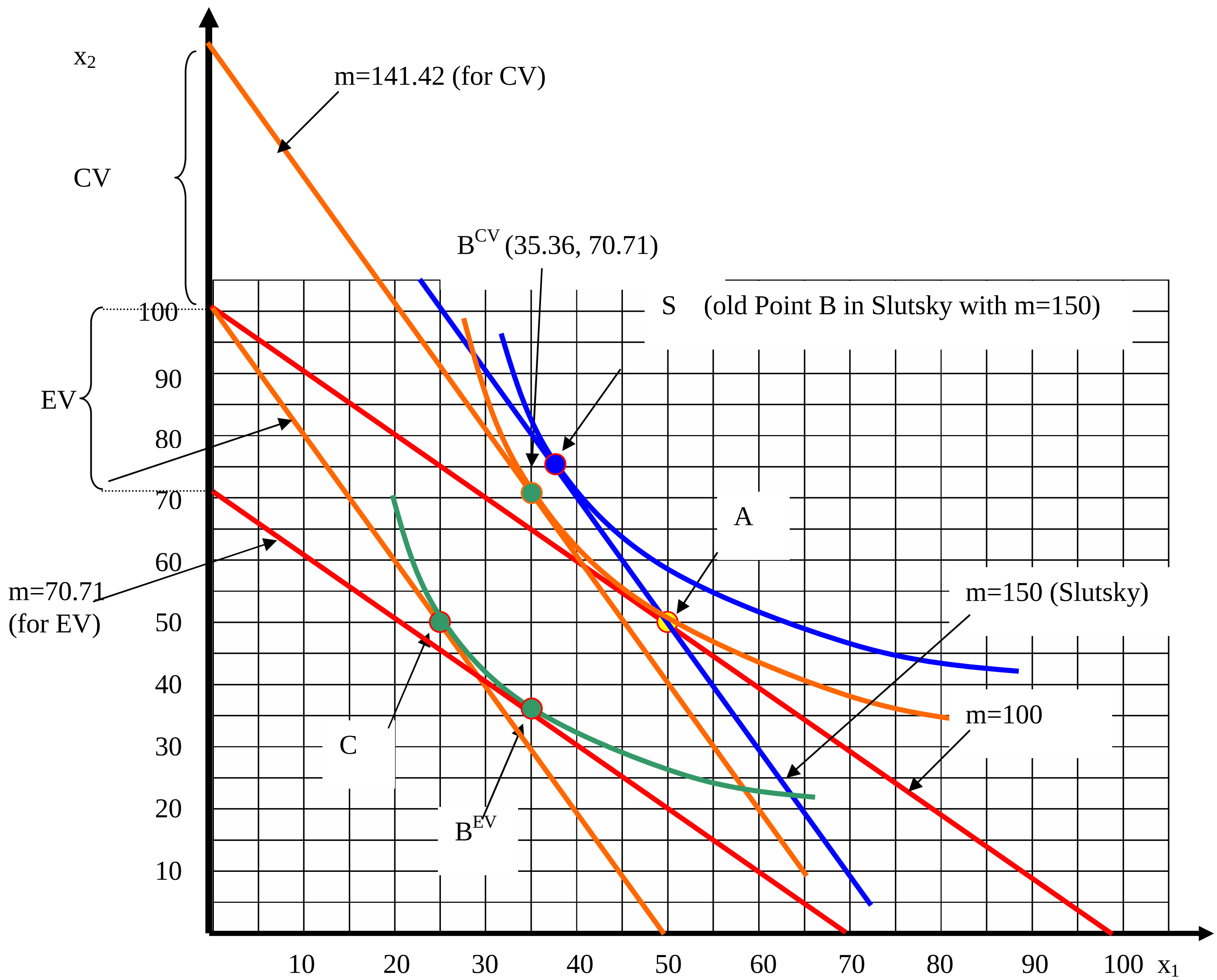
b) Now use the following initial numeric values: price of gasoline is $p_1 = 1$, budget is $m = 100$. Compute demands and utility for this initial bundle. Graphically illustrate the solution by drawing the budget line and this allocation (Point A) into the chart below.

From individual demands $x_1^* = \frac{m}{2p_1}, x_2^* = \frac{m}{p_2}$ we reach $x_1^* = 50$, $x_2^* = 50$, and

$u^*=50 \cdot 50=2500$.

c) Government is preparing a quantity tax on gasoline of $t = 1$. Assume the representative consumer carries the entire tax burden. Compute quantity of gasoline consumed and utility after the tax and illustrate this change in the diagram.

$$t = 1$$
$$p_1^{'} = p_1 + t = 2.$$
$$\Rightarrow x_1^{*'} = 25, x_2^{*'} = 50,$$
$$u' = 25 \cdot 50 = 1250.$$



d) Finding Slutsky Substitution effect mathematically and illustrate graphically (Point S above)

$$m' = 2 \cdot 50 + 50 = 150.$$

$$x_1'' = \frac{150}{2 \cdot 2} = 37.5$$

$$x_2'' = \frac{150}{2} = 75$$

$$\Delta x_1^s = x_1'' - x_1^* = 37.5 - 50 = -12.5$$

Income effect:

$$\Delta x_1^n = x_1^{*'} - x_1^* = 25 - 37.5 = -12.5$$

f) Mathematically find equivalent and compensating variation (EV and CV) to the tax, as well as net consumer surplus.

EV: We look for the income that needs to be taken off from the consumer such that she reaches the same utility level as after tax but at old prices:

u'=1250

At old prices and new (lower) utility we get the allocation Point B^{EV}:

$$x_1^{B^{EV}} = \frac{m''}{2p_1} = \frac{m''}{2} \qquad 1250 = \frac{m''}{2} \cdot \frac{m''}{2} = \frac{m''^2}{4} \Leftrightarrow m''^2 = 5000 \Leftrightarrow m'' = 70.71$$

$$EV = m - m'' = 100 - 70.71 = 29.29 \ .$$

$$x_2^{B^{EV}} = \frac{m''}{2}$$

Note that p. 257 has EXACTLY the numeric values of our example.

CV: We look for the income that needs to be added to the consumer such that she reaches the old utility level as after tax at new prices:

At new prices and old utility we get the allocation in Point B^{CV}:

$$x_1^{B^{CV}} = \frac{m'''}{2p_1'} = \frac{m'''}{4}$$

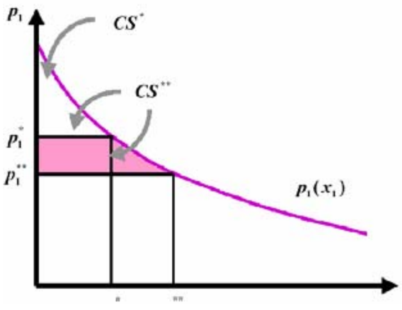
$$x_2^{B^{CV}} = \frac{m'''}{2}$$

Textbook version:

$$2500 = \frac{m'''}{4} \cdot \frac{m'''}{2} = \frac{m'''^2}{8} \Leftrightarrow m'''^2 = 20{,}000 \Leftrightarrow m''' = 141.42$$

$$CV = 141.42 - 100 = 41.42 \ .$$

Net Consumer Surplus: Recall that this is the pink area below: We integrate the demand curve over the interval $p_1^{**}=1$ and $p_1^{*}=2$ as shown below:



We know demand and that $p_1 = 1, p_1' = p_1 + t = 2$

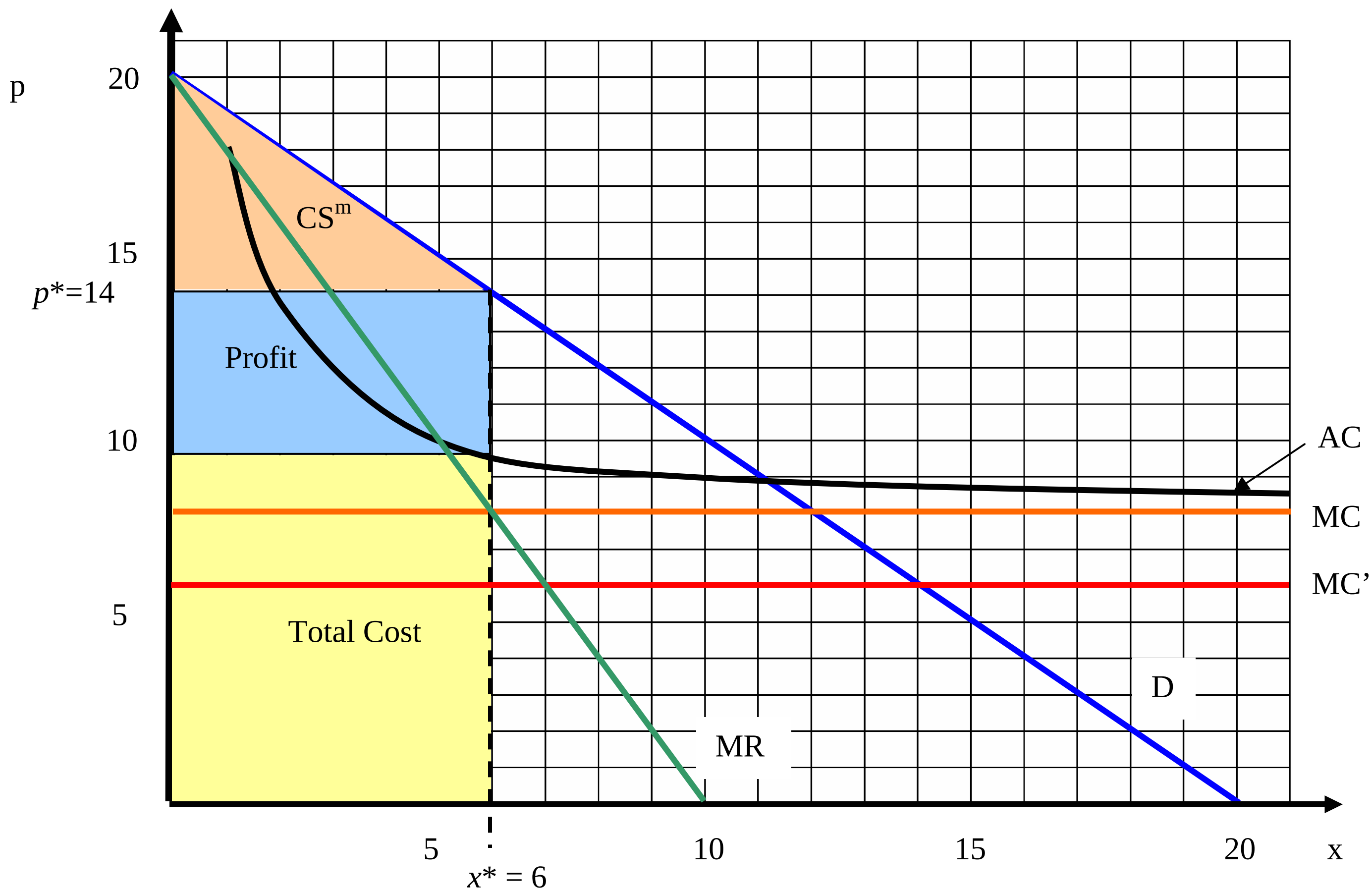
Demand is $x_1 = \frac{m}{2p_1}$.

$$NCS = \int_1^2 x_1(p_1)\,dp_1 = \int_1^2 \frac{m}{2p_1}dp_1 = \frac{m}{2}\frac{1}{p_1}dp_1 = \frac{m}{2}\left[\ln p_1\right]_1^2 = \frac{m}{2}\left[\ln 2 - \ln 1\right] = \frac{m}{2}(0.69 - 0) = 34.66$$

Comparing our findings, we have CV>NCS>EV.

(3) A monopolist faces an inverse demand curve given by $p = 20 - x$. The cost function is $C(x)=10+8x$.

(a) Derive and graph demand curve, marginal revenue, average cost and marginal cost. Find and mark profit-maximizing quantity, price, total cost, and the profit. Compute the Lerner index.



Given: demand curve $p(x)=20-x$

Cost curve: $C(x)$=10+8x

MC=$C'(x)$= 8 (const.)

$$AC = \frac{C(x)}{x} = \frac{10+8x}{x} = \frac{10}{x} + 8.$$

x	AC
1	18
2	13
5	10
10	9
20	8.5

R(x) = $p(x)x$ = (20-x)x

MR = R'(x) = 20-2x.

We find the profit maximizing output where MC=MR:

$$MC \overset{!}{=} MR \Leftrightarrow 8 = 20 - 2x \Leftrightarrow x^* = 6.$$
$$p^* = 20 - 6 = 14.$$

$$\pi = R(x) - C(x) = p^* x^* - AC \cdot x^* = 6 \cdot 14 - (10 + 8 \cdot 6) = 84 - 58 = 26.$$

$$LI = \frac{p^* - MC}{p^*} = \frac{14-8}{14} = \frac{6}{14} \approx 0.43 \ .$$

(b) Find the price elasticity of demand for the profit maximizing quantity.
Compute consumer surplus and compare the consumer surplus of the monopoly with the one under perfect competition (polypolists using the same technology). Mark both areas in the chart.

$$\varepsilon_{p,x} = \frac{dx}{dp}\frac{p^*}{x^*} = (-1)\frac{14}{6} \approx -2.3 \ .$$

The consumer surplus is the area below demand and above price p^*. It is the area of the triangle with the base equal to x^* and the height of the difference between the demand's p-intercept and p^*:

$$CS^m = \frac{(20 - p^*)(x^*)}{2} = \frac{(20-14) \cdot 6}{2} = 18.$$

In case of perfect competition, the consumer surplus increases to cover the entire triangle between MC and demand. Note that under perfect competition, x = 12 and p = MC.

$$CS^{pc} = \frac{(20-8)(12)}{2} = 72.$$

(c) Assume the monopolist introduces a new technology of production that reduces marginal cost to MC=6. Find the new profit maximizing quantity and the new price. Did the Lerner index change? Did the consumer surplus change?

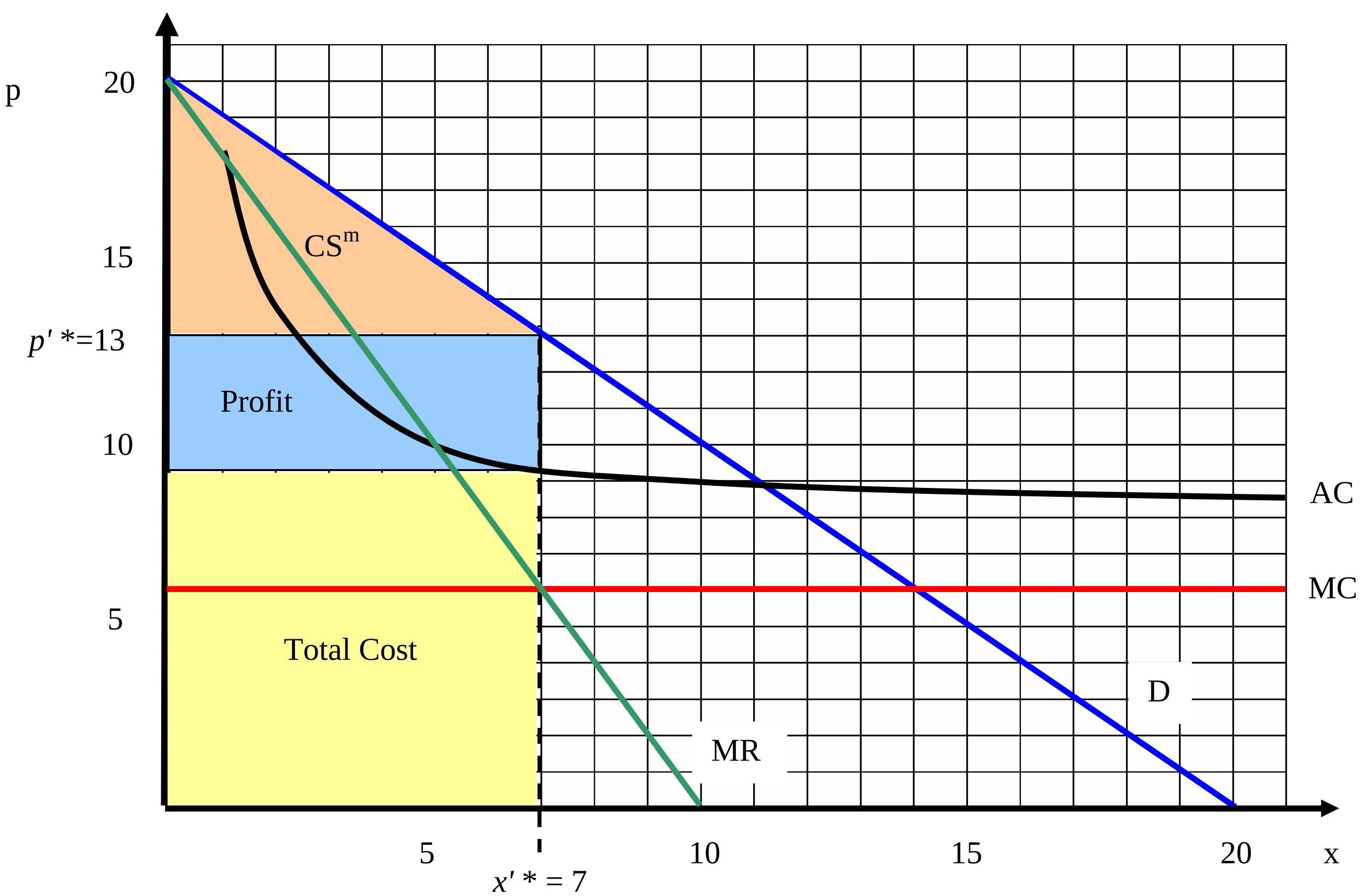
The new profit maximizing quantity and the new price denote:

$$MC' \overset{!}{=} MR \Leftrightarrow 6 = 20 - 2x \Leftrightarrow x'^* = 7.$$
$$p^* = 20 - 7 = 13.$$

Lerner index changes to $LI' = \frac{p^* - MC}{p^*} = \frac{13-6}{13} = \frac{7}{13} \approx 0.53$.

The consumer surplus increases to $CS'^m = \frac{(20-13)(7)}{2} = \frac{49}{2} = 24.5$.

New graph:



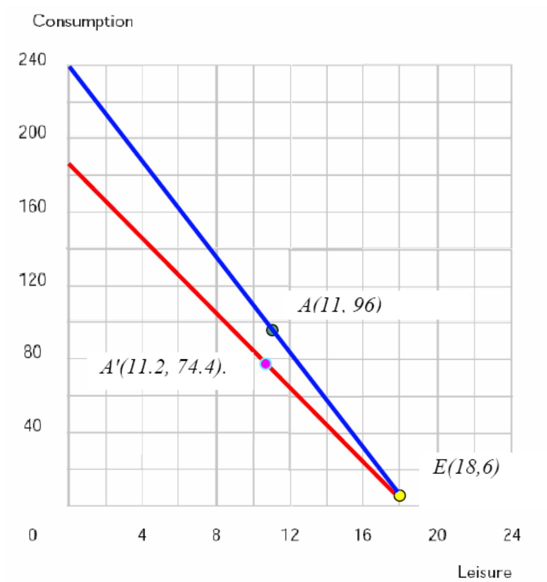
(4) Bill Brown works at Jenny's Roadside Cafe. He can work as many hours per day as he wishes at a wage rate of *w*. Let C be the number of dollars he spends on consumer goods and let R be the number of hours of leisure that he chooses.

(a) Bill is paid $13 an hour and has 18 hours per day to devote to labor or leisure. He furthermore receives a daily nonlabor income of $ 6 from dividends (his portfolio management was ok, but not that good in the past). Price of Consumption is $1. Write down the equation for his budget between consumption and leisure and draw this line into the chart below. Mark his endowment point.

Good 1 is Leisure (R), Good 2 is Consumption (C).
Budget line reads $C+wR=m+w\overline{R}$ *. That is, what Bill can buy of R (priced at w) and C (p=1) is equal to his nonlabor income times the value of his time endowment of 16 hours. Using numeric values Plugging in values we get:*

$$C+wR=m+w\overline{R}=6+234=240$$

We draw the budget line into the graph below (red) and mark the Endowment Point E(18,6):



b) Assume that Bill has Cobb-Douglas preferences for leisure and consumption of the style $U(R,C)=2R^3C^2$. How many hours will he work (you may round the result)? What will be his amount of Consumption worth in $?

We use the short version of the Lagrangian to derive the tangency condition for the function U(R,C)= $2R^3C^2$

$$L=2R^3C^2-\lambda(m+w\overline{R}-wR-C)$$

$$\left.\begin{aligned}\frac{\partial L}{\partial R}&=6R^2C^2-\lambda w=0\\ \frac{\partial L}{\partial C}&=4R^3C-\lambda=0\end{aligned}\right\}\frac{3}{2}\frac{C}{R}=w\Leftrightarrow C=\frac{2}{3}wR.$$

Substituting the tangency condition $C=\frac{2}{3}wR$ *into the above budget line, we have an equilibrium consumption of Leisure, which is:*

$$\frac{2}{3}wR+wR=240 \Leftrightarrow \frac{5}{3}wR=240 \Leftrightarrow R=\frac{3}{5}\frac{240}{13}11.07\approx 11$$

Consumption is either found by simply plugging into tangency: $C=\frac{2}{3}wR$ *=96 or by using the budget line C=240-13(11.07)=96.*

Last, his labor supply is the difference from the time endowment minus leisure: L=18-11 = 7 hours.

c) Write down Bill's Slutsky equation for labor and leisure and mark the signs of each term. Please use the correct notation for the variables. Can you predict without knowing the magnitude of each effect whether he will reduce or increase labor supply? Why or why not? Explain.

$$\frac{dR}{dw}=\underbrace{\frac{\partial R^s}{\partial w}}_{-}+\underbrace{(\overline{R}-R)}_{L\geq 0}\underbrace{\frac{\partial R}{\partial m}}_{+}$$

The first effect is the substitition effect, which is negative (implying an increase of relaxation with a decrease in wage). The term in parenthesis is positive or zero (labor). Since relaxation is a normal good (because of Cobb-Douglas), the income derivative $\partial R/\partial m$ *is positive. Therefore, we need to know the magnitude to tell whether Bill will reduce or increase his labor supply if wage increses (part d below).*

d) Jenny's Roadside Cafe receives competition from a new chic Seattle-based coffee chain. To save the business, all employees agree to work henceforth at a lower wage of *w*=10, so does Bill. Write down the equation for the new budget line and draw this new budget line into the graph on page 1. Mathematically find his new allocation. Does he increase or decrease labor supply? Thus, which effect has dominated, the substitution or the (total) income effect? Explain.

The budget line changes into C+wR= m+ w$\overline{R}=6+10\cdot 18=186$. *We again find the allocation by substituting C through the right-hand side of the tangency condition:*

$$\frac{2}{3}w'R+w'R=186 \Leftrightarrow \frac{5}{3}w'R=186 \Leftrightarrow R=\frac{3}{5}\frac{186}{10}=11.16\approx 11.2.$$

Thus, Bill has (slightly) increased his amount of leisure, and simultaneous decreased his labor supply to 18-11.2=6.8 hours of labor. His new consumption C is found again by using the tangency condition $C=\frac{2}{3}w'R=74.4$ *(or by using the budget line again, leading to the same result). The new budget line and new allocation is drawn (blue line) into the graph above, together with the new allocation A'(11.2, 74.4).*

We conclude that his labor supply has decreased, and therefore the substitution effect was larger in magnitude, compared to the joint income effect.

5) Tom's inverse demand function for running shoes is $p(q)=\frac{1}{8q^3}$.

a) Find his price elasticity of demand for price *p*.

Answer:

$$\varepsilon_{q,p}=\frac{dq}{dp}\frac{p}{q}.$$

We first find the price derivative $\frac{dq}{dp}$. *To derive, we find demand from inverse demand:*

$$p(q)=\frac{1}{8q^3}\Leftrightarrow q^3=\frac{1}{8p}, or \ \ q=\frac{1}{2\sqrt[3]{p}}. \ \ \textit{This can be simplified into:} \ \ q=\frac{1}{2}p^{-\frac{1}{3}}.$$

$$\frac{dq}{dp}=-\frac{1}{6}p^{-\frac{4}{3}}.$$

We plug into the elasticity formula:

$$\varepsilon_{q,p}=\left(-\frac{1}{6}p^{-\frac{4}{3}}\right)\frac{p}{q}=\left(-\frac{1}{6}p^{-\frac{4}{3}}\right)p\cdot\frac{1}{\frac{1}{2}p^{-\frac{1}{3}}}=\left(-\frac{1}{6p^{\frac{4}{3}}}\right)p\cdot\frac{2}{p^{\frac{1}{3}}}=\left(-\frac{1}{6p^{\frac{4}{3}}}\right)2p^{\frac{3}{3}-\frac{2}{3}}=-\frac{1}{3}.$$

b) If the price of running shoes is \$1, what is his price elasticity of demand?

Since price elasticity is a constant, the result does not change with the price. Thus,

$$\varepsilon_{q,p}\Big|_{p=1}=-\frac{1}{3}.$$

(6)** A monopolist faces a revenue function of $R(p)=\frac{p}{(p+1)^2}$. There are no marginal costs, so there is no difference between the monopolist finding the revenue maximizing price and the profit maximizing price.
Find the revenue maximizing price by setting *MR*(*p*) to zero (first-order condition).
Check the second order condition to see that by setting *MR*(*p*) to zero you indeed find a maximum.

Solution:

$$R(p)=\frac{p}{(p+1)^2}$$

First-order condition:

$$MR=0.$$

$$\frac{d\frac{p}{(p+1)^2}}{dp}=?$$

Using quotient rule:

Let p=f, let $(p+1)^2 = g$

$$For\ y = \frac{f}{g}\ y' = \frac{f' \cdot g - g' \cdot f}{g^2}$$

here:

$$\frac{d\frac{p}{(p+1)^2}}{dp} = MR = \frac{(p+1)^2 - 2(p+1)\cdot p}{(p+1)^4}\left[= \frac{1}{(p+1)^2} - \frac{2p}{(p+1)^3} \right]$$

$Setting\ \ MR = 0$

$$\frac{(p+1)^2 - 2(p+1)\cdot p}{(p+1)^4} = 0$$

A fraction is zero if the numerator is zero:

$$(p+1)^2 - 2(p+1)\cdot p = 0$$
$$(p+1)^2 = 2(p+1)\cdot p$$
$$p+1 = 2p$$
$$p = 1.$$

$We\,need\ to\ check\,the\,SOC: R'' < 0$ to have a maximum: in other words: the MR needs to have a negative slope:

Deriving $\frac{1}{(p+1)^2} - \frac{2p}{(p+1)^3}\ w.r.\ to\ p$:

$$R'' = \frac{6p}{(p+1)^4} - \frac{4}{(p+1)^3}$$

Plugging in the solution p=1 this is

$$R'' = \frac{6}{(2)^4} - \frac{4}{(2)^3} = \frac{6}{16} - \frac{4}{8} = \frac{3}{8} - \frac{4}{8} < 0$$

Thus, the SOC is fulfilled and revenue is indeed maximized at p=1.