Ch. 4: Utility

The Victorian view: Utility is an indicator of a person's overall well-being. It is as if we can compare individuals by their visible well-being.

Conceptual problems: Can we quantify the amount of utility associated with different choices? Particularly difficult: Interpersonal comparisons.

Q: How would you know 1 candy gives the same utility to Person A as it gives to Person B?

Utility function as a way of assigning a number

A utility function U(x) represents a preference relation $\succsim$ if and only if:

$x' \succ x''$	$U(x') > U(x'')$
$x' \prec x''$	$U(x') < U(x'')$
$x' \sim x''$	$U(x') = U(x'')$.

Ordinal utility

Utility is an ordinal (i.e. ordering) concept.

E.g. if U(x) = 6 and U(y) = 2 then bundle x is strictly preferred to bundle y.

But x is not preferred three times as much as is y.

Utility functions and indifference curves

Consider the bundles (4,1), (2,3) and (2,2).

Suppose $(2,3) \succ (4,1) \sim (2,2)$.

Assign to these bundles any numbers that preserve the preference ordering;

e.g. U(2,3) = 6 > U(4,1) = U(2,2) = 4.

Call these numbers utility levels.

Utility functions and indifference curves

An indifference curve contains equally preferred bundles.

Equal preference $\Rightarrow$ same utility level.

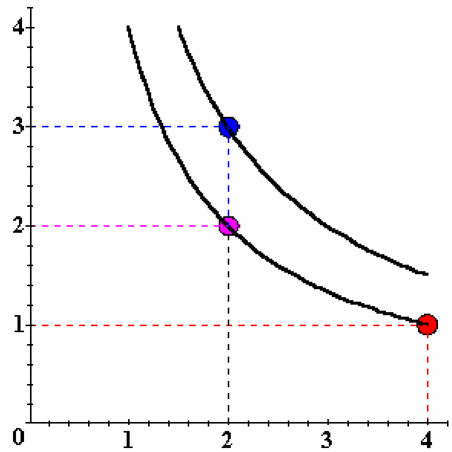
Therefore, all bundles in an indifference curve have the same utility level.

Utility functions and indifference curves

So the bundles (4,1) and (2,2) are in the indifference curve with utility level $U \equiv 4$

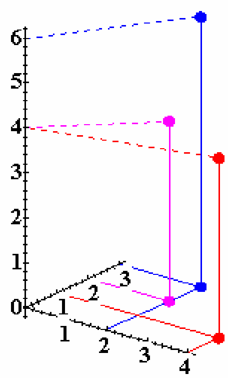
But the bundle (2,3) is in the indifference curve with utility level $U \equiv 6$.

Utility functions and indifference curves



Visualizing this same information:

Plotting the utility level on a vertical axis



Higher indifference curves contain “more preferred” bundles.

Utility Functions

Comparing all possible consumption bundles gives the complete collection of the consumer's indifference curves, each with its assigned utility level.

This complete collection of indifference curves completely represents the consumer's preferences.

Continuous Utility Functions

A preference relation that is complete, reflexive, transitive and continuous can be represented by a continuous utility function.

Continuity means that small changes to a consumption bundle cause only small changes to the preference level.

Utility Functions

The collection of all indifference curves for a given preference relation is an indifference map.

An indifference map is equivalent to a utility function; each is the other.

There is no unique utility function representation of a preference relation.

Suppose $U(x_1,x_2) = x_1x_2$ represents a preference relation.

Utility Functions

Again consider the bundles (4,1), (2,3) and (2,2).

$U(x_1,x_2) = x_1x_2$ so

$U(2,3) = 6 > U(4,1) = U(2,2) = 4;$

that is, $(2,3) \succ (4,1) \sim (2,2)$.

Preserving the same order

$U(x_1,x_2) = x_1x_2 \Rightarrow (2,3) \succ (4,1) \sim (2,2)$.

Now define $V = U^2$.

Then, $V(x_1,x_2) = x_1^2x_2^2$ and

$V(2,3) = 36 > V(4,1) = V(2,2) = 16$

so again

$(2,3) \succ (4,1) \sim (2,2)$.

$\Rightarrow$ V preserves the same order as U and so represents the same preferences.

Summing up

If

– U is a utility function that represents a preference relation $\succsim$

and

– f is a strictly increasing function,

then V = f(U) is also a utility function representing $\succsim$.

Page 55: Three valid utility functions

Table 4.1:

Bundle	U1	U2	U3
A	3	17	-1
B	2	10	-2
C	1	0.002	-3

Monotonic Transformation :

$f(u)$ transforms u in a sense that $u_1 > u_2$ implies $f(u_1) > f(u_2)$

Note that this also implies that the difference $f(u_2) - f(u_1)$ must have the same *sign* as the difference $u_2 - u_1$.

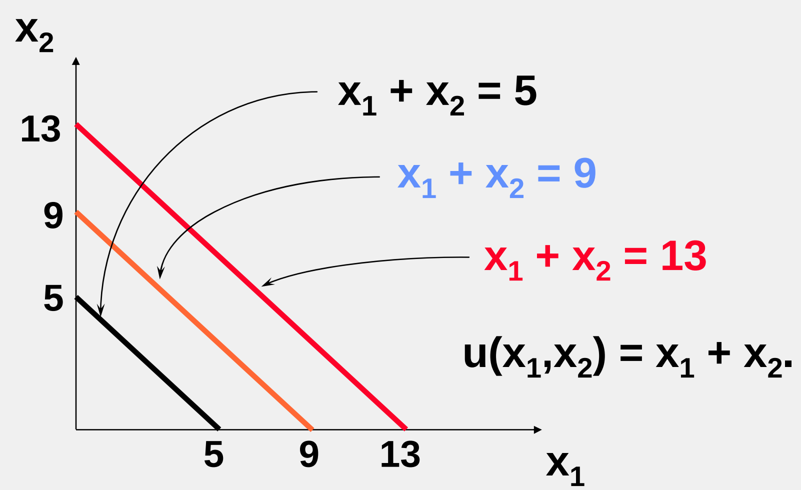
Monotonic Transformation

1 Assume $(x_1, x_2) \succ (y_1, y_2)$ with $u(x_1, x_2)$ representing $u(x_1, x_2) > u(y_1, y_2)$.

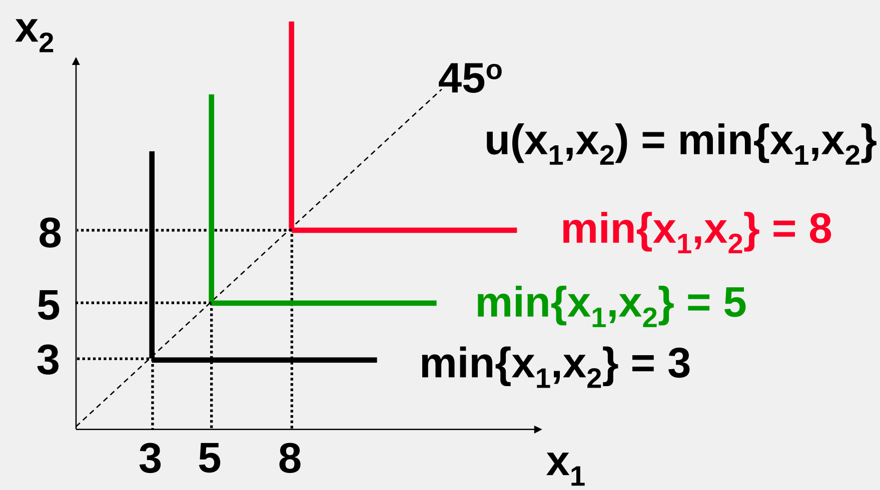
2 Now assume an $f(u)$ (monotonic transformation) that whenever $f(u(x_1, x_2)) > f(u(y_1, y_2))$ also $u(x_1, x_2) > u(y_1, y_2)$ holds.

3 If $(x_1, x_2) \succ (y_1, y_2)$ then $f(u(x_1, x_2)) > f(u(y_1, y_2))$.
$f(u)$ represents the preferences in the same way as the original utility function $u(x_1, x_2)$.

Perfect Substitutes: Notation of u



Perfect Complements: Notation of u



Utility $u(x_1,x_2)$: Two important properties

1. Positive marginal utility

$$MU_i > 0, \quad i = 1,2$$

2. Decreasing marginal utility

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial MU_i}{\partial x_i} < 0, \quad i = 1,2$$

Marginal utilities as a ratio

$$MU_1 = \frac{\Delta u}{\Delta x_1} = \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1}$$

$$MU_2 = \frac{\Delta u}{\Delta x_2} = \frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{\Delta x_2}$$

Marginal Utilities

$$\Delta x_1 \to 0 \quad \Rightarrow MU_1 = \frac{\partial u}{\partial x_1}$$

$$\Delta x_2 \to 0 \quad \Rightarrow MU_2 = \frac{\partial u}{\partial x_2}$$

Marginal Utilities

The marginal utility of commodity i is the rate-of-change of total utility as the quantity of commodity i consumed changes; i.e.

$$MU_i = \frac{\partial U}{\partial x_i}$$

E.g. *if $U(x_1,x_2) = x_1^{1/2} x_2^2$ then*

$$MU_1 = \frac{\partial U}{\partial x_1} = \frac{1}{2} x_1^{-1/2} x_2^2 \qquad \textbf{and} \qquad MU_2 = \frac{\partial U}{\partial x_2} = 2x_1^{1/2} x_2$$

Marginal Utilities and MRS

Discrete observation: ***MRS is the rate at which a consumer is just willing to substitute a small amount of good 2 for good 1.***

$$MU_1 \cdot \Delta x_1 + MU_2 \cdot \Delta x_2 = 0$$

$$\frac{\Delta x_2}{\Delta x_1} = MRS = -\frac{MU_1}{MU_2}$$

Marginal Utilities and MRS

Continuous observation:

$U(x_1,x_2) \equiv k$, a constant.

Totally differentiating this identity yields

$$\frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0$$

$$\frac{\partial U}{\partial x_2} dx_2 = -\frac{\partial U}{\partial x_1} dx_1$$

Marginal Utilities and MRS

And

$$\frac{\partial U}{\partial x_2} dx_2 = -\frac{\partial U}{\partial x_1} dx_1$$

The “per-unit change in utility” =MU

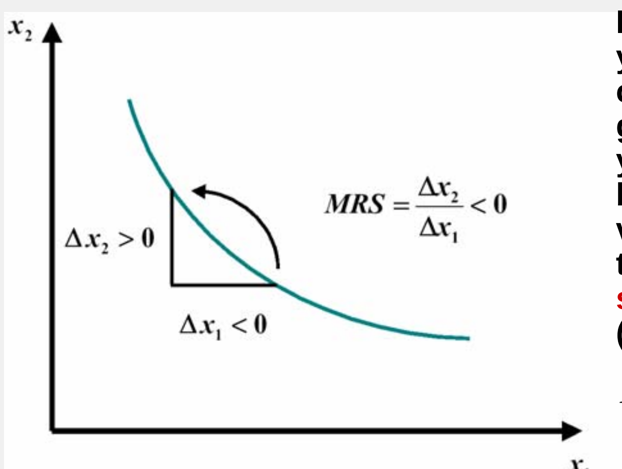
Changes in units of goods

rearranged is

$$\frac{d\,x_2}{d\,x_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2}.$$

This is the MRS.

MRS: Interpretation

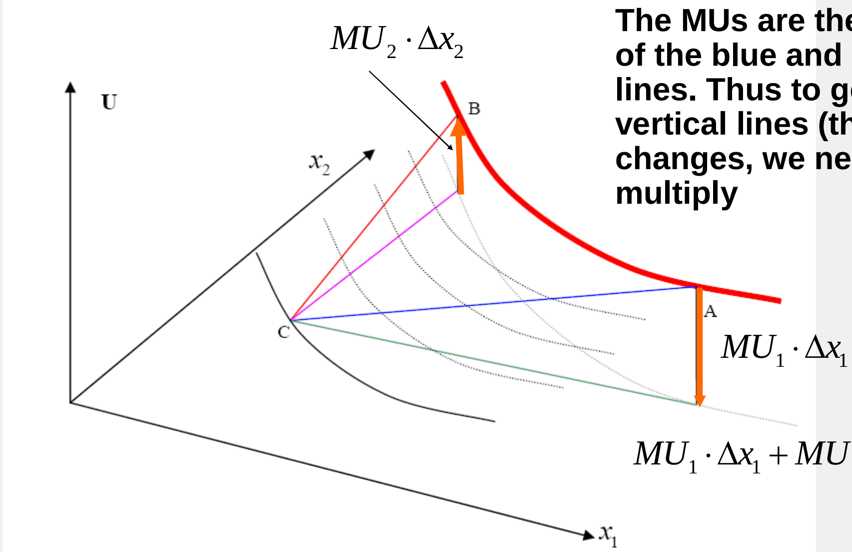


Intuition of MRS: What you take away from the consumer as units of good 1 worth in utils, you need to give him back in units of good 2, valued in utils, should the consumer stay at the same utility level (indifference curve).

$$MU_1 \cdot \Delta x_1 + MU_2 \cdot \Delta x_2 = 0$$

Problem: What you see here is only the two goods, not the differences in utility. This is because the graph only shows a horizontal section of the utility function.

MRS: Interpretation



The MUs are the slope of the blue and red lines. Thus to get the vertical lines (the utility changes, we need to multiply

$$MU_1 \cdot \Delta x_1 + MU_2 \cdot \Delta x_2 = 0$$

Example 1

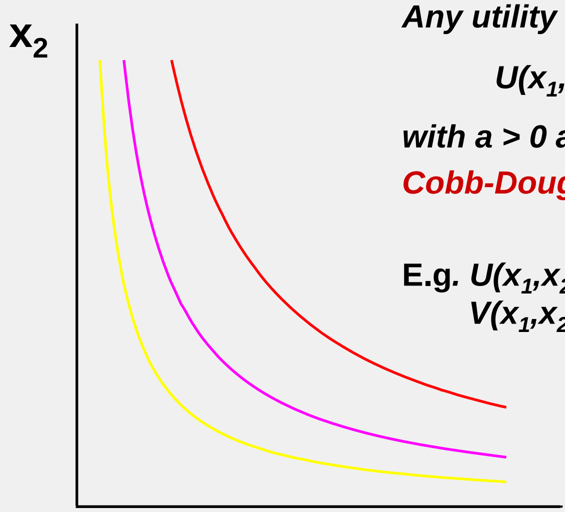
Suppose $U(x_1,x_2) = x_1x_2$. Then

$$\frac{\partial U}{\partial x_1} = (1)(x_2) = x_2$$

$$\frac{\partial U}{\partial x_2} = (x_1)(1) = x_1$$

so $$MRS = \frac{dx_2}{dx_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2} = -\frac{x_2}{x_1}.$$

Cobb-Douglas Indifference Curves



Any utility function of the form

$U(x_1,x_2) = x_1^a x_2^b$

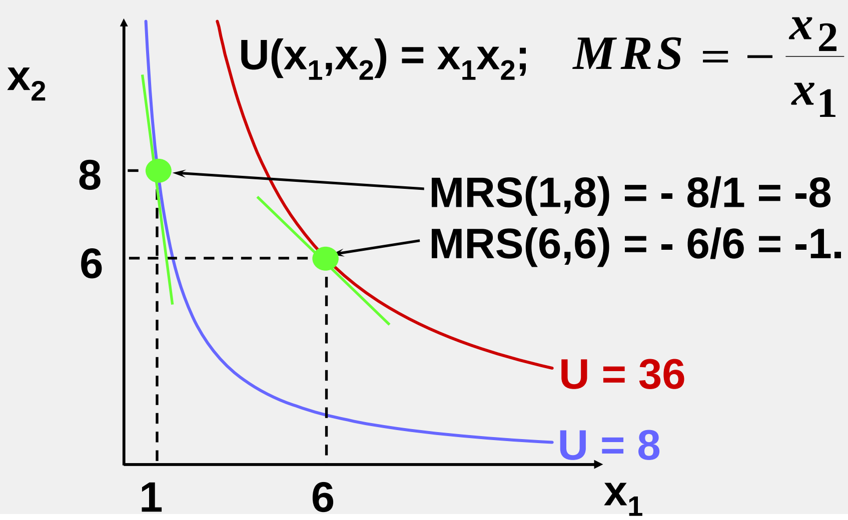
with a > 0 and b > 0 is called a ***Cobb-Douglas utility function.***

E.g. $U(x_1,x_2) = x_1^{1/2} x_2^{1/2}$ (a = b = 1/2)

$V(x_1,x_2) = x_1 x_2^3$ (a = 1, b = 3)

Example 1

$U(x_1,x_2) = x_1x_2$; $MRS = -\frac{x_2}{x_1}$



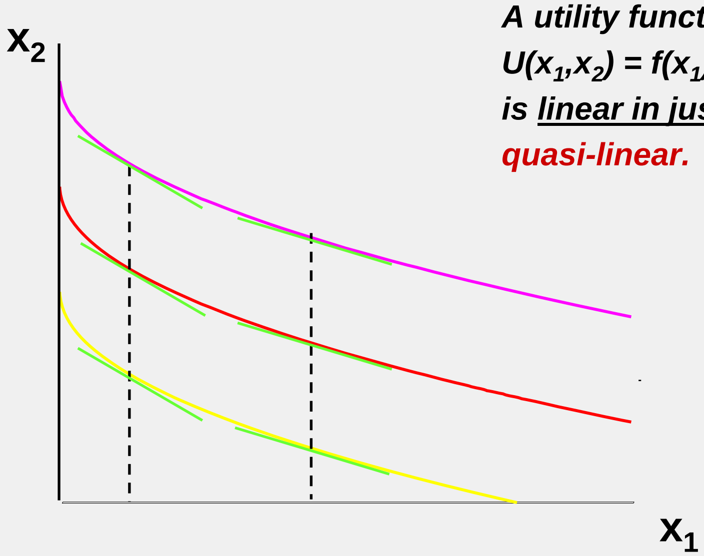
Example 2: Quasi-linear Utility Functions

$U(x_1,x_2) = f(x_1) + x_2$.

$$\frac{\partial U}{\partial x_1} = f'(x_1) \qquad \frac{\partial U}{\partial x_2} = 1$$

so $$MRS = \frac{d\,x_2}{d\,x_1} = -\frac{\partial U / \partial x_1}{\partial U / \partial x_2} = -f'(x_1).$$

Quasi-linear Indifference Curves



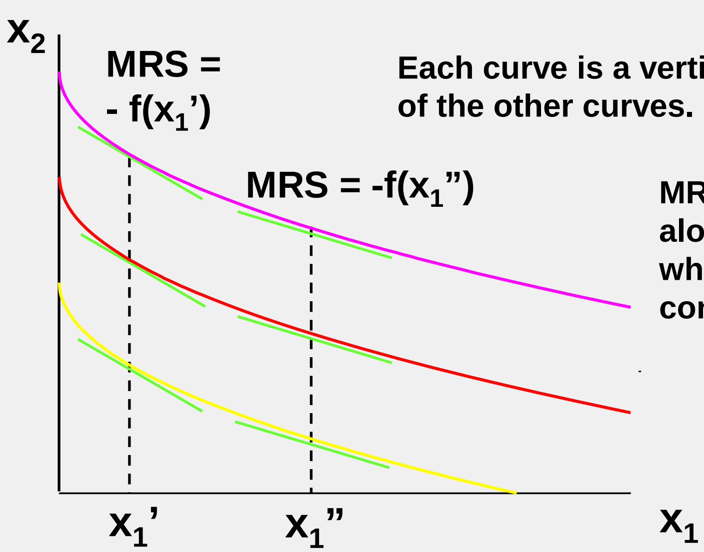
*A utility function of the form $U(x_1,x_2) = f(x_1) + x_2$ is linear in just x_2 and is called **quasi-linear.***

Why we use quasi-linear Utility Functions

MRS = - f ’ (x_1) does not depend on x_2 so the slope of indifference curves for a quasi-linear utility function is constant along any line for which x_1 is constant.

What does that make the indifference map for a quasi-linear utility function look like?

MRS for Quasi-linear Utility Functions



Each curve is a vertically shifted copy of the other curves.

MRS is a constant along any line for which x_1 is constant.