

## Midterm Exam 2 – Solutions

(1) Kelly Umpleby has a sister, Kate. Kate grows strawberries (good 1) and rhubarb (good 2). Like all the Umpleby's, Kate is a Cobb-Douglas consumer and has preferences that follow the utility function  $u(x_1, x_2) = x_1 x_2^{\frac{1}{3}} = x_1 \sqrt[3]{x_2}$ . Strawberries and rhubarb have positive prices.

(a) Use Lagrangian to compute Kate's individual demands for strawberries and rhubarb. For simplicity we assume throughout this problem that  $p_2 = 1$ .

*Suggested answer:*

$$u(x_1, x_2) = x_1 x_2^{\frac{1}{3}}$$

*Setting up the Lagrangian leads to*

$$L(x_1, x_2, \lambda) = x_1 x_2^{\frac{1}{3}} - \lambda(p_1 x_1 + x_2 - m)$$

$$\frac{dL}{dx_1} = x_2^{\frac{1}{3}} - \lambda p_1 \doteq 0 \quad (1)$$

$$\frac{dL}{dx_2} = \frac{1}{3} x_1 x_2^{-\frac{2}{3}} - \lambda \doteq 0 \quad (2)$$

$$\frac{dL}{d\lambda} = m - p_1 x_1 - x_2 \doteq 0 \quad (3)$$

Dividing (1) by (2) leads to the tangency condition:

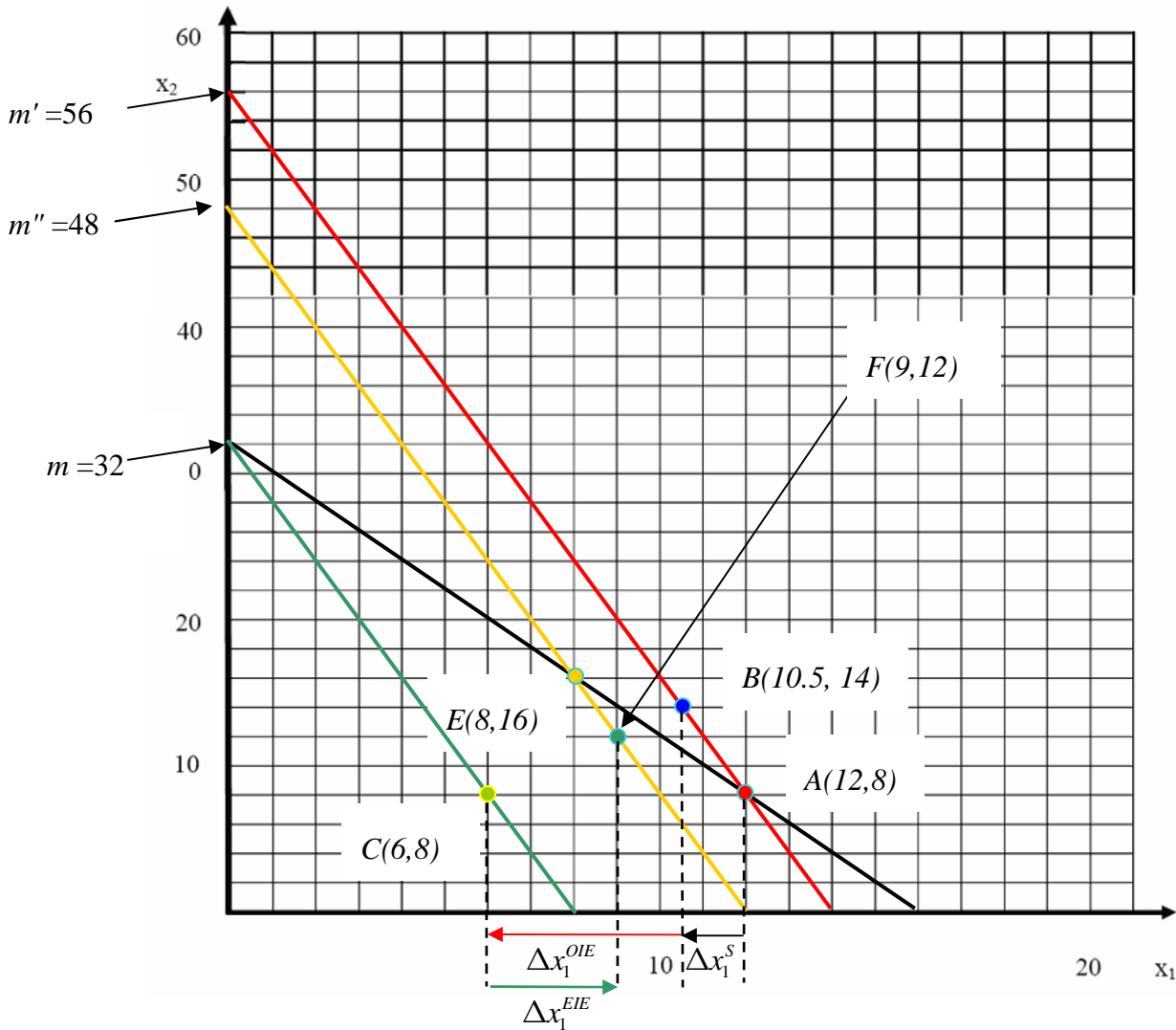
$$\frac{x_2^{\frac{1}{3}}}{\frac{1}{3} x_1 x_2^{-\frac{2}{3}}} = p_1 \Leftrightarrow \frac{3x_2^{\frac{1}{3}} x_2^{\frac{2}{3}}}{x_1}, \text{ yielding } x_2 = \frac{1}{3} p_1 x_1.$$

Plugging the right-hand side into the budget constraint (3) yields individual demand for  $x_1$ :

$$p_1 x_1 + \frac{1}{3} p_1 x_1 = \frac{4}{3} p_1 x_1 = m \Leftrightarrow x_1 = \frac{3m}{4p_1}.$$

$$x_2 = \frac{1}{3} p_1 x_1 = \frac{1}{3} p_1 \frac{3m}{4p_1} = \frac{m}{4}.$$

(b) Kate harvests 8 baskets of strawberries and 16 baskets of rhubarb. Mark her endowment bundle E in the graph below. The market price for strawberries is \$2. Find Kate's original income and draw the budget line into the graph (mark it with "m=...").



Her endowment point is given:  $E(8,16)$ . Initial income is found by using

$$m = p_1\omega_1 + p_2\omega_2 = 2 \cdot 8 + 16 = 32.$$

(c) By using individual demands find Kate's original consumption bundle A and mark it in the graph. Compute Kate's utility in A.

$$\text{Using } m=32 \text{ we have } x_1^A = \frac{3 \cdot 32}{4 \cdot 2} = 12, \quad x_2^A = \frac{32}{4} = 8, \quad \Rightarrow A(12,8)$$

$$u^A = x_1^A \sqrt[3]{x_2^A} = 12 \sqrt[3]{8} = 24.$$

(d) Now assume that the market price for strawberries increases to  $p_1' = 4$ . Find Slutsky demand and her compensated bundle. Draw the new compensated budget line into the graph

graph as well and mark it with "  $m'=. .$  ". Find the Slutsky substitution effect. Indicate the numeric value of the substitution effect along the  $x_1$ -axis.

*We start by computing Slutsky income compensation: How much money does Kate need should she still be able at new prices to still be able to buy 12 units of good 1 after the price increase?*

$$\text{This is } \Delta m = \Delta p_1 x_1^A = (4 - 2) \cdot 12 = 24.$$

$$m' = m + \Delta m = 32 + 24 = 56.$$

*(See red budget line).*

*We compute the compensated bundle B at new income and new prices:*

$$\left. \begin{aligned} x_1^B &= x_1(m', p_1') = \frac{3 \cdot 56}{4 \cdot 4} = 10.5 \\ x_2^B &= x_2(m') = \frac{56}{4} = 14 \end{aligned} \right\} B(10.5, 14)$$

*The value for the substitution effect is*

$$\Delta x_1^s = x_1^B - x_1^A = 10.5 - 12 = -1.5 \quad . \quad (\text{see black arrow on graph})$$

(e) Find her consumption bundle using the original income  $m$  and at new prices. Label this bundle "C" and compute Kate's utility in C. Find the value her ordinary income effect and mark it in the graph as well. Also, draw the budget line that goes through C into the diagram.

*Giving back the original  $\Delta m$  of compensation lifts income back to  $m$ , but at new prices, with the following demands:*

$$\left. \begin{aligned} x_1^C &= x_1(m, p_1') = \frac{3 \cdot 32}{4 \cdot 4} = 6 \quad . \\ x_2^C &= x_2(m) = \frac{32}{4} = 8 = x_2^A. \end{aligned} \right\} C(6, 8)$$

$$\text{Her utility in C is } u^C = x_1 \sqrt[3]{x_2} = 6 \sqrt[3]{8} = 12.$$

*The value of the ordinary income effect is*

$$\Delta x_1^{OIE} = x_1^C - x_1^B = 6 - 10.5 = -4.5 \quad . \quad (\text{see red arrow on graph}).$$

(f) Recalculate her income at new prices and draw the new budget line through E into the diagram and mark it with "  $m''$  ". Find her final consumption bundle and the endowment income effect and mark it in the graph as well.

While price increase reduces consumption of good 1 following the ordinary income effect, it benefits her by increasing her final income in E. Her endowment at new prices is worth more, namely  $m'' = p_1' \omega_1 + \omega_2 = 4 \cdot 8 + 16 = 48$ .

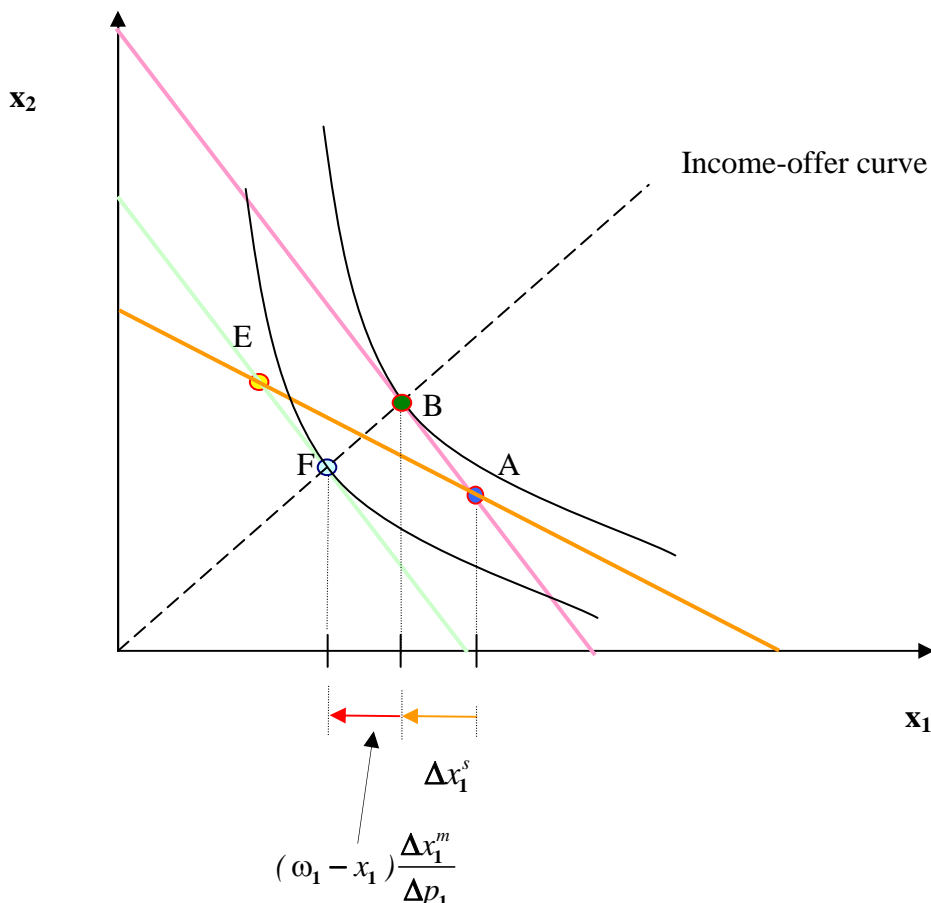
We find individual demands for her final consumption bundle F to be:

$$\left. \begin{aligned} x_1^F &= x_1(m'', p_1') = \frac{3 \cdot 48}{4 \cdot 4} = 9 \\ x_2^F &= x_2(m'') = \frac{48}{4} = 12 \end{aligned} \right\} F(9, 12)$$

The value of the endowment income effect is

$$\Delta x_1^{EIE} = x_1^F - x_1^C = 12 - 9 = 3. \quad (\text{see green arrow on graph})$$

(g) Write down Kate's Slutsky equation for buying and selling. Can you tell from the sign of the substitution effect and the (combined) income effect that she will reduce consumption because of the price increase? Refer to each part of the equation and explain what direction of demand change the equation predicts in this case, both verbally and graphically. Use the graph below to illustrate, sketching the original budget line plus the budget lines at new prices that as well go through A and E. Mark the income offer curve into the diagram and the substitution effect as well as the combined income effect.



Comments:

The substitution effect is the reduction in  $x_1$ -consumption from A to B, the net income effect is  $(\omega_1 - x_1) \frac{\Delta x_1^m}{\Delta p_1}$  and thus the reduction in  $x_1$ -consumption from B to F. It is easy to see that **as long as she is a net buyer, F is always left of B**, and the both substitution and net income effect point into the same direction as indicated in the Slutsky equation.

(h) Find her compensating variation (CV) and her equivalent variation (EV) following the price increase. You may either use

- Hicks demand through finding  $B^{CV}$  and  $B^{EV}$  and compute the changed income in these points, *or*

- the "Varian method" that involves plugging in demand into utility.

Verbally explain the meaning of CV and EV in this context.

- *Version 1: We find Hicks demand:*

$u = x_1 x_2^{\frac{1}{3}}$ . Using tangency condition we eliminate  $x_2$  and solve for  $x_1$ :

$$u = x_1 \left(\frac{1}{3} p_1 x_1\right)^{\frac{1}{3}} \Leftrightarrow u^3 = \frac{1}{3} p_1 x_1^{\frac{4}{3}}$$

$$x_1 = \sqrt[4]{\frac{3u^3}{p_1}}.$$

Finding CV: From the first part we know that  $u^A = x_1^A \sqrt[3]{x_2^A} = 12 \sqrt[3]{8} = 24$ .

Point  $B^{CV}$  has the the coordinates:

$$x_1 = \sqrt[4]{\frac{3(u^A)^3}{p_1'}} = \sqrt[4]{\frac{3(24)^3}{4}} = \sqrt[4]{10368} \approx 10.09.$$

$$x_2 = \frac{1}{3} p_1' x_1 = \frac{4}{3} x_1 = 13.45.$$

$$\Rightarrow B^{CV} = (10.09, 13.45)$$

$$m^{CV} = 4(10.09) + 13.45 = 53.81$$

$$CV = m^{CV} - m = 53.81 - 32 = 21.81$$

Finding EV: From the first part we know that  $u^C = x_1^C \sqrt[3]{x_2^C} = 16 \sqrt[3]{8} = 12$ .

Point  $B^{EV}$  has the the coordinates:

$$x_1 = \sqrt[4]{\frac{3(u^C)^3}{p_1}} = \sqrt[4]{\frac{3(12)^3}{2}} = \sqrt[4]{2592} \approx 7.13 .$$

$$x_2 = \frac{1}{3} p_1 x_1 = \frac{2}{3} x_1 \approx 4.75 .$$

$$\Rightarrow B^{EV} = (7.13, 4.75)$$

$$m^{EV} = 2(7.13) + 4.75 \approx 19 .$$

$$EV = 32 - 19 = 13.$$

**- Version 2: "Varian Method"**

$$CV: u^A = x_1 \sqrt[3]{x_2} = 12 \sqrt[3]{8} = 24.$$

$$\left( \frac{3m^{CV}}{4p_1'} \right) \left( \frac{m^{CV}}{4} \right)^{\frac{1}{3}} = 24$$

$$\left( \frac{3m^{CV}}{16} \right) \left( \frac{m^{CV}}{4} \right)^{\frac{1}{3}} = 24$$

$$\frac{3m^{CV \frac{4}{3}}}{16(4)^{\frac{1}{3}}} = 24$$

$$m^{CV \frac{4}{3}} = 128(4)^{\frac{1}{3}}$$

$$m^{CV} = \left( 128(4)^{\frac{1}{3}} \right)^{\frac{3}{4}} = 128^{\frac{3}{4}} \sqrt{2} = 53.817$$

$$CV = m^{CV} - m = 21.8$$

$$EV: u^C = x_1 \sqrt[3]{x_2} = 6 \sqrt[3]{8} = 12.$$

$$\left( \frac{3m^{EV}}{4p_1} \right) \left( \frac{m^{EV}}{4} \right)^{\frac{1}{3}} = 12$$

$$\frac{3m^{EV \frac{4}{3}}}{8(4)^{\frac{1}{3}}} = 12$$

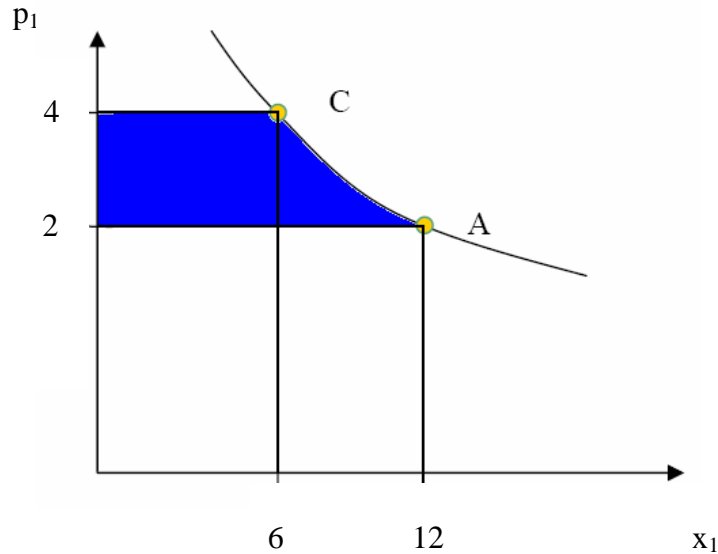
$$m^{EV \frac{4}{3}} = 32(4)^{\frac{1}{3}}$$

$$m^{EV} = \left( 32(4)^{\frac{1}{3}} \right)^{\frac{3}{4}} = 32^{\frac{3}{4}} \sqrt{2} = 19$$

$$EV = m - m^{EV} = 13.$$

(i) Mathematically find the change in consumer surplus (or net consumer surplus, NCS) for good 1, graphing original demand and original price into the diagram below, as well as demand and price in Bundle C. Label the axes. Compare CV, the change in consumer surplus, and EV. Explain verbally: why do they differ?

$$\begin{aligned} NCS &= \int_2^4 x_1(p_1) dp_1 = \int_2^4 \frac{3m}{4p_1} dp_1 = 24 \int_2^4 \frac{1}{p_1} dp_1 = 24 [\ln p_1]_2^4 = 16 [\ln 4 - \ln 2] = \\ &= 24(1.386 - 0.693) = 24(0.693) = 16.63 . \end{aligned}$$

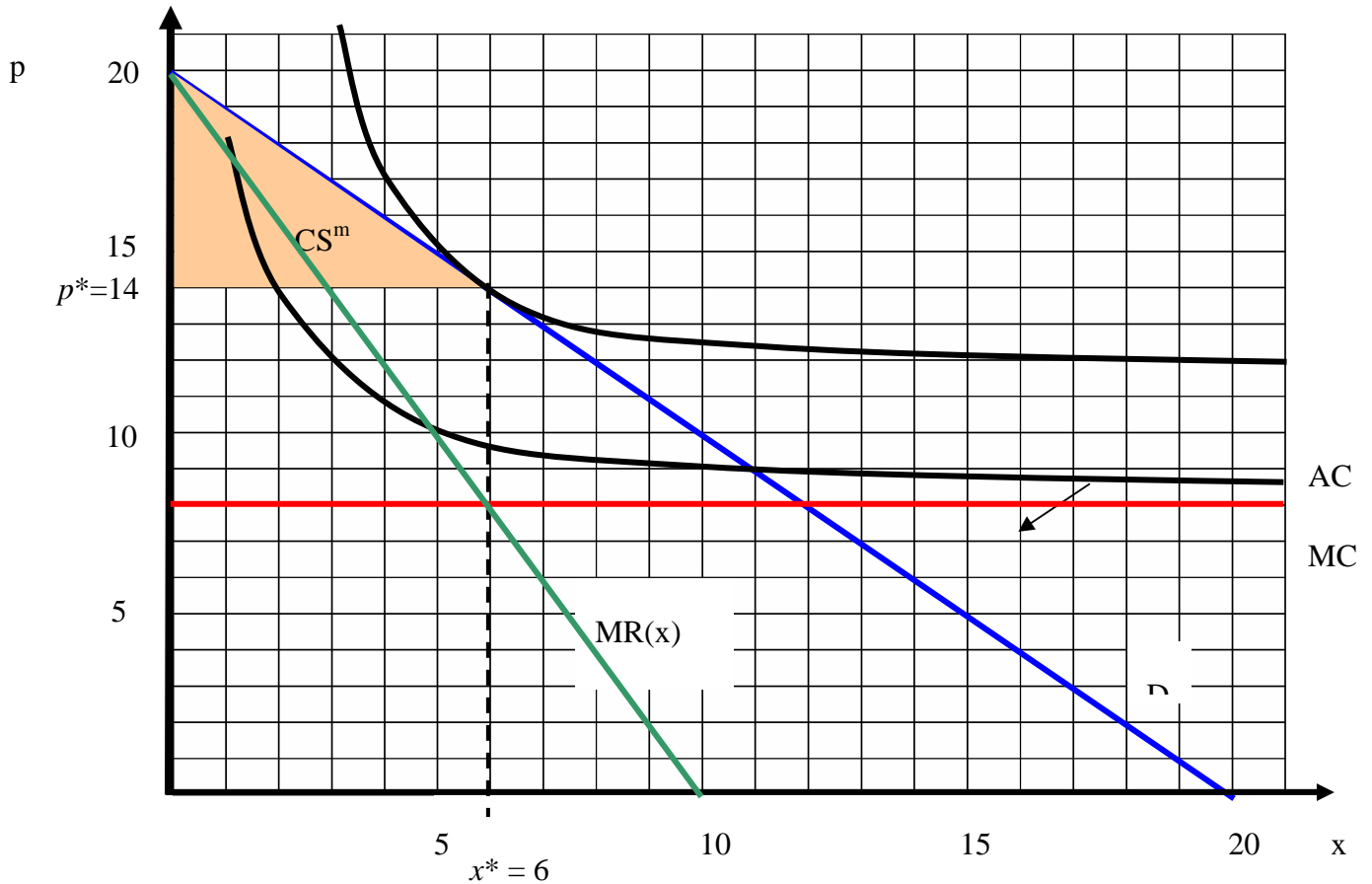


Comparison  $EV < NCS < CV$ :  $EV=13$ ,  $NCS=16.63$ ,  $CV=21.81$  .

We reach different results since the preferences for good 1 are not quasilinear. Thus, an increase in

(2) A monopolist faces an inverse demand curve given by  $p = 20 - x$ . The cost function is  $C(x) = 10 + 8x$ .

(a) Derive and graph demand curve, marginal revenue, average cost and marginal cost. Find and mark profit-maximizing quantity, price, total cost, and the monopolist's profit. To draw the average cost function  $AC(x)$ , draw a table and use 5 value pairs.



Demand curve:  $p(x) = 20 - x$

Cost curve:  $C(x) = 10 + 8x$

$MC = C'(x) = 8$  (const.)

$$AC = \frac{C(x)}{x} = \frac{10 + 8x}{x} = \frac{10}{x} + 8.$$

x	AC1
1	18
2	13
5	10
10	9
20	8.5

$$R(x) = p(x)x = (20-x)x$$

$$MR = R'(x) = 20-2x.$$

We find the profit maximizing output where  $MC=MR$ :

$$MC = MR \Leftrightarrow 8 = 20 - 2x \Leftrightarrow x^* = 6.$$

$$p^* = 20 - 6 = 14.$$

$$\pi = R(x) - C(x) = p^* x^* - AC \cdot x^* = 6 \cdot 14 - (10 + 8 \cdot 6) = 84 - 58 = 26.$$

(b) Find the price elasticity of demand for the profit maximizing quantity. Show that

$$\frac{p^* - MC}{p^*} = \frac{1}{|\varepsilon|}.$$

Elasticity for the profit maximizing price and quantity is found by using

$$\varepsilon_{p,x} = \frac{dx}{dp} \frac{p^*}{x^*} = (-1) \frac{14}{6} = -\frac{7}{3}.$$

To show that  $\frac{p^* - MC}{p^*} = \frac{1}{|\varepsilon|}$ , we compute both sides:

$$\frac{p^* - MC}{p^*} = \frac{14 - 8}{14} = \frac{3}{7}.$$

The inverse of the absolute value of elasticity is  $\frac{1}{|\varepsilon|} = \frac{1}{\frac{7}{3}} = \frac{3}{7}$ .

(c) Assume the monopolist's fixed cost increase from 10 to 36. Again compute five value pairs for the new average cost function  $AC(x)$  and draw the function into the diagram. Does the profit maximizing quantity change? Find and mark the new total cost and the new profit of the monopolist. Comment on the new profit situation. What do you observe?

We first compute the new AC curve (graphed above).

x	AC2
1	44
2	26
3	20
6	14
9	12
12	11

The profit-maximizing quantity is computed by setting  $MR=MC$ . A change in fixed cost does not affect the profit-maximizing quantity. It however affects total cost and thus profit: At  $x=6$ , total cost now equate total revenue, and the profit is zero.

3) Bill Brown works at Jenny's Roadside Cafe. He can work as many hours per day as he wishes at a wage rate of  $w$ . Let  $C$  be the number of dollars he spends on consumer goods and let  $R$  be the number of hours of leisure that he chooses.

(a) Bill is paid \$13 an hour and has 18 hours per day to devote to labor or leisure. He furthermore receives a daily nonlabor income of \$ 6 from dividends (his portfolio management was ok, but not that good in the past). Price of Consumption is \$1. Write down the equation for his budget between consumption and leisure and draw this line into the chart below. Mark his endowment point.

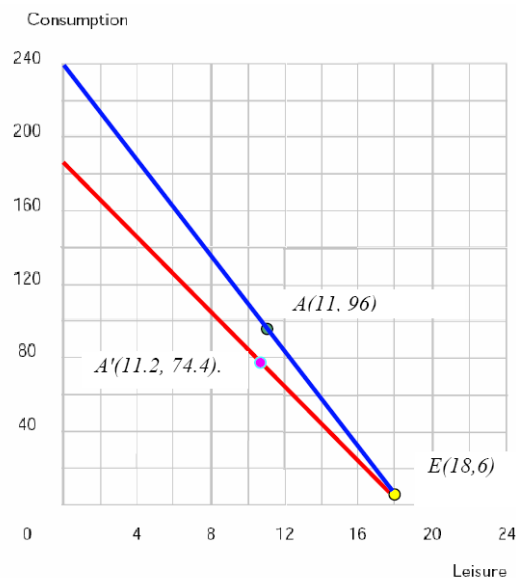
*Good 1 is Leisure (R), Good 2 is Consumption (C).*

*Budget line reads  $C+wR=m+ w\bar{R}$ . That is, what Bill can buy of R (priced at w) and C ( $p=1$ ) is equal to his nonlabor income times the value of his time endowment of 16 hours. Using numeric values Plugging in values we get:*

$$C+wR= m+ w\bar{R} = 6+ 234 = 240$$

*We draw the budget line into the graph below (red) and mark the Endowment Point*

*E(18,6):*



b) Assume that Bill has Cobb-Douglas preferences for leisure and consumption of the style  $U(R,C)=2R^3C^2$ . How many hours will he work (you may round the result)? What will be his amount of Consumption worth in \$?

*We use the short version of the Lagrangian to derive the tangency condition for the function  $U(R,C)= 2R^3C^2$*

$$L = 2R^3C^2 - \lambda(m + w\bar{R} - wR - C)$$

$$\left. \begin{array}{l} \frac{\partial L}{\partial R} = 6R^2C^2 - \lambda w = 0 \\ \frac{\partial L}{\partial C} = 4R^3C - \lambda = 0 \end{array} \right\} \frac{3}{2} \frac{C}{R} = w \Leftrightarrow C = \frac{2}{3} wR.$$

Substituting the tangency condition  $C = \frac{2}{3}wR$  into the above budget line, we have an equilibrium consumption of Leisure, which is:

$$\frac{2}{3}wR + wR = 240 \Leftrightarrow \frac{5}{3}wR = 240 \Leftrightarrow R = \frac{3}{5} \frac{240}{13} 11.07 \approx 11$$

Consumption is either found by simply plugging into tangency:  $C = \frac{2}{3}wR = 96$  or by using the budget line  $C = 240 - 13(11.07) = 96$ .

Last, his labor supply is the difference from the time endowment minus leisure:  $L = 18 - 11 = 7$  hours.

c) Write down Bill's Slutsky equation for labor and leisure and mark the signs of each term. Please use the correct notation for the variables. Can you predict without knowing the magnitude of each effect whether he will reduce or increase labor supply? Why or why not? Explain.

$$\frac{dR}{dw} = \underbrace{\frac{\partial R^s}{\partial w}}_{-} + \underbrace{(\bar{R} - R)}_{L \geq 0} \underbrace{\frac{\partial R}{\partial m}}_{+}$$

The first effect is the substitution effect, which is negative (implying an increase of relaxation with a decrease in wage). The term in parenthesis is positive or zero (labor). Since relaxation is a normal good (because of Cobb-Douglas), the income derivative  $\partial R / \partial m$  is positive. Therefore, we need to know the magnitude to tell whether Bill will reduce or increase his labor supply if wage increases (part d below).

d) Jenny's Roadside Cafe receives competition from a new chic Seattle-based coffee chain. To save the business, all employees agree to work henceforth at a lower wage of  $w=10$ , so does Bill. Write down the equation for the new budget line and draw this new budget line into the graph on page 1. Mathematically find his new allocation. Does he increase or decrease labor supply? Thus, which effect has dominated, the substitution or the (total) income effect? Explain.

The budget line changes into  $C + wR = m + w\bar{R} = 6 + 10 \cdot 18 = 186$ . We again find the allocation by substituting  $C$  through the right-hand side of the tangency condition:

$$\frac{2}{3}w'R + w'R = 186 \Leftrightarrow \frac{5}{3}w'R = 186 \Leftrightarrow R = \frac{3}{5} \frac{186}{10} = 11.16 \approx 11.2.$$

Thus, Bill has (slightly) increased his amount of leisure, and simultaneously decreased his labor supply to  $18 - 11.2 = 6.8$  hours of labor. His new consumption  $C$  is found again by using the tangency condition  $C = \frac{2}{3}w'R = 74.4$  (or by using the budget line again, lead-

ing to the same result). The new budget line and new allocation is drawn (blue line) into the graph above, together with the new allocation  $A'(11.2, 74.4)$ .

We conclude that his labor supply has decreased, and therefore the substitution effect was larger in magnitude, compared to the joint income effect.

4) Elvira consumes mint chocolates and other goods. Her utility function for mint chocolates  $x_1$  and money to spend on other goods  $x_2$  is given by

$$u(x_1, x_2) = 20x_1 - x_1^2 + x_2$$

a) Find Elvira's inverse demand curve for mint chocolates using Lagrange. You may assume the price of other goods to be 1.

*Solution: We set price of good 2 to 1 and use Lagrangian:*

$$L(x_1, x_2, \lambda) = 20x_1 - x_1^2 + x_2 - \lambda(p_1x_1 + x_2 - m)$$

$$\frac{dL}{dx} = 20 - 2x_1 - \lambda p_1 \doteq 0 \quad (1)$$

$$\frac{dL}{dy} = 1 - \lambda \doteq 0 \quad (2)$$

$$\frac{dL}{d\lambda} = m - p_1x_1 - x_2 \doteq 0 \quad (3)$$

By substituting, we can use (2) to set  $\lambda = 1$

and reach inverse demand by plugging into (1):

$$20 - 2x_1 - \lambda p_1 = 0$$

$$20 - 2x_1 - (1)p_1 = 0$$

$$20 - 2x_1 = p_1$$

$$\text{or } p_1(x_1) = 20 - 2x_1.$$

Remark: That this is indeed inverse demand can be checked by finding  $MU_1$ . This is the same expression. Since utility is quasilinear, this  $MU_1$  is inverse demand.

b) If the price of mint chocolate is \$12, how many mint chocolates will she consume?

$$p_1(x_1) = 20 - 2x_1 \Leftrightarrow x_1 = 10 - 0.5p_1$$

$$x_1|_{p_1=12} = 10 - 6 = 4.$$

She will consume 4 mint chocolates.

c) If the price of mint chocolate is \$8, how many mint chocolates will she consume?

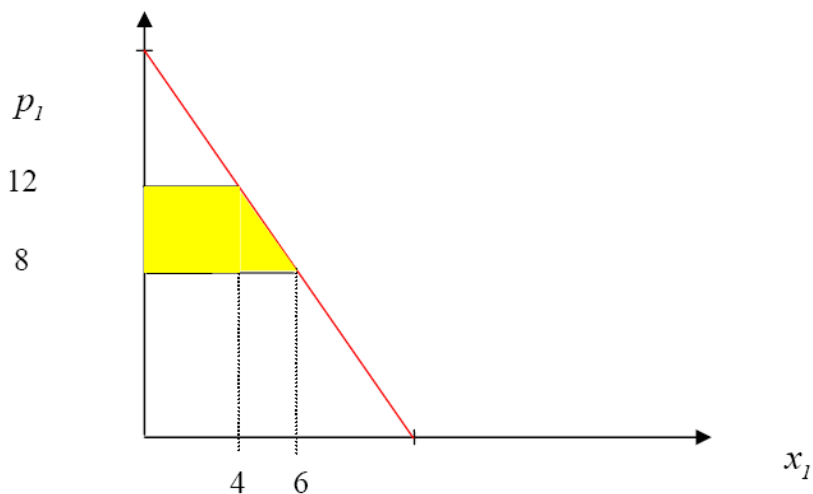
$$p_1(x_1) = 20 - 2x_1 \Leftrightarrow x_1 = 10 - 0.5p_1$$

$$x_1|_{p_1=8} = 10 - 4 = 6.$$

She will consume 6 mint chocolates.

Find her increase in consumer surplus if the price drops from \$12 to \$8. Sketch the market diagram in the space beginning of next page. Label the axes and mark the two price/qty pairs as well as the change in consumer surplus.

ket diagram in the space beginning of next page. Label the axes and mark the two



Elvira's increase in consumer surplus is the shaded area. It is the area of the rectangle plus the area of the triangle, thus

$$\Delta CS = (12 - 8)4 + \frac{1}{2}(6 - 4)(12 - 8) = 20.$$

d) Suppose Elvira has a budget of \$500. What is her total utility for mint chocolates and for  $x_2$  if the price of mint chocolates is \$8?

To find  $u$ , we first need to compute the amount of good 2 consumed. This is found by plugging in good 1-demand into the budget line:

$$x_1 = 10 - 0.5p_1, \text{ and } m = p_1x_1 + x_2$$

$$x_2 = m - p_1x_1 = 500 - p_1x_1.$$

$$x_1 \Big|_{p_1=8} = 10 - 4 = 6.$$

$$x_2 = m - p_1x_1 = 500 - 6(8) = 452.$$

$$\text{Then, } u = 20(6) - 6^2 + 452 = 120 - 36 + 452 = 536.$$

e) What is her total utility for mint chocolates and for  $x_2$  if the price of mint chocolate is \$12?

$$\text{Again, } x_2 = m - p_1x_1 = 500 - p_1x_1.$$

$$x_1 \Big|_{p_1=12} = 10 - 6 = 4.$$

$$x_2 = m - p_1x_1 = 500 - 12(4) = 452$$

$$\text{Then, } u = 20(4) - 4^2 + 452 = 80 - 16 + 452 = 516.$$

f) Find the difference in utility that follows the price decrease and compare it with the consumer surplus. What do you observe? Explain the result.

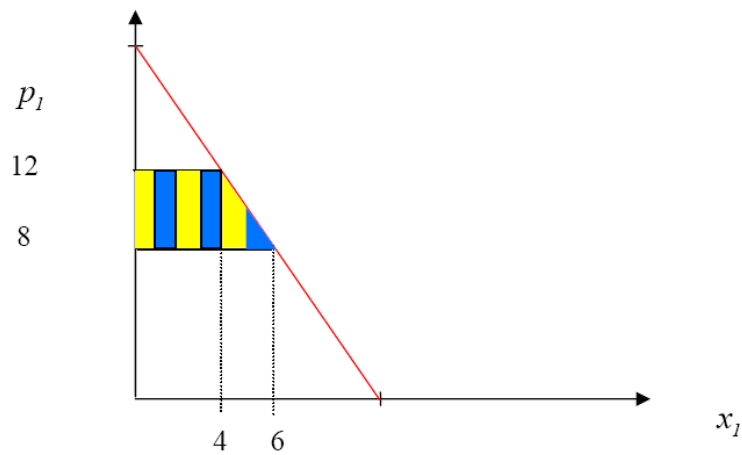
The difference is

$$\Delta U = 536 - 516 = 20.$$

This is exactly the same as her change in consumer surplus, which is

$$\Delta CS = 20.$$

The explanation is straightforward: Since we have a quasilinear utility function, consumer surplus is a direct measure of the change in utility. This also holds for the change in consumer surplus. One can graphically illustrate through slicing the difference in CS into portions of 1 unit (as done below with alternating bars of a base of 1):



Note that per unit, we have a marginal utility increase of 4 if price drops to \$8. This marginal utility follows demand. Summing up the bars yields the change in total CS for all units consumed, and the total utility change, which is the same.