Midterm Exam 1 – July 18, 2007 - Solutions

1 a) Gary has Cobb-Douglas preferences of $u(x_1, x_2) = \frac{1}{2} x_1^2 x_2^5$. By using the Lagrange method, find his individual demands for the two goods.

$$L(x_1, x_2, \lambda) = \frac{1}{2} x_1^2 x_2^5 + \lambda (m - p_1 x_1 - p_2 x_2)$$

$$\frac{\partial L}{\partial x_1} = x_1 x_2^5 - \lambda p_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = \frac{5}{2} x_1^2 x_2^4 - \lambda p_2 = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0. \qquad (3)$$

Dividing (1) by (2) yields the tangency condition $\frac{2x_2}{5x_1} = \frac{p_1}{p_2}$, solved for x_2 yields $x_2 = \frac{5}{2}\frac{p_1}{p_2} x_1$.

Using budget line we find demand for good 1: $p_1 x_1 + p_2 \left(\frac{5}{2}\frac{p_1}{p_2} x_1 \right) = m$.

This leads to $x_1 = \frac{2}{7}\frac{m}{p_1}$, plugged into tangency condition yields demand for $x_2 = \frac{5}{7}\frac{m}{p_2}$.

b) Find his initial consumption bundle for the two goods, assuming $m=70$, $p_1 = 4$ and $p_2 = 1$. Draw his original budget line and his bundle into the graph below. Mark the bundle with an "A" and the budget line using the budget equation with the values for m and the prices.

$$x_1^A = x_1(p_1 = 4, m = 70) = \frac{2}{7}\frac{70}{4} = 5 .$$

$$x_2^A = x_2(p_2 = 1, m = 70) = \frac{5}{7}\frac{70}{1} = 50.$$

Point A has the coordinates A(5,50) (see graph below).

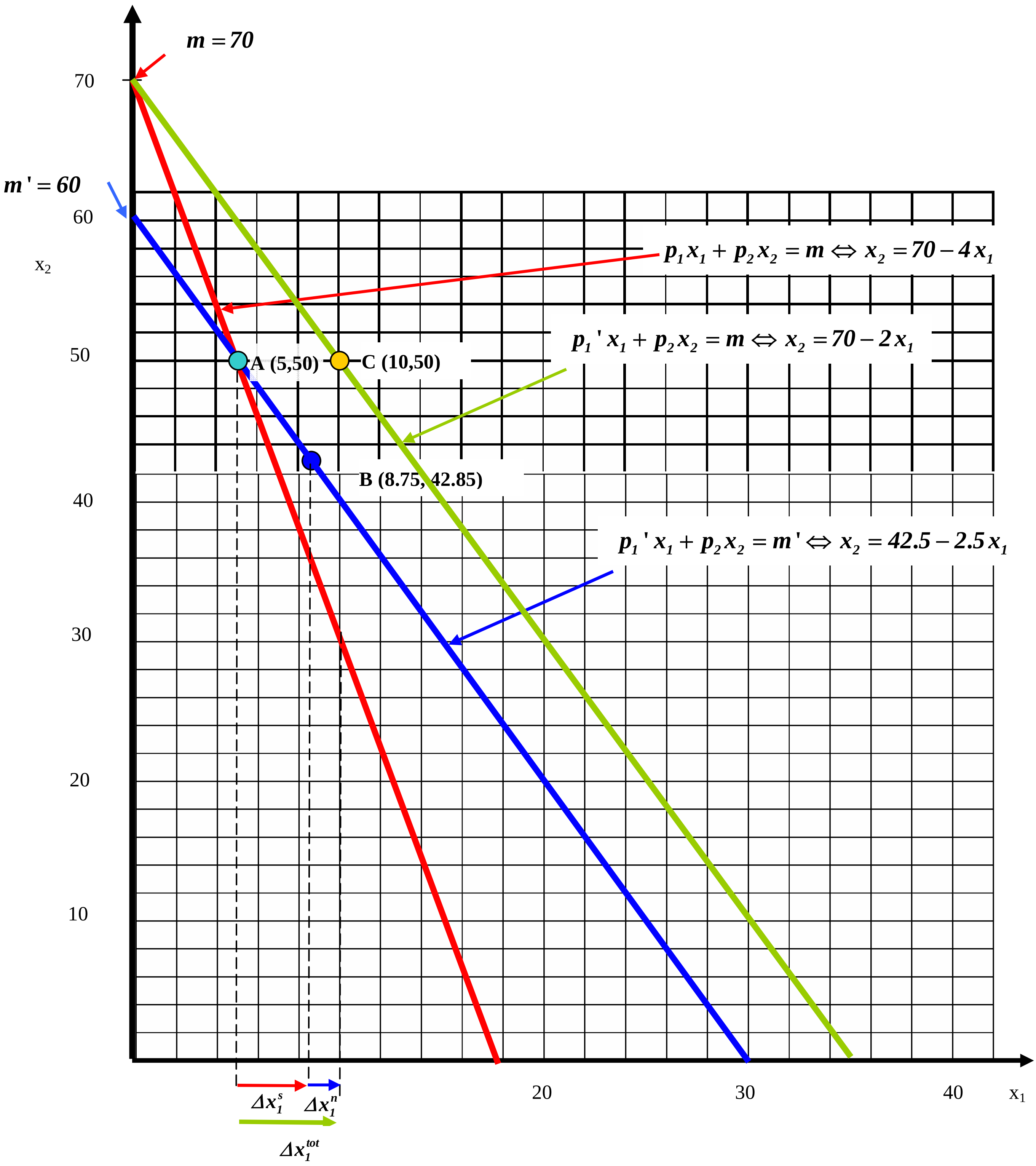
c) Assume that the price for good 1 decreases to $p_1' = 2$. Find his income compensation in A and compute the new compensated equilibrium, and mark it with a "B." Draw the new budget line into the above diagram and label it with the right expression using new prices and new income. Mathematically find his substitution effect and mark it in the diagram above as well.

We first find the Slutsky income compensation to Point A at new prices. First we follow the intuition: We know that he consumer 5 units of good 1 and pays $4 each. If he pays half of it, namely $2 each, he must save 5($2)=$10 when having new prices and (hypothetically) still staying put in Point A. This is his purchasing power gain that we first take away.

Using the formula we find

$$\Delta m = \Delta p_1 x_1^A = (p_1' - p_1) x_1^A = (2-4)5 = -10 .$$

Thus, his original income m changes into $m' = m + \Delta m = 70 - 10 = 60$. (see graph).



Finding the compensated equilibrium (Point B):

$$x_1^B = x_1(p_1' = 2, m' = 60) = \frac{2}{7}\frac{60}{2} = 8.57$$
$$x_2^B = x_2(p_2 = 1, m' = 60) = \frac{5}{7}\frac{60}{1} = 42.85.$$

Finding the Substitution Effect Δx_1^s:

$\Delta x_1^s = x_1^B - x_1^A = 8.57 - 5 = 3.57$ (see graph).

d) Find his final consumption bundle at new prices and mark it with a "C." Draw the final budget line into the above diagram and label it with the right expression using the right values for income and prices. Mathematically find his income effect and mark it in the diagram above as well.

Finding the final consumption bundle (Point C):

$$x_1^C = x_1(p_1' = 2, m = 70) = \frac{2}{7}\frac{70}{2} = 10.$$
$$x_2^C = x_2(p_2 = 1, m = 70) = \frac{5}{7}\frac{70}{1} = 50.$$

Finding the Substitution Effect Δx_1^s:

$\Delta x_1^n = x_1^C - x_1^B = 10 - 8.57 = 1.43$ (see graph).

e) Show mathematically that the total effect is composed of the two single effects and mark the total effect into the graph as well.

$\Delta x_1^{tot} = \Delta x_1^s + \Delta x_1^n$. We add:

$\Delta x_1^s = x_1(p_1', m') - x_1(p_1, m)$

$\Delta x_1^n = x_1(p_1', m) - x_1(p_1', m')$.

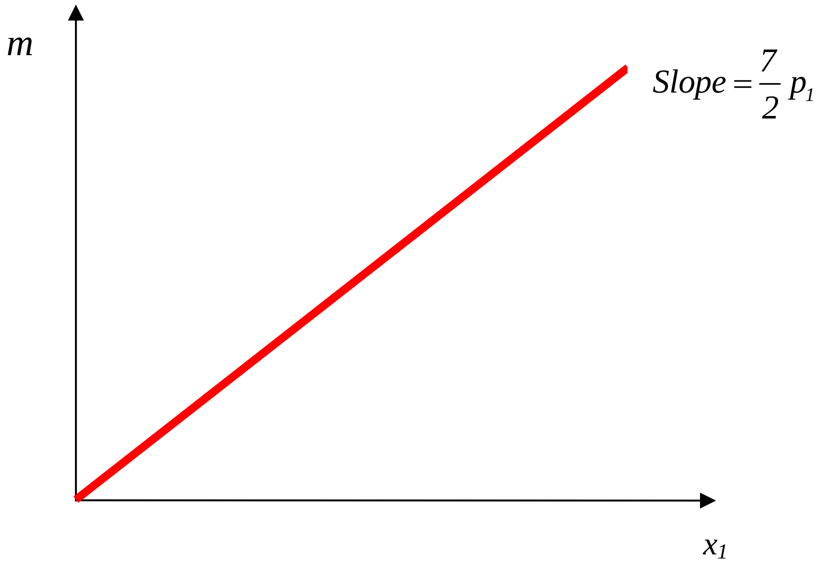
$\Delta x_1^{tot} = \Delta x_1^s + \Delta x_1^n = x_1(p_1', m' x_1(p_1', m) + x_1(p_1', m) - x_1(p_1', m') = x_1(p_1', m) - x_1(p_1, m).$

This can be computed as follows:

$x_1(p_1', m) - x_1(p_1, m) = x_1^C - x_1^A$ = 10 – 5 = 5 (see green arrow above).

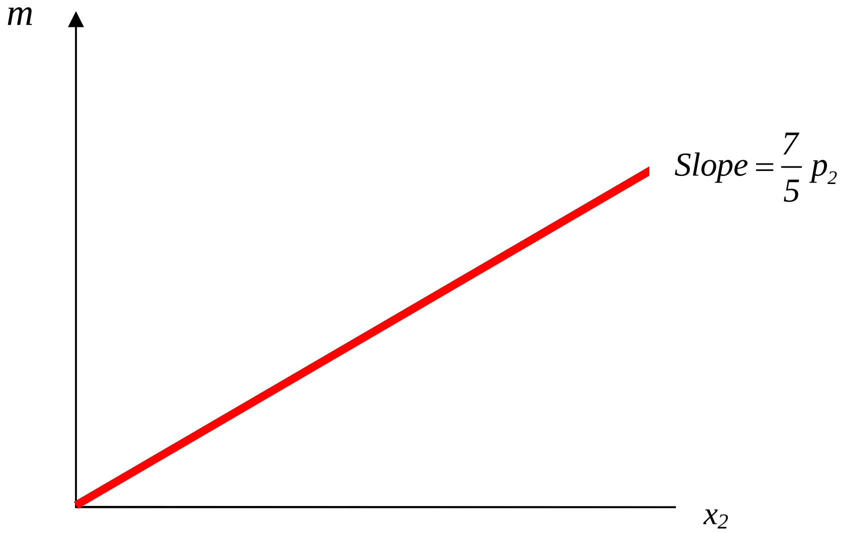
f) Compute and draw the Engel curves for each good. Use one for each good and label the axes. What is the intercept, what is the slope of each Engel curve? Explain the properties you observe by using the fact that Gary has Cobb-Douglas preferences.

We compute the Engel curve for each good by solving individual demand for *m:* From $x_1 = \frac{2}{7}\frac{m}{p_1}$ we reach $m = \frac{7}{2} p_1 x_1$. This is graphed below:



That is, the Engel curve for good 1 has a slope of $\frac{7}{2}p_1$, and no intercept. This follows from Gary's assumed Cobb-Douglas preferences that are homothetic (Engel curve and Income Offer Curves are linear and go through the origin, which means that the goods are consumed in a constant proportion that does not change with the income).

The same holds for good 2, only that the slope now reads $\frac{7}{5}p_2$. See graph below.



(2) A consumer has preferences following the utility function $u(x_1, x_2) = \frac{1}{2}\ln x_1 + \frac{1}{3}\ln x_2$.

(a) Derive the marginal utilities of the consumer and the MRS.

$$MU_1 = \frac{\partial u}{\partial x_1} = \frac{1}{2x_1}$$
$$MU_2 = \frac{\partial u}{\partial x_2} = \frac{1}{3x_2}.$$

$$MRS = -\frac{MU_1}{MU_2} = -\frac{\frac{1}{2x_1}}{\frac{1}{3x_2}} = -\frac{3x_2}{2x_1}.$$

(b) Transform this utility function into any function $v(x_1, x_2)$ in which no logarithmic terms appear any more. Use a transformation that preserves the same order of preferences. Show that the MRS is the same as in (a).

The transformation uses the inverse function to the ln (x) function, which is the e-function:

$$v(x_1, x_2) = e^{\frac{1}{2}\ln x_1 + \frac{1}{3}\ln x_2} = e^{\ln\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{3}}\right)} = x_1^{\frac{1}{2}} x_2^{\frac{1}{3}}$$

Deriving the MUs and MRS yields now:

$$MU_1 = \frac{\partial v}{\partial x_1} = \tfrac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{3}} = \frac{\sqrt[3]{x_2}}{2\sqrt{x_1}}$$
$$MU_2 = \frac{\partial v}{\partial x_2} = \tfrac{1}{3} x_1^{\frac{1}{2}} x_2^{-\frac{2}{3}} = \frac{\sqrt{x_1}}{3\sqrt[3]{x_2^2}}$$

While the MUs differ, the MRS remains the same after transformation. This can be either shown by using broken exponents or (as shown here) by algebraically transforming the root expressions:

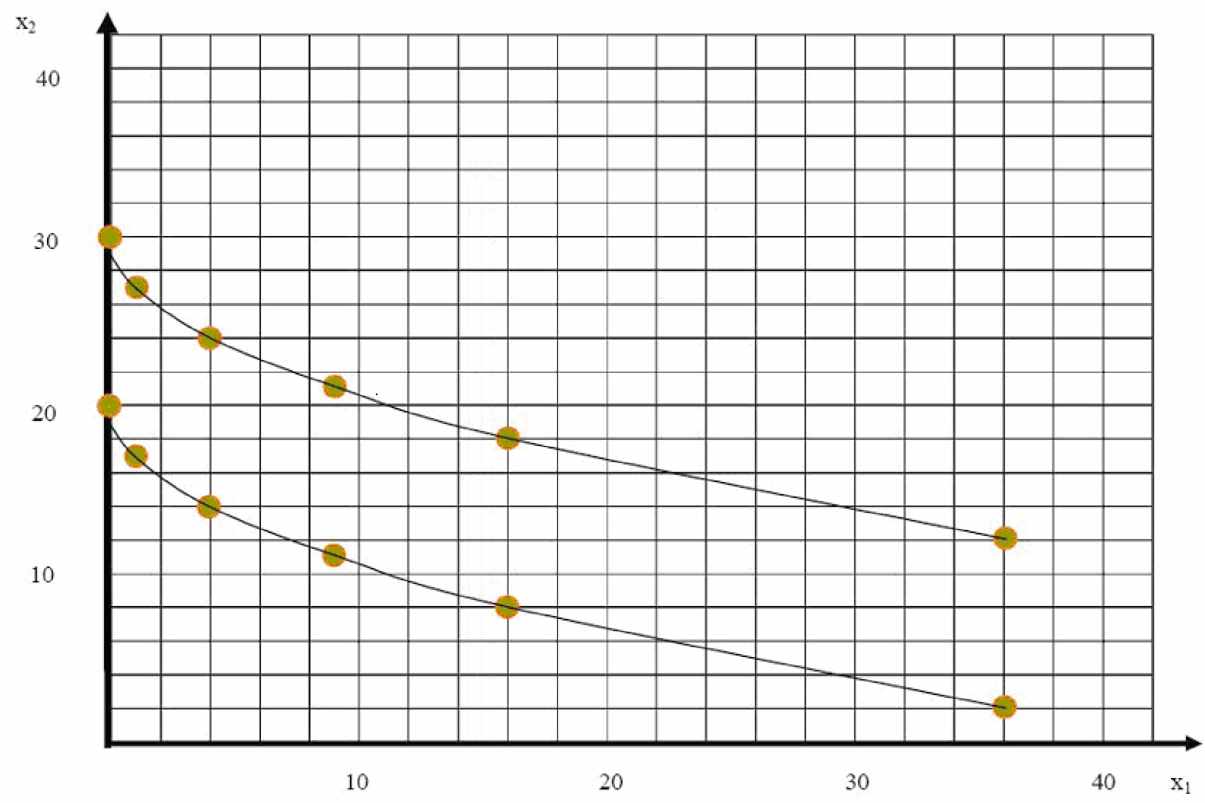
$$MRS = -\frac{MU_1}{MU_2} = -\frac{\frac{\sqrt[3]{x_2}}{2\sqrt{x_1}}}{\frac{\sqrt{x_1}}{3\sqrt[3]{x_2^2}}} = -\frac{\sqrt[3]{x_2}}{2\sqrt{x_1}} \cdot \frac{3\sqrt[3]{x_2^2}}{\sqrt{x_1}} = -\frac{3x_2}{2x_1}.$$

Comparing with the solution in part a) it is easy to see that the MRS is the same. ■

(3) Consider the utility function $u(x_1, x_2) = 3\sqrt{x_1} + x_2$

(a) Plot the indifference curves for $u = 20$ and $u = 30$ into the chart below by computing 4-5 value pairs for each indifference curve you are free to choose.

	u=20	u=30
x1	x2	x2
0	20	30
1	17	27
4	14	24
9	11	21
16	8	18
36	2	12



b) Find individual demands for each good using the Lagrange method.

$$L(x_1, x_2, \lambda) = 3\sqrt{x_1} + x_2 + \lambda(m - p_1x_1 - p_2x_2)$$

$$\frac{\partial L}{\partial x_1} = \frac{3}{2\sqrt{x_1}} - \lambda p_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = 1 - \lambda p_2 = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = m - p_1x_1 - p_2x_2 = 0. \quad (3)$$

From (2) we have: $\lambda = \frac{1}{p_2}$; plugged into (1) yields the tangency condition

$$\frac{3}{2\sqrt{x_1}} = \frac{p_1}{p_2} \Leftrightarrow \sqrt{x_1} = \frac{3}{2}\frac{p_2}{p_1} \Leftrightarrow x_1 = \frac{9}{4}\left(\frac{p_2}{p_1}\right)^2.$$

Inserting into (3) yields $x_2 = \frac{m}{p_2} - \frac{9}{4}\frac{p_2}{p_1}$.

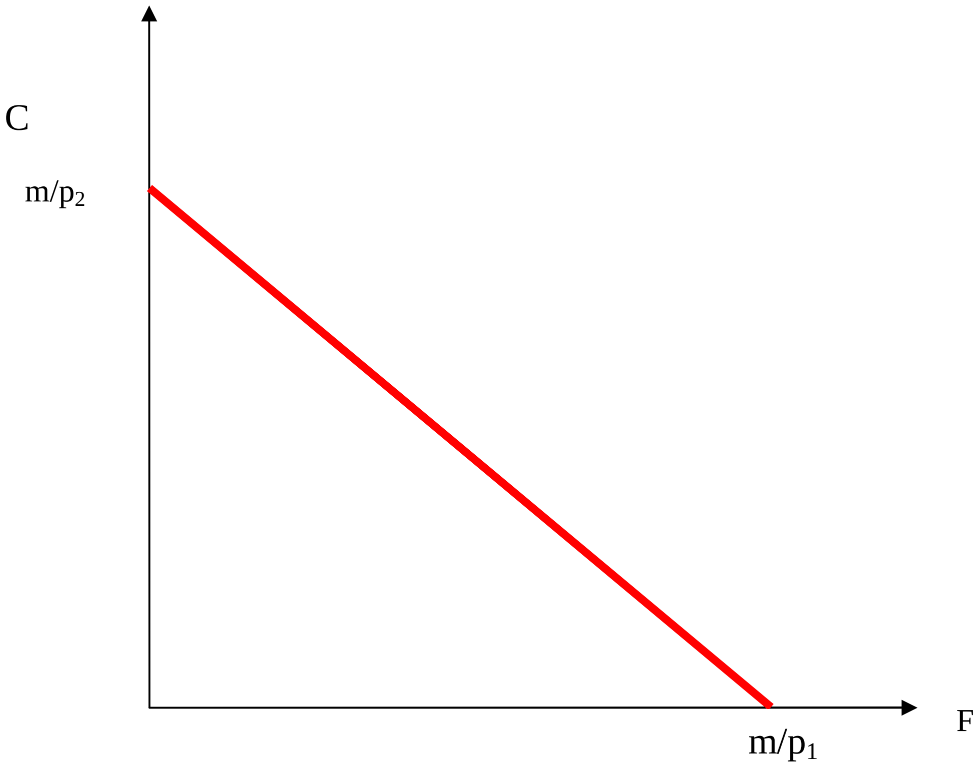
c) Using individual demand for good 1, check if the two goods are substitutes or complements. Explain.

From $x_1 = \frac{9}{4}\left(\frac{p_2}{p_1}\right)^2$ we already see that demand for good 1 increases (quadratically) with the price of good 2. To follow the textbook, we find the cross-price derivative

$$\frac{d\left(\frac{3}{2}\frac{p_2}{p_1}\right)^2}{dp_2} = 2\cdot\left(\frac{3}{2}\frac{p_2}{p_1}\right)\left(\frac{3}{2p_1}\right) = \frac{9}{2}\frac{p_2}{p_1^2}$$, which is positive.

Interpretation: any increase in the price of good 2 will therefore increase demand for good 1. The goods are substitutes.

4) The following figure shows Denise's budget line for food (F) and all other goods (C).



Assume that p_2 is set to one and that the price for food is \$2.75 . Her initial budget is m = \$ 82.50.

a) Would she spend all her money on food, how many units can she consume? Show your work.

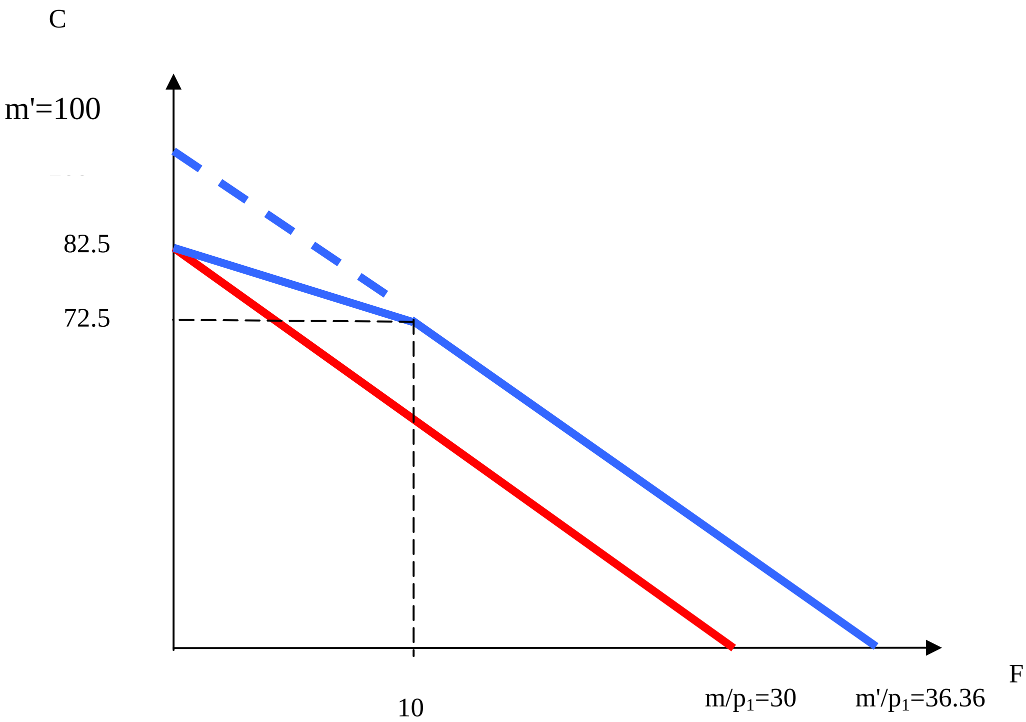
This part asks for the numeric value of the F-intercept: This is m/p_1 or 82.50/2.5=\$30

b) Assume now that Denise becomes eligible for a state-wide food stamps program. This program works as follows. Denise can buy food stamps for \$1, and for up to 10 units of food. Find her new budget line if he buys food stamps for 10 units. Compute the coordinates of the "kink" in his budget line. How many units of food could he consume now if he would spend all his money on food? Find the values of the two intercepts and mark them in the graph above.

Solution:

Would she buy 10 units of food, she would pay $10 for it. Her money at F=10 for all other goods is thus reduced by $10. The coordinates of the kink are (10, 72.5).

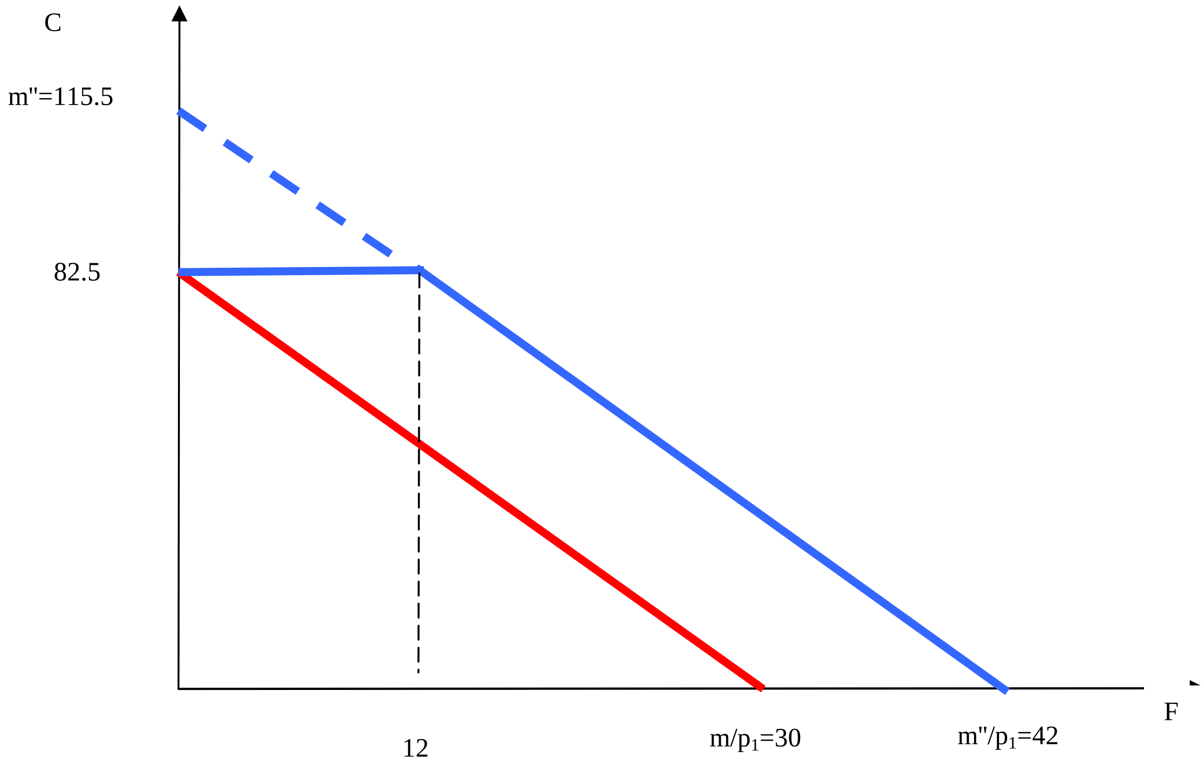
To compute the intercepts, note the following: As long as she consumes at least 10 units of food this program makes her richer by what she saves, namely 10($1.75)=$17.50. Right of the kink, her money m'=100 and her new F-intercept m'/p_1 increases to 100/2.75 = $36.36 . See illustration below.



c) Similar to the change in the U.S. in 1979, assume that the food stamps program changes in that it gives Denise 12 food stamps for free that she can spend on 12 units of food. Assume that she is again eligible for the program: how many units of food could she consume now if she would spend all her money on food? Again, draw the new budget line and find the values of the intercepts and mark them in the graph as well. For part c) use the graph below.

The solution is similar as in b). Everything is as if the red budget line is rightward shifted by 12 units of F. In this case, the C-coordinate of the kink is again 82.5, the F-coordinate is now 12. By the same argument as before, Denise is now virtually richer, exactly by what he saves, which is 12 units times $2.75=$33. Her new money m" is now $115.5, and her new F-intercept is 30+12.5=m"/p1=42 units. In other words: everything is as if his budget line starts at 115.5 (C-intercept) and goes down with the same slope as before (-2.75).

See graph below:


C
m"=115.5
82.5
12
m/p1=30
m"/p1=42
F