Solutions to Exercise using Lagrangian   Econ21   W. Gick

A consumer has preferences of

a) $u(x_1, x_2) = 3x_1^2 x_2^3$ (Cobb-Douglas)

b) $u(x_1, x_2) = 3\sqrt{x_1} + x_2$ (quasilinear).

In both cases, her budget line is $p_1 x_1 + p_2 x_2 = m$

Find her individual demands in each case by using the Lagrange method.

Solution:

a)

Budget constraint is $p_1 x_1 + p_2 x_2$=m.

max $u(x_1, x_2) = 3x_1^2 x_2^3 \quad s.t. \quad p_1 x_1 + p_2 x_2 = m.$

Lagrangian:

$$L(x_1, x_2, \lambda) = 3x_1^2 x_2^3 + \lambda(m - p_1 x_1 - p_2 x_2)$$

$$\frac{\partial L}{\partial x_1} = 6x_1 x_2^3 - \lambda p_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = 9x_1^2 x_2^2 - \lambda p_2 = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0. \qquad (3)$$

We use (1) and (2): first we isolate the x-terms:

$$6x_1 x_2^3 = \lambda p_1 \qquad (1)$$

$$9x_1^2 x_2^2 = \lambda p_2 \qquad (2)$$

Dividing (1)/(2):

$\frac{6x_1 x_2^3}{9x_1^2 x_2^2} = \frac{p_1}{p_2}$. Simplifying yields the tangency condition (T.C.) $\frac{2x_2}{3x_1} = \frac{p_1}{p_2}$.

It is helpful to solve for x_2: $x_2 = \frac{3}{2}\frac{p_1}{p_2} x_1$.

Next step: We find the individual demand for x_1 by plugging into (3) (=budget line):

$$p_1 x_1 + p_2\left(\frac{3}{2}\frac{p_1}{p_2} x_1\right) = m \Leftrightarrow p_1 x_1 + \frac{3}{2} p_1 x_1 = m \Leftrightarrow \frac{5}{2} p_1 x_1 = m \Leftrightarrow x_1 = \frac{2m}{5p_1}.$$

x_2- demand is found by plugging this result into T.C.:

$x_2 = \frac{3}{2}\frac{p_1}{p_2} x_1 = \frac{3}{2}\frac{p_1}{p_2} \cdot \frac{2m}{5p_1} = \frac{3m}{5p_2}$. Individual demands read: $x_1 = \frac{2m}{5p_1}; x_2 = \frac{3m}{5p_2}$.

b)

max $u(x_1, x_2) = 3\sqrt{x_1} + x_2 \quad s.t. \quad p_1x_1 + p_2x_2 = m.$

Lagrangian:

$$L(x_1, x_2, \lambda) = 3\sqrt{x_1} + x_2 + \lambda(m - p_1x_1 - p_2x_2)$$

$$\frac{\partial L}{\partial x_1} = \frac{3}{2}x_1^{-\frac{1}{2}} - \lambda p_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = 1 - \lambda p_2 = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = m - p_1x_1 - p_2x_2 = 0. \qquad (3)$$

We use (1) and (2): first we again bring the lambda-terms to the right:

$$\frac{3}{2}x_1^{-\frac{1}{2}} = \lambda p_1 \qquad (1)$$

$$1 = \lambda p_2 \qquad (2)$$

Dividing (1)/(2):

$\frac{3}{2}x_1^{-\frac{1}{2}} = \frac{p_1}{p_2}$. Simplifying yields x_1-demand if utility is quasilinear (because this tangency condition does not contain any x_2-term!!):

$$\frac{2}{3}x_1^{\frac{1}{2}} = \frac{p_2}{p_1} \Leftrightarrow \frac{2}{3}\sqrt{x_1} = \frac{p_2}{p_1} \Leftrightarrow \sqrt{x_1} = \frac{3}{2}\frac{p_2}{p_1} \Leftrightarrow x_1 = \left(\frac{3}{2}\frac{p_2}{p_1}\right)^2 = \frac{9p_2^2}{4p_1^2}.$$

To find individual demand for x_2 we need the budget line (3):

$$p_1\frac{9p_2^2}{4p_1^2} + p_2x_2 = m \Leftrightarrow p_2x_2 = m - \frac{9p_2^2}{4p_1} \Leftrightarrow x_2 = \frac{m - \frac{9p_2^2}{4p_1}}{p_2}.$$

Thus, individual demands read $x_1 = \frac{9p_2^2}{4p_1^2}; x_2 = \frac{m - \frac{9p_2^2}{4p_1}}{p_2}$.