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# Mathematics, the Cubic, and Me

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One afternoon nine years ago, I complained to my older brother that there were too many questions and too little time on the previous day's math exam. Bolaji was the math guy, and in an instant, he asked to see the questions. I took them out and said, "Look at this: here they asked us to compute the product of 1,000,001 and 999,999. And this was supposed to take us under thirty seconds. Bolaji moaned with characteristic impatience and asked, "Can't you see that 1,000,001 is  $1,000,000 + 1$  and that 999,999 is  $1,000,000 - 1$ ? So take  $a = 1,000,000$  and  $b = 1$ . As you know, for any numbers  $a$  and  $b$ , the product of  $(a + b)$  and  $(a - b)$  is  $a^2 - b^2$ . The answer is  $1,000,000,000,000 - 1$ ." It was both stunning and shameful. How many such shortcuts could come from recognizable patterns like this? I was to find out that there was virtually no limit to their abundance, and no questions that could not be asked.

Here is one example: take a number - positive, negative, fractional, whatever. Multiply it by itself. Multiply the result by the original number. Then add the original number to what you have. Now, if you tell me what you have, I can tell you what your original number was. How? The solution to this magic trick is a particular instance of an equation I got to know very well: *the cubic equation*.

As I matured, my interest in math grew, but the advent of the brain-devouring monster called calculus was enough to disillusion me towards higher mathematics. You probably remember what it was like: one step after the other, cut and dry. I liked literature a lot more—largely because in it I could see patterns; I could see the magic. But it was not until I was done with three years of college and a "practical" Computer Science major that I began to take a look at math beyond calculus and was exposed to the vast array of structure and theme within. I was not really surprised to find out in my studies that Gauss, Lagrange and Hamilton, three of the greatest mathematicians of all time, all had literary interests (often in the classics), especially early in life.

The main problem I encountered in math pursuits was a depressing lack of skill. This was mostly owing to a paucity of experience and practice. However, I learned that even a giant like Isaac Newton started out in mathematics relatively late in life. (I hope that these comparisons to legends do not make me appear too unrealistic in my outlook. I merely take inspiration from the best sources.) And yet, one problem that even Newton did not face is the abundance of new material to learn due to the rapid growth of math and science particularly within these last two centuries. So, I decided that in order to develop my "artistic" approach to mathematics (without encountering the problem of wading through tons of papers written in the past) I would try a few problems not requiring much previous knowledge but some measure of ingenuity.

The first of these I came across last term in the elementary course in Abstract Algebra at Dartmouth, Math 31. The professor discussed the inspiration for the material taught in the course, , saying that it had come out of efforts of mathematicians such as Lagrange, Abel and Galois to discover aspects of "solutions to

general polynomial equations using addition, subtraction, multiplication, division, and the extraction of roots only." In other words, they wanted to explore the existence of simple formulas (like the familiar formula for quadratic equations) for solving the most complex mathematical equations like the cubic " $x^3 + 5x^2 + 1 = 0$ ".

The professor handed out an excerpt from a textbook, which gave the historical background on how the cubic equation was solved in the 16th century. The cubic is like the quadratic equation, which is something like "what  $x$  can you pick so that  $2x^2 + 3x + 7 = 0$ ?" The difference is that the cubic can be, instead, something like  $5x^3 + 2x^2 + 3x + 7 = 0$ . The idea is to find out what  $x$  could be. For example, you can check that our earlier "pick a number" question really means that if your final result was 9, then I would have to solve for  $x$  in  $x^3 + x = 9$ . The ancient Greeks had taken care of the quadratic before the birth of Christ, but mathematics was not done in Europe. For many subsequent centuries, the Romans and the early Christians were not particularly interested in real science and mathematics. But when the Renaissance arrived, the Italians revived the Greek legacy, so their "algebraists" got down to the problem of the cubic. It took them a while, partly because they were still acting like competing alchemists, and ideas were not freely exchanged. Also, they were naturally primitive in their outlook. They believed that numbers "existed" or did not, and that they were not just abstractions—hence some kinds of numbers, like negative numbers, did not exist. Nevertheless, Scipio Del Ferro and then Niccolò Tartaglia came up with the solution, independently.

At the beginning of Winter term 1999, I got a book from the Storage Library written by Lagrange called "Lectures On Elementary Mathematics." It was the book that contained the cubic solution. Right before I was to read the solution, I stopped and decided I would attempt the problem myself. I did not really expect to arrive at the answer, but I thought it would at least help me to understand what a researcher does. I began by simplifying the equation to remove the  $x^2$  term and got a new equation that expresses the same problem. This was quite easy, merely a generalization of the "completing the squares" method to cubes—it took me a few hours before I stumbled upon it. The real work now lay ahead.

Subsequently, I encountered what is probably the most unfortunate part of research - the sheer amount of junk that is created and waded through before the researcher gets on the right track. Strangely I came to enjoy the independence associated with lying on my back in bed and scribbling what were probably worthless ideas on a piece of paper. I began to skip lectures to do this, and much later saw the repercussions in the grade report for that term. However, the project increasingly came to serve as a kind of altar I returned to whenever I felt bothered by the affairs of the world. After a week, I told a professor what I was studying. He advised me to discontinue my efforts, being of the opinion that the cubic was "not a trivial problem" and ought not be

attempted, especially by a student with only Math 24 to his credit.

I decided to take a unique philosophical standpoint on the problem. First, I started to see that I was making essentially the same kind of mistake on each attempt at the problem: I would reduce the cubic equation to another cubic equation. I had tried various approaches, but with that same result. One approach had been based on the observation that when you try to guess a “form” for the quadratic’s formula—assuming that the solution might turn out to be something like “a plus the square root of b” where a and b depend on the coefficients of  $x^2$  and  $x$ , respectively—you get a system of equations which leads you to the quadratic’s formula. Therefore, the form you guessed is “conceptually close” to the actual form (because it, like the actual form, involves the presence of a square root). So I tried to guess the form for the cubic, using different combinations of addition, multiplication and roots. I was unsuccessful; partly I think, because the form for the cubic is so complex. So the equations I ended up with were no simpler than the one I was trying to simplify.

I tried many other things, but I soon came to the conclusion that whenever I failed, I needed to go beyond just saying, “That didn’t work. I’ll try something else.” I needed to see exactly what aspects made it fail, then avoid any ideas involving those aspects. This cut my “bad ideas” pool substantially and finally led me to a stage of deeper conceptual reflection. Another improvement I implemented was that I forced myself to remember what I had previously done. I have a bad memory, and I imagine it would be terrible to be on the verge of a breakthrough idea and to miss it, because you forgot something.

The real leap I made was a pattern of thinking that I call “constructive conjecture”, which I think some lucky people learn at an early age. The primitive problem-solver (no doubt in part to the plenitude of calculus classes thrust upon to him in the past), when confronted with a logical conundrum, looks for “step one.” What is step one? He thinks, “Well, I just learned about improper integrals in class today, so I’d better look for the improper integral in here.” He is at a loss in dealing with problems when no ready-made logical framework of concepts exists; which is to say, when he has to actually invent the notion of improper integrals.

In *The Murder In The Rue Morgue*, the world’s first detective story, Edgar Allan Poe writes, “Such mysteries as what songs the Sirens sang, or what name Achilles assumed when living among the women, though veritable conundrums, are not beyond all conjecture.” It turns out that when a problem-solver liberates his mind from the fact that he does not know certain information and instead makes deductions concerning the information he does not

have, often he can arrive at the path to the solution.

I sat on my bed and asked what a solution, if it existed, would consist of. I knew that since the problem consisted of unknowns, it might be simplified to a set of equations. What kind of equations would these be? Certainly not a cubic or higher (at least, so I supposed). Probably a set of simultaneous equations, or a quadratic one. The quadratic possibility seemed more feasible, especially since a set of simultaneous equations would be linked, and might simplify to a quadratic. So I thought, supposing there was a way to simplify a cubic to a quadratic, what would it look like? It would need the substitution of a new variable. In the quadratic case, one observes that if, say in the equation  $x^2 + 2x + 1 = 0$ , you solve for the number that is 1 bigger than  $x$  (call it  $y$ ), you get  $(y - 1)^2 + 2(y - 1) + 1 = 0$ . This reduces to a much simpler equation, which is  $y^2 = 0$ . So  $y = 0$ , and  $x$  must be one less than  $y$ , which is  $-1$ . But the point is that the equation could be simplified by substituting some expression, which was  $(y - 1)$ , in place of  $x$ .

It turned out that, for the cubic, I needed to set up an equation to find out what  $p$  would be, if I were to use a  $(y + p)$  expression. This new equation was simpler: a quadratic! I came up with a method (although I was pessimistic about its chances), then went out to get a bite to eat. I was sitting in the lobby of Food Court waiting for an acquaintance, when I began to impatiently scribble on the back of a poster. I made a few of my characteristic computation errors, but I soon found that I had essentially solved the cubic! It dawned on me that anyone with a little patience and a little free time can discover a whole world of patterns.

*Author’s solution is on the back cover*

## ABOUT THE AUTHOR

Ajibayo Ogunshola ‘99 is a computer science major with future plans in software development and graduate school work. He has aspiring research goals in subjects pertaining to algebra, topology, geometry, and physics. Throughout his years at Dartmouth, Ajibayo’s true love for both the sciences and the arts has shaped his experience and outlook on life.