Structural Change with Long-run Income and Price Effects

Diego Comin  
Dartmouth  
Danial Lashkari  
Harvard  
Martí Mestieri  
Northwestern  

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Abstract

We present a multi-sector growth model that accommodates long-run demand and supply drivers of structural change. The model generates nonhomothetic Engel curves at all levels of development and is consistent with the decline in agriculture, the hump-shaped evolution of manufacturing and the rise of services over time. The economy converges to a constant aggregate growth rate that depends on sectoral income elasticities, capital intensities and rates of technological progress. We estimate the demand system derived from the model using historical data on sectoral shares from 25 different countries and household survey data. We show that our model parsimoniously accounts for the broad patterns of sectoral reallocation observed among rich, miracle and developing economies in the post-war period.

Keywords: Structural Transformation, Nonhomothetic preferences

JEL Classification: E2, O1, O4, O5.

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1 Introduction

Economies undergo large-scale sectoral reallocations of employment and capital as they develop, a process commonly known as structural change (Kuznets, 1973; Maddison, 1980; Vries et al., 2014). These reallocations lead to a gradual fall in the relative size of the agricultural sector and a corresponding rise in manufacturing. As the process of growth continues, services eventually emerge as the largest sector in the economy (Herrendorf et al., 2014). Understanding the origins of these sweeping changes in the sectoral composition of economies along the path of development constitutes a core theme of economic growth. Leading theories of structural transformation may be classified based on whether they consider mechanisms involving production or demand. Production-side theories focus on differences across sectors in the rates of technological growth and capital intensities, both of which are ultimately reflected in differential sectoral price trends (Baumol, 1967; Ngai and Pissarides, 2007; Acemoglu and Guerrieri, 2008). Demand-side theories, in contrast, emphasize the role of heterogeneity in income elasticities of demand across sectors (non-homotheticity in preferences) in driving the observed reallocations accompanying income growth (Kongsamut et al., 2001).

The shapes of sectoral Engel curves play a crucial role in determining the contribution of supply and demand channels to structural transformation.\(^1\) If the differences in the slopes of Engel curves are large and persistent, we can readily explain reallocation of resources toward sectors with higher income elasticities. For instance, steep upward Engel curves for services, flat Engel curves for manufacturing, and steep downward Engel curves for agricultural products can give rise to sizable shifts of employment from agriculture toward services. However, demand-side theories have generally relied on a specific class of nonhomothetic preferences, including, e.g., generalized Stone-Geary preferences, that imply Engel curves level off quickly as income grows. Such specifications for preferences crucially limit the explanatory power of the demand channel in the long run.

Nevertheless, the existing evidence on the shape of Engel curves points toward fairly robust heterogeneity in the income elasticity of demand across sectors. For instance, Aguiar and Bils (forthcoming) use the U.S. Consumer Expenditure Survey (CEX) to estimate Engel curves for 20 different consumption categories. Their estimates for the income elasticities are different from unity and vary significantly across consumption categories.\(^2\) In particular, they suggest that Engel curves may be well approximated by log-linear functions with stable slopes. As another example, Young (2012) employs the Demographic and Health Survey (DHS) to infer the elasticity of real consumption of 26 goods and services with respect to income for

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\(^1\)We define Engel curves as the relationship between sectoral consumption shares and aggregate real consumption holding prices constant.

\(^2\)The items with lower elasticity are tobacco and other smoking (-0.26) and food at home (0.37), and those with highest are entertainment fees, admission and reading (1.74), cash contributions (1.81) and education (1.6).
Figure 1: Partial Correlations of Sectoral Expenditure and Aggregate Consumption

(a) Agriculture relative to Manufacturing

(b) Services Relative to Manufacturing

Notes: Data for OECD countries, 1970-2005. Each point corresponds to a country-year observation after partialling-out sectoral prices and country fixed effects. The red line depicts the OLS fit, the shaded regions, the 95% confidence interval.

29 sub-Saharan and 27 other developing countries. Young also uses a log-linear Engel curve formulation and finds that the slopes of Engel curves greatly differ across consumption categories but appear stable over time. Below, we further complement this body of evidence on the approximate log-linearity of Engel curves by showing that income elasticities stay quite similar across different income brackets using the CEX.

Log-linear Engel curves also provide reasonable approximations to aggregate consumption variables. Figure (1a) plots the relationship between the residual (log) expenditure share in agriculture relative to manufacturing and residual (log) income after filtering the effects of relative prices for OECD countries. Figure (1b) plots the residual (log) expenditure share for services (relative to manufacturing) also against residual (log) income. We see that residual relative expenditure shares and residual income are strongly negatively correlated in agricultural goods and positively correlated in services. The depicted log-linear fit shows that Engel curves with constant slope capture a great part of the variation in the data. In fact, if we split the sample into observations below and above the median income in the sample and estimate the Engel curves separately, we cannot reject the hypothesis of identical slopes of the Engel curves (see Table D.1 in the online appendix). This suggests that Engel curves remain constant at different income levels and do not level off as income grows.

3Specifically, he estimates the elasticity of consumption for the different categories with respect to the education of the household head and then uses the estimates of the return to education from Mincerian regressions to back out the income elasticity of consumption.

4Olken (2010) discusses Young’s exercise using Indonesia survey data and finds similar results for a small sample of three goods and services he studies. Young (2013) also makes use of log-linear Engel curves to infer consumption inequality.

5The $R^2$ of the regressions shown in Figure 1 are 27% and 20%, respectively.
Motivated by this evidence, we develop a multi-sector model of structural transformation that accommodates for nonhomotheticity in the form of log-linear Engel curves, as well as trends in relative prices. The model builds on the standard framework used in recent empirical work on structural transformation (e.g., Herrendorf et al., 2014). The key departure from the standard framework is that we introduce a class of utility functions that generates heterogeneous, non-homothetic sectoral demands for all levels of income, including when income grows toward infinity. These preferences, which we will refer to as nonhomothetic CES preferences, have been studied by Gorman (1965), Hanoch (1975), Sato (1975), and Blackorby and Russell (1981) in the context of static, partial-equilibrium models.

Nonhomothetic CES preferences generate Engel curves for different sectors that match the evidence discussed above: the logarithm of relative demand for the output of each sector has an approximately linear relationship with the logarithm of income. More specifically, this relationship is characterized by a sector-specific income elasticity parameter. Crucially, these income elasticity parameters are independent of the elasticity of substitution between sectors, a feature that is unique to our choice of preferences. Since our framework does not impose functional relationships between income and substitution elasticities, it lends itself to the task of decomposing the contributions of the demand and supply channels to structural change. In addition, our framework can accommodate an arbitrary number of sectors with heterogeneous income elasticities. We take advantage of this feature to first consider a standard three-sector setting with agriculture, manufacturing and services. We then extend our analysis to a thinner disaggregation (ten sectors) to explore sectoral reallocation patterns within manufacturing and services.

Our theory of structural change yields a number of theoretical and empirical results. First, the equilibrium in our model asymptotically converges to a path of constant real consumption growth. The asymptotic growth rate of real consumption depends on parameters characterizing both the supply and demand channels; it is a function of the sectoral income elasticities as well as sectoral growth rates of TFP and sectoral factor intensities. In this respect, our model differs from standard models using Stone-Geary preferences in which long-run growth is pinned down solely by the growth rate of TFP, generalizing the findings of Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008). Second, our theory can produce similar evolutions for nominal and real sectoral measures of economic activity, which is a robust feature of the data. This is a consequence of the role of income elasticities in generating sectoral reallocation patterns. Third, our framework can generate hump-shaped patterns for the evolution of manufacturing consumption shares both in nominal and real terms, which is a well-documented feature in the data (Buera and Kaboski, 2012a).

To evaluate the model empirically, we use structural equations derived from our theory to

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6Herrendorf et al., 2014 show that supply-side driven structural transformation cannot account for the similar evolution of nominal and real sectoral measures of activity.
estimate the elasticities that characterize our utility function. We use data on cross-country, historical sectoral shares that vary in the geographies and periods covered and in the measures of economic activity used to capture the structural transformation. A major finding is that the estimates of the elasticity of substitution and the relative slopes of the Engel curves across sectors are robust to the sample of countries, time periods and economic measures of sectoral activity. This demonstrates that the patterns presented in Figure 1 not only characterize the Engel curves in the OECD but also apply more broadly to countries at other stages of development. We take this ability to parsimoniously account for structural transformation in a variety of contexts as evidence in favor of our model.\footnote{A key parameter singled out in the literature is the price elasticity of substitution between consumption of different goods and services. Our baseline estimate of the elasticity of substitution is around 0.7. We find a very similar estimate using household level data from the Consumer Expenditure Survey (CEX), for which we can directly control for sectoral demand shocks and use an IV strategy. We also find that, compared to previous estimates based on Stone-Geary preferences (Herrendorf et al., 2013), the estimate of the elasticity of substitution is more robust to using either value added or expenditure measures.}

Finally, we use our model to study the drivers of structural transformation. Both relative prices and income effects turn out to be significant contributors. However, in contrast to several studies (e.g., Dennis and Iscan, 2009, Boppart, 2013), we find that income effects are more important than sectoral substitution driven by relative price trends. A key reason for this discrepancy may be that in our framework income effects are not hard-wired to have only transitory effects on the structural transformation or to be correlated with price effects. Once we do not impose these arbitrary constraints on income effects, our estimates are consistent with a predominant role of income effects in accounting for the structural transformation observed during the postwar period in a large sample of countries at very different stages of development.

Our paper relates to a large literature aiming at quantifying the role of non-homotheticities on growth and development (see, among others, Echevarria, 1997, Gollin et al., 2002, Duarte and Restuccia, 2010, Alvarez-Cuadrado and Poschke, 2011).\footnote{An alternative formulation that can reconcile demand being asymptotically non-homothetic with balanced growth path is given by hierarchical preferences (e.g., Foellmi and Zweimüller, 2008 and Foellmi et al., 2014). This formulation, however, abstracts from different production sectors. Rather, it focuses on a product cycle for products whereby they start as luxury goods, to eventually become a necessity (see, among others, Matsuyama, 2000, 2002 and Saint-Paul, 2006).} Buera and Kaboski (2009) and Dennis and Iscan (2009) have noted the limits of the Stone-Geary utility function to match long time series or cross-sections of countries with different income levels.\footnote{In recent work, Święcki (2014) uses an indirect utility function that also features non-vanishing income effects. Święcki estimates a static structural model in which he allows for international trade, intersectoral wedges and non-constant elasticities of substitution across goods. He finds that income and price effects account for most of the variation in his panel of countries for 1970-2005.}

The paper that is the closest to ours is Boppart (2013). Boppart studies the evolution of consumption of goods relative to services by introducing a sub-class of price-independent-generalized-linear (PIGL) preferences that also yield income effects in the long-run. There are
several important differences between the PIGL preferences and the nonhomothetic Constant Elasticity of Substitution (CES) preferences that we use. First, just like explicitly separable preferences such as Stone-Geary, PIGL preferences also presuppose specific parametric correlations for the evolution of income and price elasticities over time (Gorman, 1965). In contrast, Nonhomothetic CES preferences do not build in any connection between price and income effects, allowing the data to determine the relative importance of each. Second, PIGL preferences can only accommodate two sectors with distinct income elasticities. In contrast, our framework allows for an arbitrary number of sectors. ¹⁰ These differences between the two models are reflected in the empirical results. Whereas we find a larger contribution for demand nonhomotheticity in accounting for structural change, Boppart concludes that supply and demand make roughly similar contributions. ¹¹

The rest of the paper is organized as follows. Section 2 presents the basic ingredients of the model that we use to build the estimation equations for our empirical exercise. Section 3 contains the estimation and model evaluation for a panel of 25 countries for the period 1947-2005. Section 4 analyzes household expenditure data and aggregate macroeconomic time series for the United States. Section 5 studies the dynamics and asymptotic properties of our model. Notably, this section provides conditions under which the equilibrium in our model is unique and converges asymptotically to a path of constant real consumption growth. Section 6 concludes the paper.

2 Basic Model

In this section, we provide a theoretical framework that guides our empirical investigation of structural transformation in Sections 3 and 4. For concreteness, we first focus on the introduction of nonhomothetic CES preferences into a multi-sector growth model to highlight the empirically relevant equations that govern the equilibrium of our model. We leave a full-fledged presentation of our theory and the study of the asymptotic properties of our model to Section 5.

The model presented in this section closely follows workhorse models of structural transformation, (e.g., Buera and Kaboski, 2009; Herrendorf et al., 2013, 2014). The model essentially replaces the standard aggregators of sectoral consumption goods with a nonhomothetic CES

¹⁰Boppart notes that he can extend PIGL utility to three sectors, to have agriculture, manufacturing and services. The idea is to make one of the two goods that enter the utility function a constant-elasticity-of-substitution composite of agriculture and manufacturing. However, the method he proposes is not amenable to our purposes because it imposes the same income elasticity for agriculture and manufacturing. He also shows that additional sectors can be added as long as they have constant expenditure shares and unitary income elasticities, which also is not amenable to our purposes.

¹¹In terms of the scope of the empirical exercise, while Boppart (2013) estimates his model with U.S. data and considers two sectors, the empirical evaluation of our model includes, in addition to the U.S., a wide range of other rich and developing countries and more than two sectors.
aggregator. Crucially, this single departure from the standard workhorse model delivers the main theoretical results of the paper and the demand system later used in the estimation of our data. On the production side, the model combines two distinct potential drivers of sectoral reallocation previously highlighted in the literature: heterogeneous rates of technological growth (Ngai and Pissarides, 2007) and heterogeneous capital-intensity across sectors (Acemoglu and Guerrieri, 2008). We show that our framework can account for both of these supply-side channels through the price effect.

2.1 Preferences and the Household Problem

A representative household has the following intertemporal preferences over goods and services produced in $I$ different sectors

$$\sum_{t=0}^{\infty} \beta^t \left( \frac{C_t^{1-\theta} - 1}{1 - \theta} \right),$$

where $\beta \in (0, 1)$ is the discount factor, and $\theta$ is the reciprocal of the elasticity of intertemporal substitution. Aggregate consumption, $C_t$, combines sectoral goods, $\{C_{it}\}_{i=1}^I$, according to the implicitly defined function

$$\sum_{i=1}^{I} \Omega_i^{\frac{1}{\sigma}} C_t^{\frac{\epsilon_i - \sigma}{\sigma}} C_{it}^{\frac{\sigma - 1}{\sigma}} = 1,$$

where $\sigma \in (0, 1)$ is the elasticity of substitution, and $\Omega_i$’s are constant weights. Each sectoral good $i$ is identified with a parameter $\epsilon_i$, which is a measure of the income elasticity of demand for that good. Equation (2) introduces a nonhomothetic generalization of the standard Constant Elasticity of Substitution (CES) aggregator, which corresponds to the special case where $\epsilon_i \equiv 1$ for all sectors. Intuitively, as the aggregate income $C_t$ increases, the weight given to the consumption of good $i$ varies at a rate controlled by parameter $\epsilon_i$. As a result, the household’s demand for sectoral goods $i$ features a constant elasticity in terms of the aggregate consumption $C_t$, which is in turn determined by household income.

A number of unique features of the nonhomothetic CES aggregator makes it a natural choice for our model. In particular, consider the static expenditure minimization problem with sectoral prices $\{p_i\}_{i=1}^I$ and aggregate consumption (i.e., utility) defined as in equation (2). The resulting Hicksian demand function has the following properties.

1. The elasticity of the relative demand for two different goods with respect to aggregate

\[ \text{If } \sigma > 0 \text{ and } \Omega_i > 0 \text{ for all } i \in I \equiv \{1, \ldots, I\} \text{ and if } \epsilon_i > \sigma \text{ when } \sigma < 1, \text{ or } \epsilon_i < \sigma \text{ when } \sigma > 1, \text{ then the aggregator } C_t \text{ introduced in equation (2) is globally monotonically increasing and quasi-concave, defining a well-defined utility function over the bundle of goods } (C_{1t}, \cdots, C_{It}) \text{ (Hanoch, 1975).} \]
consumption is constant, i.e.,
\[
\frac{\partial \log (C_i/C_j)}{\partial \log C} = \epsilon_i - \epsilon_j. \tag{3}
\]

2. The elasticity of substitution between goods of different sectors is uniquely defined and constant\(^{13}\)
\[
\frac{\partial \log (C_i/C_j)}{\partial \log (P_j/P_i)} = \sigma. \tag{4}
\]

The first property ensures that the nonhomothetic features of these preferences do not systematically vary as income grows. As discussed in the introduction and in Section 4, available data on sectoral consumption, both at the macro and micro levels, suggest stable and heterogeneous income elasticities across sectors. Therefore, we find it reasonable to specify preferences that do not result in systematically vanishing patterns of nonhomotheticity, as, for instance, would be implied by the choice of Stone-Geary preferences. Similarly, the second property ensures that the patterns of inter-sectoral substitution do not systematically vary as the income grows. As one would expect based on their name, this property is indeed unique to this class of nonhomothetic CES preferences.\(^{14}\) The combination of these two properties uniquely defines the aggregator in equation (2).

To complete the characterization of the household behavior, we assume that the representative household inelastically supplies one unit of labor and owns all capital in the economy. It takes the sequence of wages, rental prices of capital, and prices of goods and services \(\left\{w_t, R_t, \{p_{it}\}_{i=1}^I\right\}_{t=0}^\infty\) as given, and chooses a sequence of capital stocks \(\{K_t\}_{t=0}^\infty\) and aggregate consumption \(\{C_t\}_{t=0}^\infty\) to maximize its utility defined in equation (1), subject to the per-period budget constraint
\[
K_{t+1} + \sum_{i=1}^I p_{it}C_{it} \leq w_t + K_t (1 - \delta + R_t), \tag{5}
\]
where we have normalized the price of capital to 1. The next lemma characterizes necessary conditions for a sequence of sectoral consumption goods to be a solution to the household problem.

\(^{13}\)Note that for preferences defined over \(I\) goods when \(I > 2\), alternative definitions for elasticity of substitution do not necessarily coincide. In particular, equation (4) defines the so-called Morishima elasticity of substitution, which is not in general symmetric. This definition may be contrasted from the Allen-Ozawa elasticity of substitution defined as \(\frac{E \cdot \partial C_i/\partial P_j}{C_i/C_j}\), where \(E\) is the corresponding value of expenditure. Blackorby and Russell (1981) prove that the only preferences for which the Morishima elasticities of substitution between any two goods are symmetric, constant, and identical to Allen-Ozawa elasticities have the form of equation (2), albeit with a more general dependence of weights on \(C\).

\(^{14}\)Hanoch (1975) shows that any preferences that are explicitly additive in sectoral goods imply parametric links between income and substitution elasticities. Appendix A illustrates how such links appears in specific case of Stone-Geary and price-independent generalize linear (PIGL) preferences, two types of specifications recently used in studies of structural change.
Lemma 1. (Household Behavior) Consider a household with preferences and budget constraint as described by equations (1), (2), and (5). Given a sequence of prices \(\{w_t, R_t, \{p_{it}\}_{i=1}^I\}_{t=0}^\infty\), any interior solution satisfies the following intratemporal and intertemporal conditions.

1. The intratemporal allocation of consumption goods satisfies

\[
C_{it} = \Omega_i \left( \frac{p_{it}}{P_t} \right)^{-\sigma} C_t^{\bar{\epsilon} i},
\]

where \(P_t\) is the aggregate price index

\[
P_t \equiv E_t = \frac{1}{C_t} \left[ \sum_{i=1}^I \Omega_i C_t^{\epsilon_i} - \sigma \right]^\frac{1}{1-\sigma},
\]

and \(E_t \equiv \sum_i p_i C_i\) denotes total expenditure.

2. The intertemporal allocation of real aggregate consumption satisfies the Euler equation

\[
C_t^{-\theta} = (1 - \delta + R_t) \frac{P_t}{P_{t+1}} \left( \frac{\sum_{i=1}^I \omega_{it} \epsilon_i - \sigma}{\sum_{i=1}^I \omega_{i,t+1} \epsilon_i - \sigma} \right) C_{t+1}^{-\theta},
\]

where \(\omega_{it}\) is the expenditure share in sector \(i\), \(p_{it} C_{it}/E_t\).

Proof. See Appendix B. \qed

The key insight from Lemma 1 is that the household problem can be decomposed into two sub-problems: one involving the allocation of consumption across sectors, and one involving the allocation of consumption over time. In the latter, the consumer solves the intertemporal consumption-savings problem, finding the sequence of \(\{K_t, C_t\}_{t=0}^\infty\) to maximize expected utility (1) subject to the constraint

\[
K_{t+1} + E \left( C_t, \{p_{it}\}_{i=1}^I \right) \leq w_t + K_t (1 - \delta + R_t),
\]

where \(E \left( C_t, \{p_{it}\}_{i=1}^I \right)\) is the expenditure function for the nonhomothetic CES preferences, defined in equation (7). Because of nonhomotheticity, consumption expenditure is a nonlinear function of real aggregate consumption, and the price index reflects changes in the sectoral composition of consumption as income grows. The household incorporates this relationship in its Euler equation, (8), where we see a wedge between the marginal cost of real consumption and the aggregate price index. The size of this wedge, given by \((\bar{\epsilon}_t - \sigma) / (1 - \sigma)\), depends on

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\(^{15}\)With a slight abuse of notation, here we have equated household assets and capital stock of the economy. Accordingly, the sequence of aggregate consumption and capital investments have to further satisfy the No-Ponzi condition \(\lim_{t\to\infty} K_t \left( \prod_{t'=1}^{t-1} \frac{1}{1-\delta + R_{t'}} \right) = 0\) and the corresponding transversality condition.
the average income elasticities across sectors, \( \bar{\epsilon}_t = \sum_{i=1}^{I} \omega_{it} \epsilon_i \), and varies over time. In the case of homothetic CES where \( \epsilon_i \equiv 1 \), this wedge disappears.

The second part of the household problem involves the *intratemporal* problem of allocating consumption across different goods. Equation (6) is identical to the Hicksian demand for nonhomothetic CES utility with the corresponding sectoral prices and aggregate consumption. The lemma establishes that given aggregate consumption \( C_t \) allocated to period \( t \), sectoral demand simply follows the solution to the static allocation problem.

Note that Equation (6) restates the two main features of the nonhomothetic CES aggregator expressed in equations (3) and (4): constant and independent elasticities of income and substitution for different goods. We can rewrite this relation in terms of the logarithm of relative real consumption and consumption expenditure shares between sectors \( i \) and \( j \) as a function of the logarithm of relative prices and aggregate consumption

\[
\log \left( \frac{C_{it}}{C_{jt}} \right) = -\sigma \log \left( \frac{p_{it}}{p_{jt}} \right) + (\epsilon_i - \epsilon_j) \log C_t + \log \left( \frac{\Omega_i}{\Omega_j} \right), \quad (9)
\]

\[
\log \left( \frac{\omega_{it}}{\omega_{jt}} \right) = (1 - \sigma) \log \left( \frac{p_{it}}{p_{jt}} \right) + (\epsilon_i - \epsilon_j) \log C_t + \log \left( \frac{\Omega_i}{\Omega_j} \right), \quad (10)
\]

where the second equation simply states the relationship in terms of expenditure shares, which play a key role in our theory as we will see later. Equation (9) once again highlights the key features of the demand system implied by this nonhomothetic CES preferences. Interpreting \( C_{it} \) as the Hicksian demand for good \( i \) with aggregate consumption \( C_t \) under prices \( p_{it} \)'s, we find a *perfect separation of the price and the income effects*. The first term on the right hand side shows the price effects characterized by constant elasticity of substitution \( \sigma \). More interestingly, the second term on the right hand side shows the change in relative sectoral demand as consumers move across indifference curves.\(^{16}\) This income effect is characterized by constant sectoral income elasticity parameters \( \epsilon_i \)'s. If \( \epsilon_i > \epsilon_j \), demand for good \( i \) rises relative to good \( j \) as consumers become wealthier.\(^{17}\)

Equations (9) and (10) also show how our model can generate a positive correlation be-

\(^{16}\)Nonhomothetic CES preferences inherit this property because they belong to the class of Implicitly Additive preferences (Hanoch, 1975). In contrast, Explicitly Additive preferences such as Stone-Geary preferences do not allow separation of income and price effects (see Appendix A).

\(^{17}\)The expenditure elasticity of demand for sectoral good \( i \) is given by

\[
\eta_{it} = \frac{\partial \log C_{it}}{\partial \log E_t} = 1 + \frac{1 - \sigma}{\bar{\epsilon}_t - \sigma} (\epsilon_i - \bar{\epsilon}_t), \quad (11)
\]

which, as Engel aggregation requires, averages to 1 when sectoral weights are given by expenditure shares. Sector \( i \) with an income elasticity parameter \( \epsilon_i \) that exceeds the economy-wide average elasticity parameter \( \bar{\epsilon}_t \) at time \( t \) has an expenditure elasticity greater than 1 at that point in time. Equation (11) further shows that the expenditure elasticity is invariant to a re-scaling of all sectoral income elasticity parameters \( \epsilon_i \)'s. We can express the expenditure elasticity of relative demand as:

\[
\frac{\partial \log (C_{it}/C_{jt})}{\partial E_t} = \frac{\epsilon_i - \epsilon_j}{\bar{\epsilon}_t - \sigma}, \quad (12)
\]
tween relative sectoral consumption in real and expenditure terms, as it is observed in the data. As in the case with homothetic CES aggregators, the combination of the price effect and gross complementarity \((\sigma < 1)\) imply that relative real sectoral consumption should negatively correlate with relative sectoral prices. To see why, note that relative real consumption is decreasing in relative prices with an elasticity of \(-\sigma\), while relative expenditure is increasing with an elasticity of \(1 - \sigma\). However, our demand system has an additional force, income effects, which makes both time series co-move in aggregate consumption. Thus, if income effects are sufficiently strong, both time series can be positively correlated. In Section 3.4 we show that this is the case when we estimate our demand system.

### 2.2 Production and Competitive Equilibrium

The supply side of the economy allows for two distinct sources of heterogeneity in sectoral production. The model combines the heterogenous sectoral productivity growth framework of Ngai and Pissarides (2007) with heterogenous sectoral factor intensity model of Acemoglu and Guerrieri (2008). A representative firm in each consumption sector produces sectoral output under perfect competition. In addition, a representative firm in an investment sector produces investment good \(Y_{0t}\) that is employed in the process of capital accumulation, again under perfect competition. We assume Cobb-Douglas production functions with time-varying Hicks-neutral sector-specific productivities

\[
Y_{it} = A_{it} K_{it}^{\alpha_i} L_{it}^{1-\alpha_i}, \quad i \in \{0\} \cup \mathcal{I},
\]

where \(K_{it}\) and \(L_{it}\) are capital and labor used in the production of sector \(i\) at time \(t\) (we have identified the sector producing investment good as \(i = 0\)) and \(0 < \alpha_i < 1\) denotes sector-specific capital intensity. The aggregate capital stock of the economy, \(K_t\), accumulates using investment goods and depreciates at the rate \(\delta\), \(Y_{0t} = K_{t+1} - (1 - \delta) K_t\).

We focus on the features of the competitive equilibrium of this economy that motivate our empirical specifications.\(^{19}\) Firm profit maximization and equalization of the prices of labor which is the parallel to equation (3) now expressed in terms of expenditure, rather than real aggregate consumption. To connect the model to our empirical analysis in Section 3 we use the specification in Equation (10) in terms of real consumption, as will be discussed in detail in Section 3 and the online Appendix.

\(^{18}\)With homothetic CES aggregators of complementary goods, i.e., \(\sigma < 1\), the correlation between relative sectoral consumption in real and expenditure terms is negative, while it is positive in the data. See Herrendorf et al. (2014) for further discussion.

\(^{19}\)Given initial stock of capital \(K_0\) and a sequence of sectoral productivities \(\{A_{it}\}_{t=1}^T\), a competitive equilibrium is defined as a sequence of allocations \(\{C_t, K_{t+1}, Y_{0t}, L_{0t}, K_{0t}, \{Y_{it}, C_{it}, K_{it}, L_{it}\}_{i=1}^T\}_{t=1}^T\), a sequence of prices \(\{w_t, R_t, \{p_{it}\}_{i=1}^I\}_{t=1}^T\), and a sequence of aggregate price indices \(\{P_t\}_{t=1}^T\) such that (i) agents maximize the present discounted value of their utility given their budget constraint, equation (1), (ii) firms maximize profits and (iii) markets clear.
and capital across sectors pin down prices of sectoral consumption goods

\[ p_{it} = \frac{p_{it}}{p_{0t}} = \alpha_0^\alpha (1 - \alpha_0)^{1-\alpha_0} \left( \frac{w_t}{R_t} \right)^{\alpha_0 - \alpha_i} \frac{A_{0t}}{A_{it}}, \tag{13} \]

where, since the units of investment good and capital are the same, we normalize the price of investment good, \( p_{0t} \equiv 1 \). Equation (13) illustrates that variations in sectoral prices stem either from differences in sectoral factor intensities (second term on the right hand side) or productivities (third term on the right hand side).

Goods market clearing ensures that household sectoral consumption expenditure equals the value of sectoral production output, \( P_{it} C_{it} = P_{it} Y_{it} \). Competitive goods markets and profit maximization together imply that a constant share of sectoral output is spent on the wage bill,

\[ L_{it} = (1 - \alpha_i) \frac{P_{it} C_{it}}{w_t} \omega_{it}, \tag{14} \]

where \( \omega_{it} \) is the share of sector \( i \) in household consumption expenditure.

Equation (14), together with equations (10) and (13) summarize the main insights from the theory that we employ in our empirical strategy in Section (3). First, equation (14) implies

\[ \frac{L_{it}}{L_{jt}} = \frac{1 - \alpha_i}{1 - \alpha_j} \frac{\omega_{it}}{\omega_{jt}}, \quad i, j \in \mathcal{I}. \tag{15} \]

Equation (15) shows that the paths of relative sectoral employment shares follow those of relative consumption expenditure shares. Second, Equation (10) characterizes the paths of relative consumption expenditure shares as a function of relative prices and aggregate real consumption. Finally, equation (13) highlights the fact that price effects capture both supply-side drivers of sectoral reallocations, that is, the heterogeneity in productivity growth rates and in capital intensities. In summary, in presence of reliable indices for the paths of sectoral prices and aggregate consumption expenditure, equations (15) and (10) together predict the evolution of relative employment shares across sectors. We will use these two equations extensively in our empirical exercise in the next two sections.

3 Quantitative Exploration of a Cross-Country Panel

In this section we explore the ability of nonhomothetic CES preferences to account for the broad patterns of structural transformation observed across countries during the postwar period. We discipline our model by using the same parameters of the utility function \( \{\sigma, \epsilon_i\}_{i \in \mathcal{I}} \) for all countries. After estimating these parameters, we gauge the ability of our model to account for the very different experiences of advanced, miracle and developing economies. We

\[ ^{20}\text{In our empirical applications, we account for sectoral trade flows.} \]
conclude the section by conducting a battery of exercises that revisit critical findings in the structural transformation literature through the lens of our model.

3.1 Empirical Strategy

Our empirical strategy uses the solution of the intratemporal problem and the production decisions of firms to estimate the preference parameters of utility function (2). Taking the logarithms of the sectoral demands (10) and using that the ratio of sectoral expenditures is proportional in equilibrium to the ratio of sectoral labor allocations, we obtain that

\[
\log \left( \frac{L_{a,t}^c}{L_{m,t}^c} \right) = \zeta_{am} + (1 - \sigma) \log \left( \frac{p_{a,t}^c}{p_{m,t}^c} \right) + (\epsilon_a - \epsilon_m) \log C_t^c + \nu_{am,t}^c,
\]

(16)

\[
\log \left( \frac{L_{s,t}^c}{L_{m,t}^c} \right) = \zeta_{sm} + (1 - \sigma) \log \left( \frac{p_{s,t}^c}{p_{m,t}^c} \right) + (\epsilon_s - \epsilon_m) \log C_t^c + \nu_{sm,t}^c,
\]

(17)

where \(a, m\) and \(s\) denote agriculture, manufacturing and services, respectively, and \(t\), time. The superscript \(c\) denotes a country, and \(\nu_{am,t}^c\) and \(\nu_{sm,t}^c\) are the error terms. We allow for country-sector dyad fixed effects, \(\zeta_{am}^c\) and \(\zeta_{sm}^c\), as there may be systematic differences in measurement across countries. These country-sector dyad fixed effects also absorb cross-country differences in sectoral taste parameters, \(\Omega_i^c\), and differences in factor shares in the production function, \(\alpha_i\). Note that there are two cross-equation restrictions. The price elasticity \(\sigma\) is restricted to be the same across sectors and countries. Income elasticities, \(\epsilon_i\), are also restricted to be the same across countries for a given sector \(s\). Thus, our structural model assumes that income and price elasticities are constant and identical across countries.

Finally, according to the model developed in Section 2, we could either use expenditure shares in the left-hand-side of our regression or relate expenditure shares to employment shares using the market clearing conditions. We prefer to use employment shares because, in order to construct the price indices, we use both nominal and real sectoral consumption data. As a result, if we used expenditures in the left-hand-side of our estimation equations, we would potentially obtain biased estimates of the income and price elasticities. However, as we discuss below, we obtain similar estimates in practice.\(^{21}\) The use of employment shares introduces an additional complication. In contrast to relative expenditures, it is important to account for the fact that some goods can be imported and exported, thus affecting the sectoral

---

\(^{21}\)Nominal consumption expenditure has to be deflated by the ideal price index to obtain the correct measure of real consumption. We use chained Fisher indices as price deflators because they approximate to second order the ideal price index (online Appendix C contains further discussion on the topic). We have run Monte-Carlo simulations to assess the extent of the bias induced by using this approximation around the estimated parameters. We find that the error is less than 1% of the estimated parameters (Online Appendix B contains a sample code). Deaton and Muellbauer (1980) find the same result for the AIDS demand system: using the exact form of the price index or an approximation by a superlative price index makes little difference for the estimation results.
employment composition. Accordingly, we control for sectoral net exports in our regressions. We control for the share of net exports over total production in sector $i$, time $t$ and country $c$. This particular functional form for the controls follows from our theoretical model.\textsuperscript{22,23}

**Identification** The identification strategy relies on the intra-period allocation of consumption that follows from the solution of the intratemporal allocation problem (10). That is, conditional on the observed levels of aggregate consumption $C^c_t$ and sectoral prices $p^c_{it}$, we use our demand system to estimate relative consumption across sectors. Given the presence of country-sector dyad fixed effects, $\zeta_{sam}, \zeta_{sm}^c$, the relevant variation used to identify the elasticities is the within country-sector time variation. To the extent that we have a long time series for $C^c_t$ and $p^c_{it}$, we have a super-consistent estimator of the elasticities (Hamilton, 1994).

Changes over time in aggregate and sectoral productivity contribute to the identification of the price and income elasticities. For example, sectoral productivity shocks, such as an increase in relative productivity of one sector, affect relative prices and introduce variation in the estimating equations, (16) - (17).\textsuperscript{24} Sectoral and aggregate productivity shocks can also affect the level of total consumption, thereby introducing additional variation in the estimation through $C^c_t$.

Our structural model and corresponding estimation method assume that taste parameters are constant over time. However, as this may not be true in the data, it is important to discuss the role of demand shocks. Aggregate demand shocks, such an increase in the propensity to spend, are captured through the aggregate expenditure term $\log C^c_t$ in (16) - (17) and also contribute to the identification of our demand system. However, sectoral taste shocks that induce consumers to spend more in one sector for a given level of aggregate expenditure and sectoral prices are not well-captured in this specification.\textsuperscript{25} Given that we have already country-sector dyad fixed effects, we cannot add an additional time fixed effect that would control for preference shocks. To the extent that these shocks are uncorrelated with other type of shocks, they enter as classical measurement error the estimating equations (16) – (17).

\textsuperscript{22}We have also used more reduced-form controls, such as controlling directly for net exports or exports and imports separately, obtaining similar results. To derive this result, note that $p^c_{it}C^c_{it} = p^c_{it}Y^c_{it} - NX^c_{it}$, where $NX^c_{it}$ denotes the nominal value of net exports in sector $i$, time $t$ and country $c$. In a closed economy $NX^c_{it} = 0$ and expenditure shares perfectly predict employment shares. In general, if net exports are non zero, we have that employment is a function of domestic production $p^c_{it}Y^c_{it}$. Using the model in Section 2 (equation 14), the counter-factual amount of labor needed to produce the amount consumed in sector $s$ needs to be adjusted by $1 - NX^c_{it}/p^c_{it}Y^c_{it}$.

\textsuperscript{23}An additional challenge pointed out by Herrendorf et al. (2013) for the U.S. is that investment has become more service intensive over time and most national accounts attribute investment to manufacturing. We prefer to use employment shares as the lesser of two evils to avoid using the same variables in the left and right hand side. Moreover, for many of the countries of our sample this may be less of a concern than for the U.S.

\textsuperscript{24}Using the log of the ratio of sectoral prices in our estimation has the advantage that we can directly use nominal prices and any cross-country systematic difference in the measurement of prices is going to be captured in the fixed effect.

\textsuperscript{25}To be more precise, suppose that the preference term for sector $i$, $\Omega^c_{it}$, in (2) becomes stochastic rather than constant, $\Omega^c_{it}$.
and our estimates are still consistent. If, on the other hand, sectoral preference shocks are correlated with other shocks (e.g., aggregate demand or productivity shocks) our estimation is going to produce biased estimates.

We deal with potential biases coming from sectoral taste shocks in two different ways. First, we estimate the elasticities separately for OECD and non-OECD countries and show that we cannot reject the null that they are statistically the same. While the estimates could conceivably still be biased, this would require sectoral taste shocks to be correlated with aggregate demand or productivity shocks in the same way for these two groups of countries, which we deem unlikely. Second, in Section 4.1 we use household-level data, which allows us to include sector-year fixed effects that absorb sectoral demand shocks and to use an IV strategy. Reassuringly, we find that the estimates for the elasticity of substitution and of the income elasticities are similar to our estimates from aggregate data.

3.2 Data Description

We use the GGDC 10-Sector Database for sectoral value added data (Vries et al., 2014). It provides a long-run internationally comparable dataset on sectoral measures for 10 countries in Asia, 9 in Europe, 9 in Latin America, 10 in Africa and the United States. The variables covered in the data set are annual series of production value added (nominal and real) and employment for 10 broad sectors starting in 1947. In our baseline exercise we aggregate the ten sectors into agriculture, manufacturing and services. In Section 3.6 we estimate our model for 10 sectors.

For real consumption, we use the time series on consumption per capita from the Barro-Ursua Macroeconomic Data. Their data has the advantage of using the Fisher chained price index, which allows us to have a meaningful measure of real consumption. These data do not include government services into consumption, which we exclude from our estimation. Also, the only African country covered is South Africa. Our final sample consists of 25 countries that span very different growth trajectories during the postwar period.

3.3 Estimation Results

Table 1 reports the results of estimating the system of equations for the whole sample of 25 countries and for OECD and non-OECD countries separately.

26Not all time series start in 1947, Figures 5 and 6 depict the time series for each country.
27The ten sectors are agriculture, mining, manufacturing, construction, public utilities, retail and wholesale trade, transport and communication, finance and business services, other market services and government services. We lump into manufacturing mining, manufacturing and construction, while the rest are classified as services (except agriculture).
28It can be obtained at http://scholar.harvard.edu/barro/publications/barro-ursua-macroeconomic-data
29These are Denmark, France, Italy, the Netherlands, Spain, Sweden, UK, USA, West Germany, India, Indonesia, Japan, South Korea, Malaysia, Philippines, Singapore, Taiwan, Argentina, Brazil, Chile, Colombia,
Table 1: Baseline Estimates for the Cross-Country Sample

<table>
<thead>
<tr>
<th>Dep. Var.:</th>
<th>World (1)</th>
<th>OECD (2)</th>
<th>Non-OECD (3)</th>
<th>OECD (4)</th>
<th>Non-OECD (5)</th>
<th>OECD (6)</th>
<th>Non-OECD (7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rel. Emp.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>σ</td>
<td>0.66</td>
<td>0.75</td>
<td>0.75</td>
<td>0.69</td>
<td>0.71</td>
<td>0.69</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>(0.19)</td>
<td>(0.11)</td>
<td>(0.10)</td>
<td>(0.17)</td>
<td>(0.16)</td>
<td>(0.08)</td>
<td>(0.07)</td>
</tr>
<tr>
<td>εₐ - εₚₘ</td>
<td>-0.81</td>
<td>-1.09</td>
<td>-1.04</td>
<td>-0.99</td>
<td>-0.89</td>
<td>-1.16</td>
<td>-1.10</td>
</tr>
<tr>
<td></td>
<td>(0.24)</td>
<td>(0.10)</td>
<td>(0.10)</td>
<td>(0.19)</td>
<td>(0.21)</td>
<td>(0.14)</td>
<td>(0.11)</td>
</tr>
<tr>
<td>εₛ - εₚₘ</td>
<td>0.32</td>
<td>0.32</td>
<td>0.32</td>
<td>0.40</td>
<td>0.51</td>
<td>0.23</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>(0.08)</td>
<td>(0.10)</td>
<td>(0.10)</td>
<td>(0.19)</td>
<td>(0.17)</td>
<td>(0.09)</td>
<td>(0.09)</td>
</tr>
<tr>
<td>Obs.</td>
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<td>1006</td>
<td>1005</td>
<td>436</td>
<td>436</td>
<td>570</td>
<td>569</td>
</tr>
<tr>
<td>R²</td>
<td>0.14</td>
<td>0.83</td>
<td>0.83</td>
<td>0.77</td>
<td>0.80</td>
<td>0.84</td>
<td>0.85</td>
</tr>
<tr>
<td>c · sm FE</td>
<td>N</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Trade Controls</td>
<td>N</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
</tr>
</tbody>
</table>

Note: Standard errors clustered by country.

Columns (1) to (3) in Table 1 report our estimates for the entire sample of countries. Column (1) reports the estimates without using country-sector fixed effects (and thus using cross-country variation in levels to identify the parameters). Column (2) reports our estimates using country-sector fixed effects (whereby using only within country variation to identify the elasticities) and column (3) includes trade controls. Our estimates of the price elasticity of substitution across sectors ranges from 0.66 to 0.75 and is precisely estimated in all three specifications (standard errors are clustered at the country level). In fact, we cannot reject the null that these three estimates are statistically identical. However, we can reject the null that they are equal to one at 5%, implying that these three sectors are complements. The estimates for the difference of income elasticities yield sensible results that are very stable across the three specifications and significant at conventional levels. We find that the difference in income elasticities between agriculture and manufacturing, \( \epsilon_a - \epsilon_m \), is negative and ranges from -1.09 to -0.81, while the difference between services and manufacturing, \( \epsilon_s - \epsilon_m \), is positive around 0.32 in all three specifications.\(^{30}\)

As we have discussed, sectoral taste shocks that are systematically correlated with other shocks are a threat to interpret our estimates as consistent estimates of the underlying elasticities. To mitigate this concern, we report the estimated elasticities for OECD countries

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Mexico, Peru, Venezuela and South Africa.

\(^{30}\)The scale of the elasticities does not matter for the real allocation of resources. The reason is that there is one degree of freedom in the definition of real income elasticities. Dieevert (1976) shows that by using a Fisher price index to construct real income, these are elasticities of real consumption defined relative to a benchmark utility level (see online Appendix C for further details). In Section 5, we present a specific cardinalization of utility that pins down this degree of freedom so that income elasticities can be interpreted in terms of real consumption as defined by Feenstra et al. (2013).
in columns (5) and (6) and non-OECD countries in columns (7) and (8) with and without trade controls, respectively. We find that the estimates of all elasticities are similar for the two sub-samples. In fact, we cannot reject the null that the estimates for the elasticities are the same for both sub-samples at conventional levels. For example, the $p$-value of testing the joint hypothesis of price and income elasticities being different for OECD and non-OECD countries is .45. We take this as reassuring evidence, as we deem unlikely that sectoral taste shocks are correlated across these country groupings. In the next section, we estimate our demand system using household data, which allows us to have more demanding specifications, reaching a similar conclusion.

**Fit of Cross-Country Panel and Case Studies**  
Table 1 also reports the $R^2$ of these regressions. We find that including prices and aggregate consumption with the country-sector dyad fixed effects accounts for 83% of the variation in our panel. The fit is equally good if we split the sample between OECD and non-OECD countries (77% and 84% respectively). The inclusion of explicit sectoral net exports controls improves the fit to 80% for OECD countries and 85% for non-OECD countries. Note, however, that this does not mean necessarily that trade plays a minor role. The reason is that our sectoral price measures are constructed from value added measures which include imported inputs and exported outputs. Thus, part of the productivity enhancing effects of trade are subsumed in our price measures.

Figure 2 plots the actual and the predicted employment shares implied by our estimates from column (3) of Table 1 for six countries, Mexico, Colombia, Japan, Taiwan, the U.S. and Spain.\(^{31}\) This figure confirms the good fit of the model despite the parsimony of using the same price and income elasticities for all countries. In particular, the model captures well the evolution of employment shares in all sectors for countries at very different stages of development.

For Japan and Taiwan, we see that our fitted model generates a hump-shape for employment in manufacturing that tracks well its evolution observed in the data. For Taiwan, the predicted initial level of the employment share in manufacturing is 21%, it goes up to 39% and back to 35% at the end of the period. The observed levels are 20%, 43% and 37%. For Japan, the employment share in manufacturing is 26% in the initial period, it goes up to 38% in the mid 1970s and it is 30% by the end of our sample. The fitted time series starts at 26%, goes up to 35% and declines to 33% by the end of the period.\(^{32}\)

To gain better intuition on the role played by relative prices, consumption and net exports,

\(^{31}\)Figures 5 and 6 show the predicted series of the employment shares and the actual time series for all countries in our sample.

\(^{32}\)For the U.S., we see that the evolution of the employment shares in services and manufacturing are steeper than predicted by our model. This is the case also for other OECD countries. This reflects the fact that the income elasticity of services is greater for these set of countries, as column (5) in Table 1 shows. Indeed, if we plot the predicted fit using the estimates $\{\sigma, \epsilon_s - \epsilon_m, \epsilon_a - \epsilon_m\}$ for only OECD countries this problem goes entirely away, as can be seen in Figure E.4 in the online appendix.
Figure 2: Baseline fit with common preference parameters \(\{\sigma, \epsilon_a, \epsilon_m, \epsilon_s\}\) for six countries

![Graphs showing baseline fit for Mexico, Colombia, Taiwan, Japan, USA, Spain.](image)

Figure 3: Stone-Geary fit with common preference parameters \(\{\sigma, \overline{c}, \overline{c}_a, \overline{c}_s\}\) for six countries

![Graphs showing Stone-Geary fit for Mexico, Colombia, Taiwan, Japan, USA, Spain.](image)

we show in Figure 4 the case of Japan in detail. Panel (a) shows the overall fit of the data using the estimated parameters from the entire sample (column 3). Panels (b), (c) and (d) show the time series of relative prices, consumption and sectoral net exports. Then, we report the partial fit generated by each time series (and the country-sector fixed effects) at the estimated parameter values. Panel (e) shows the partial fit generated by the relative price time series. We see that, at the estimated parameter values, relative prices account for relatively little of the variation. In contrast, the evolution of aggregate consumption accounts for much of the structural transformation (see panel (f)). In particular, income effects drive the
observed hump-shape in manufacturing. Intuitively, as the Japanese economy became richer in the 1950s, it reallocated labor away from agriculture to both manufacturing and services. Subsequent income growth led to the expansion of services which absorbed employment from manufacturing. Finally, panel (g) shows that changes in sectoral net exports did not play a significant role in accounting for the structural transformation in Japan.

Contribution of Relative Price and Income Effects  After studying the case of Japan, next we explore the drivers of structural transformation in a more systematic manner. Specifically, we compute the share of the predicted variation in employment shares that is generated by price and income effects, respectively.\(^{33}\)

The variation generated by prices in the median year accounts for 14% of the employment growth in agriculture, 43% in manufacturing and 17% in services. If we restrict attention to the OECD, we find a similar pattern with prices accounting for 19%, 39% and 20% of employment share growth in agriculture, manufacturing and services, respectively. The complementary share is accounted for by income effects. Hence, we conclude that the bulk of the variation in sectoral employment growth is generated by income effects. In the online Appendix, we study the robustness of this result to using other approaches to assess the relative contribution of the drivers of structural transformations. Table D.2 shows that the likelihood-ratio tests of including price and consumption data to a model with only country-sector fixed effects are significant.\(^{34}\) However, the increase in the likelihood is much higher when we add consumption data than when we add relative price data.\(^{35}\)

Heterogeneous Price Elasticity of Substitution Across Sectors  So far, we have assumed that the price elasticities across sectors, \(\sigma\), are identical. We can test whether this is a good approximation of our data or whether allowing for differential price elasticities across

\[^{33}\text{From the log-linearized demand system (10), we have that the growth rate of employment in sector } i \text{ relative to } j \text{ is } \gamma_{Li} - \gamma_{Lj} = (1 - \sigma)(\gamma_{pi} - \gamma_{pj}) + (\epsilon_i - \epsilon_j)\gamma_C, \]  

\[^{34}\text{The first column reports the raw log-likelihood of estimating our sample with just country-sector fixed effects (first line in Table D.2), using fixed effects and the price series only (second line), using fixed effects and the consumption series only (third line), and the full specification, which contains both the price and consumption series. Table D.3 in the online appendix also reports the partial correlations.} \]  

\[^{35}\text{This conclusion differs from Boppart (2013) who studies the evolution of services relative to the rest of the economy in the U.S. during the postwar period and finds that the contribution of price and income effects are roughly of equal sizes. If we confine our analysis to the U.S. and lump together agriculture and manufacturing into one sector, we still find that price effects generate less than a third of the variation. The key difference between our analyses is in the demand systems. In our specification, the demand price elasticity is constant. In contrast, Boppart’s demand system implies that the price elasticity of services relative to the rest of consumption is declining as the economy grows. As noted by Buera and Kaboski (2009), as relative services expenditure and value added grows at a faster rate than services’ relative price, a declining price elasticity automatically increases the explanatory power of relative prices.} \]
sectors improves the fit significantly. We run the baseline regressions (16) and (17) without imposing that the coefficient on relative prices is the same across regressions,

\[
\log \left( \frac{L_{c,t}^a}{L_{m,t}^c} \right) = c_{am}^c + (1 - \sigma_{am}) \log \left( \frac{p_{c,t}^a}{p_{m,t}^c} \right) + (\epsilon_a - \epsilon_m) \log C_t^c + \nu_{am,t}^c,
\]

(19)

\[
\log \left( \frac{L_{c,t}^s}{L_{m,t}^c} \right) = c_{sm}^c + (1 - \sigma_{sm}) \log \left( \frac{p_{c,t}^s}{p_{m,t}^c} \right) + (\epsilon_s - \epsilon_m) \log C_t^c + \nu_{sm,t}^c,
\]

(20)

where \( \sigma_{sm} \) is not restricted to be equal to \( \sigma_{am} \). Table D.4 in the online appendix reports our estimates. We find that the price elasticities in the two regressions are very similar. For services relative to manufacturing, \( \sigma_{sm} \), we find a point estimate of 0.78 with standard error of 0.18 (clustered at the country level). For agriculture relative to manufacturing, \( \sigma_{am} \), we find a point estimate of 0.67 with standard error of 0.12. Thus, we cannot reject the null that the coefficients on prices are statistically different from each other. Moreover, the income elasticity estimates remain unchanged. This suggests that the CES is a good approximation for analyzing these three sectors.

**Estimation with Value Added Shares** Some statistical agencies impute all investment employment to manufacturing, while its service component has been increasing overtime (Herrendorf et al., 2013). By measuring sectoral activity by their employment shares, we are implicitly adopting this assumption. Following Herrendorf et al., we study the robustness of our findings to estimating the baseline regressions (16)- (17) using value added shares as dependent variables. Table D.5 in the online appendix reports the estimation results. The main observation is that the estimates are robust to using value added shares as dependent variable. In particular, the estimate of the price elasticity declines insignificantly to .51 (from .76 with employment shares), the income elasticity of agriculture (relative to manufacturing) is \(-1.17 \) (vs. \(-1.04 \) with employment shares) and the income elasticity of services (relative to manufacturing) is \(0.1 \) (vs. 0.32 with employment shares).

### 3.4 Correlation between Real and Nominal Value Added

A salient feature of structural transformations in the data is that the sectoral time-series patterns are similar regardless of whether we document them in nominal or real terms (Herrendorf et al., 2014). To investigate our model’s ability to account for this fact, we use our estimates of the preference parameters \( \{\sigma, \epsilon_a - \epsilon_m, \epsilon_s - \epsilon_m\} \) and the sectoral demands, equations (9) and (10), to generate the predicted evolution of nominal and real sectoral demands.

Table 2 reports the correlation between nominal and real shares both in our estimated model and in the data. We find that the model is able to generate correlations similar to the data. In particular, the correlation between the nominal and real relative demand of
Table 2: Correlation of Nominal and Real Value Added

<table>
<thead>
<tr>
<th>Data Model</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agriculture/Manufacturing</td>
<td>0.95 0.93</td>
</tr>
<tr>
<td>Services/Manufacturing</td>
<td>0.80 0.71</td>
</tr>
</tbody>
</table>

agricultural goods to manufactures is 0.93 in our model, while in the data it is .95. For services, the model generates a correlation of .71 while in the data it is .80.

The success in matching the correlation between nominal and real measures of activity is important. First, note that it is an out-of-sample check on our model/estimates since our analysis has not targeted the evolution of real sectoral shares (recall that the left-hand-side of our estimating equations (16)-(17) were employment shares). More significantly, simultaneously accounting for the evolution of real and nominal sectoral shares highlights the critical importance of using a nonhomothetic CES framework. If we had used an homothetic framework, the correlation generated by the model would have been negative because the price elasticity of substitution is smaller than one.36 In our framework, the estimated income effects are sufficiently strong to overcome the relative price effect.

3.5 Comparison with Stone-Geary Preferences

Given the prevalence of Stone-Geary-like preferences in quantitative models of structural transformation, we compare the fit of our model with one that replaces our nonhomothetic CES preferences with Stone-Geary preferences. To this end, we consider the aggregator

$$C^c_t = \left[ \Omega^c_a \left( C^c_{a,t} + \bar{c}_a \right)^{\frac{\sigma-1}{\sigma}} + \Omega^c_m \left( C^c_{m,t} \right)^{\frac{\sigma-1}{\sigma}} + \Omega^c_c \left( C^c_{s,t} + \bar{c}_s \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}},$$

where $C^c_t$ denotes aggregate consumption of country $c$ at time $t$, $C^c_{i,t}$ denotes its consumption of sector $i$, $\bar{c}_a$ and $\bar{c}_s$ are constants that govern the nonhomotheticity of these preferences, $\sigma > 0$ and $\Omega^c_i > 0$ are preference parameters that are country specific.37 We follow the estimation procedure described in Herrendorf et al. (2013).38 As with nonhomothetic CES preferences, we estimate three parameters common across countries $\{\sigma, \bar{c}_a, \bar{c}_s\}$ that govern the price and income elasticities, and $\{\Omega^c_{i,t}\}_{i \in I, c \in C}$ which are country specific parameters. As expected, we find that $\bar{c}_a < 0$ and $\bar{c}_s > 0$.

36To see that, note that the relative trend in nominal values $\frac{\omega^c_{i,t}}{\omega^c_{j,t}}$ would be proportional to $\left( \frac{p^c_{i,t}}{p^c_{j,t}} \right)^{1-\sigma}$. For real values, $\frac{C^c_{i,t}}{C^c_{j,t}}$, would be proportional to $\left( \frac{p^c_{i,t}}{p^c_{j,t}} \right)^{-\sigma}$. As $0 < \sigma < 1$, both trends would move in opposite directions.

37Since these preferences are not implicitly additive, the price and income elasticities are not independent. In Appendix A.3 we show that the elasticity of substitution between $i$ and $j$ is $\sigma_{ij} = \sigma \eta_i \eta_j$, where $\eta$’s denote income elasticities.

38See online Appendix A for further discussion on the estimation procedure and estimation results.
Table 3: Sectoral $R^2$ measures for Stone-Geary and nonhomothetic CES

<table>
<thead>
<tr>
<th></th>
<th>Stone-Geary</th>
<th>Nonhomothetic CES</th>
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</thead>
<tbody>
<tr>
<td>Agriculture</td>
<td>0.84</td>
<td>0.98</td>
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<tr>
<td>Manufacturing</td>
<td>0.75</td>
<td>0.87</td>
</tr>
<tr>
<td>Services</td>
<td>0.74</td>
<td>0.90</td>
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</tbody>
</table>

Note: $R^2$ is computed for each sector as $R^2_i = 1 - \frac{\sum_{j=1}^{N}(y_{ij} - \hat{y}_{ij})^2}{\sum_{j=1}^{N}(y_{ij} - \bar{y}_i)^2}$ where $N$ denotes the total number of observations, $\bar{y}_i$ denotes the sample average of $y_i$ and $i \in \{a, m, s\}$.

Figure 3 shows the fit of the model for the same countries as in Figure 2 using the common parameters $\{\sigma, \bar{c}_a, \bar{c}_s\}$ and the country-specific preference shifters $\{\Omega_i^c\}_{i \in I, c \in C}$.\(^{39}\) We see that the overall fit is worse than with nonhomothetic CES preferences. For example, the fitted model is not able to reproduce the hump-shaped pattern for manufacturing of Japan and Taiwan. This worse fit is confirmed for the full sample. In Table 3 we compare the sectoral $R^2$ measures generated with Stone-Geary preferences and nonhomothetic CES.\(^{40}\) We find that in all sectors the fit is worse with Stone-Geary than with nonhomothetic CES. The difference in the $R^2$s ranges from 12 to 16 percentage points.

The intuition for the worsening of the fit is that, with Stone-Geary preferences, income effects are very low for rich countries, as for high levels of income the subsistence levels responsible for introducing the nonhomotheticity $\{\sigma, \bar{c}_a, \bar{c}_s\}$ are negligible. For example, the value of $\frac{p_i \bar{c}_a}{P_t C_t}$ for the U.S. at our estimated parameters is less than .05% at any point in time. Thus, only variation in prices and trade shares are left to explain the variation in employment shares.\(^{41}\)

3.6 Beyond Three Sectors

Jorgenson and Timmer (2011) have pointed out that in order to understand how structural transformation progresses in rich countries, it is important to zoom in the service sector, as it represents the majority of rich economies’ consumption shares (see also Buera and Kaboski, 2012b). Our framework lends itself to this purpose, as it can accommodate any arbitrary number of sectors in an analogous way to homothetic CES framework. In this section, we use the richness of the GDDC database to extend our estimation to 10 sectors: (1) agriculture, forestry and fishing, (2) mining and quarrying, (3) manufacturing, (4) public utilities, (5) construction, (6) wholesale and retail trade, hotels and restaurants, (7) transport, storage, trade and services.

\(^{39}\)Figures D.2 and D.3 in the online appendix show the fit for all countries in our sample.

\(^{40}\)Formally, we compute $R^2_i = 1 - \frac{\sum_{j=1}^{N}(y_{ij} - \hat{y}_{ij})^2}{\sum_{j=1}^{N}(y_{ij} - \bar{y}_i)^2}$ where $N$ denotes the total number of observations, $\hat{y}_{ij}$ denotes the sample average of $y_i$ and $i \in \{a, m, s\}$.

\(^{41}\)An equivalent intuition provided by Dennis and Iscan (2009) is that the subsistence constants $\{\bar{c}_a, \bar{c}_s\}$ should not be stable over time to have income effects play a greater role and improve the model fit.
and communication, (8) finance, insurance, real state, (9) community, social and personal
services, (10) government services.\footnote{The data set also contains information on dwellings that are not constructed within the period, but this information is very sparse. We decided to abstract from dwellings and focus on the observations for which we have data on the other ten sectors in a given year-country. Note also that our aggregate consumption data does not contain government services. Excluding government services from our regressions does not change the other estimates.}

We estimate a demand system analogous to the one used in our baseline estimation, where we use manufacturing as a reference sector\footnote{Note that in this case, the manufacturing sector is more narrowly defined than in the baseline estimation as it excludes mining and construction.}

\[
\log \left( \frac{L_{i,t}^c}{L_{m,t}^c} \right) = \xi_{im}^c + (1 - \sigma) \log \left( \frac{P_{i,t}^c}{P_{m,t}^c} \right) + (\epsilon_i - \epsilon_m) \log C_t^c + \nu_{i,t}^c, \tag{22}
\]

with \( i \) denoting any of our sectors and \( c \), a country index. Our panel estimates are reported in Table 8. The overall fit is good, with an \( R^2 \) above 0.9 in all regressions (this includes the country-sector fixed effect). Column (1) shows that we find an elasticity of substitution of 0.82 which is reasonably close to the 0.75 we found in our baseline, three-sector, estimation. We find that the smallest income elasticities correspond to mining and agriculture, while the highest correspond to service sectors, such as finance, insurance, real state and government services. Columns (2) and (3) show that the ranking of sectors in terms of their income elasticity is very similar when we estimate OECD and non-OECD countries separately.

4 Micro and Macro Estimates for the U.S.

This section analyzes the U.S. in more depth, for which we can obtain more detailed data. We perform two exercises. First, we estimate our demand system using household data from the Consumption Expenditure Survey. This allows us to use an instrumental variable approach and control for sectoral preference shocks using time-sector fixed effects. Second, we estimate the parameters of the utility function using data on aggregate time series for the United States. Building on the work of Herrendorf et al. (2013), we specify the utility function both over final expenditure and value added and analyze the robustness of the estimates to these alternative definitions of the utility function.

4.1 Micro Estimation: Consumer Expenditure Survey

In this section we use household expenditure data to estimate our demand system. We estimate our demand system using U.S. household quarterly consumption data for the period 1980-2006. We use Consumption Expenditure Survey data as constructed in Heathcote et al. (2010). We follow Heathcote et al. and focus on a sample of households with a present
household head aged between 25 and 60. We also use the same consumption categories, except that we separate food from the rest of non-durables consumption.\textsuperscript{44} We estimate the demand system using expenditure shares for each household on the left hand side. To control for household fixed characteristics, we estimate the demand system using household fixed effects,

\[
\log \left( \frac{\omega_{h,t}^i}{\omega_{h,nd,t}^i} \right) = (1 - \sigma) \log \left( \frac{p_{i,r,t}}{p_{nd,r,t}} \right) + (\epsilon_i - \epsilon_{nd}) \log C_t^h + \zeta_t^h + \zeta_{i,t} + \nu_{i,t}.
\]

(23)

The superscript $h$ denotes a household, and $nd$ denotes non-durables—which we use as reference in our regressions. Prices $p_{i,r,t}$ come from the corresponding sectoral CPI-Us of the BLS. They vary across regions $r$ conditional on expenditure category $i$ and time $t$. Aggregate consumption expenditure $C_t^h$ is deflated using a household specific CPI, as suggested by our theory.\textsuperscript{45} Household fixed effects are denoted by $\zeta_t^h$, while time-sector fixed-effects are denoted by $\zeta_{i,t}$. Note that this time-sector fixed effect absorbs sectoral preference shocks.

In an analogous manner to the cross-country panel, the identification comes from within household variation in total consumption expenditure over time. Note that since the BLS CPI-Us vary regionally, we can identify the price elasticity of consumption shares even after including time-sector fixed effects. Variation in the prices is arguably exogenous to households.\textsuperscript{46}

**Baseline Results** Table 4 reports the results of estimating (23) with and without including consumption durables, and with and without sector-year fixed effects. The estimates are very similar across all four specifications, suggesting that sector-specific demand shocks are not a significant source of bias in our demand system. The point estimate of the price elasticity is

\textsuperscript{44} Consumption measures are divided by the number of adult equivalents in the household. We use the categorization of Heathcote et al. (2010) for expenditures. The consumption categories in non-durables are: alcoholic beverages, tobacco, personal care, fuels, utilities and public services, public transportation, gasoline and motor oil, apparel, education, reading, health services and miscellaneous. Our data for services comes from entertainment expenditures. These includes among others fees for recreational lessons, TV and music related expenditures, pet-related expenditures, toys, games. Durables comprises vehicles (purchases/services derived from it and car maintenance and repair) and household equipment. Housing comprises the rents or imputed rents (if the dwelling is owned) as well as from “other dwellings” (primarily vacation homes). For each household we have a maximum of 4 observations (one per semester). The consumption data comes from the Family Characteristics and Income files except for years 1982 and 1983 for which the Detailed Expenditures files were used. See Heathcote et al. (2010) and Krueger and Perri (2006) for further discussion on the construction of the data set and its characteristics.

\textsuperscript{45} We note that if we deflated consumption expenditures using the aggregate CPI, we obtain similar estimates. When we include durables in our regressions, we use the aggregate CPI, as we lack information on the purchase timing used in the imputation method.

\textsuperscript{46} One additional possible concern is the fact that even after controlling for household fixed characteristics there is an unobserved and persistent shock driving both aggregate consumption expenditure and some particular consumption category, most likely durable goods and housing. Following Heathcote et al. (2010), we impute the flow services obtained from housing and vehicles, which should attenuate these concerns. We also report our estimates excluding durables and show that the estimated elasticities are very similar.
Table 4: Consumer Expenditure Survey, 1980-2006

<table>
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<th>(3)</th>
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<td>0.63</td>
<td>0.63</td>
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<td>(0.14)</td>
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<td>Food</td>
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<td>-0.49</td>
<td>-0.48</td>
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<td>(0.13)</td>
<td>(0.16)</td>
<td>(0.16)</td>
</tr>
<tr>
<td>Housing</td>
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<td>-0.31</td>
<td>-0.27</td>
<td>-0.26</td>
</tr>
<tr>
<td></td>
<td>(0.16)</td>
<td>(0.15)</td>
<td>(0.18)</td>
<td>(0.17)</td>
</tr>
<tr>
<td>Services</td>
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<td>0.62</td>
<td>0.57</td>
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<td></td>
<td>(0.24)</td>
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<td>0.93</td>
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<td>Time FE</td>
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<td>Y</td>
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<td>241470</td>
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</tbody>
</table>

Note: Std. Errors clustered at the household level. Elasticity estimates are relative to non-durables consumption. Data from Heathcote et al. (2010).

around 0.64, which is within the range of estimates we obtained in the cross-country panel (Table 1). With respect to the income elasticities, food has the lowest elasticity, followed by housing, services and we find the highest income elasticity for durables. As the purchase of durables is lumpy and most of our consumption data for durables imputes the service flow (see Heathcote et al., 2010), we take this latter estimate with a grain of salt.\(^{47}\)

**IV Strategy** One possible concern with the previous regression is that total consumption expenditure is an endogenous choice of the household that may be correlated with some omitted variable. To address this concern, we use the instrumental variable approach developed by Johnson et al. (2006). The instrument is based on the fact that the timing of the 2001 Federal income tax rebates was done as a function of the last digit of the recipient’s social security number. Thus, it was effectively random.\(^{48}\) Columns (1) and (2) in Table 5 report the OLS estimated as in our baseline estimation (23) for the period of interest (2001-2002). Note that the estimates are very similar to the whole sample. Columns (3) and (4) report IV estimates excluding and including durables. We instrument consumption expenditure with a dummy for whether or not a household received the rebate at a given point in time. We note that the estimates remain very similar to the baseline estimation. The income elasticity of

\(^{47}\)Consumption expenditure is known to be noisily measured in household survey data, so to the extent that household fixed effects do not correct for mis-measurement, we would expected some downward bias in the estimation of income elasticities.

\(^{48}\)Because the data requirements to construct detailed household expenditures in Herrendorf et al. (2013) are different than in Johnson et al. (2006), we can merge only 60% of the tax rebate data with our baseline data set. We thank Bart Hobijn for suggesting this instrument to us.
Table 5: Instrumental Variables Strategy
Consumption Expenditure Survey, 2001-2002

<table>
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<td>(\sigma)</td>
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<td>Food</td>
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<td>-0.43</td>
<td>-0.45</td>
<td>-0.44</td>
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<td>(0.09)</td>
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<td>Housing</td>
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<td>(0.12)</td>
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<td>(0.12)</td>
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<td>Services</td>
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<td>0.52</td>
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<td>(0.16)</td>
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First Stage

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<td></td>
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</tr>
<tr>
<td>Time FE</td>
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<td>Y</td>
<td>Y</td>
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<tr>
<td>Observations</td>
<td>30378</td>
<td>30378</td>
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</tbody>
</table>

Note: Std. Errors clustered at the household level. Elasticity estimates are relative to non-durables consumption. All estimates contain household and time-sector fixed effects. Data from Heathcote et al. (2010) and Johnson et al. (2006).

non-durables decreases somewhat after instrumenting but the ranking of income elasticities remains the same and the magnitudes are also very similar.\(^{49}\)

**Quartiles and Time Split Estimation** A key property of the nonhomothetic CES preferences is that the income elasticity \(\varepsilon_i\) is constant at different income levels. As argued in the introduction, this property is supported by the evidence. Next, we complement the evidence provided by Aguiar and Bils (forthcoming) for the CEX. We estimate the baseline regression (23) by income quartiles for the period 1980-2006. Columns (1) to (4) in Table 7 report the elasticity estimates for the first to the fourth quartiles of the CEX. We find that the value of all elasticities is quite stable at different income levels. Likewise, splitting the sample between the pre-1993 and post-1993 periods and estimating the elasticities separately also yields very similar estimates.

\(^{49}\)If we use the value of the rebate rather than an indicator, we obtain very similar results. However, as the magnitude of the rebate was not random, we prefer to use only the indicator.
4.2 Macro Estimation: Value Added and Expenditure Measures

We conclude the empirical exploration by estimating our demand system with aggregate U.S. consumption time series. We use the data from the Bureau of Economic Analysis as constructed by Herrendorf et al. (2013) for agriculture, manufacturing and services. Aggregate consumption data are decomposed in two different ways: expenditure and value added. Consumption expenditure classifies sectors according to final good expenditure. Value added decomposes each dollar of final expenditure into the share of value added attributable to agriculture, manufacturing and services using U.S. input-output tables. For example, purchases of food from supermarkets is included in agriculture in the final expenditure computation, while it is broken down into food (agriculture), food processing (manufacturing) and distribution (services) when using the value added formulation. Thus, the final expenditure and value added representation of consumption are two alternative classifications of the same underlying data.

Herrendorf et al. (2013) estimate a Stone-Geary utility demand system specified over expenditure and value added time series. They find that this distinction yields quantitatively very different results. Using value added measures, the elasticity of substitution across sectors is not statistically different from 0. Buera and Kaboski (2009) report a similar finding for the period 1870-2000. When estimating the model with final expenditure they find that the elasticity of substitution is around 0.85. Herrendorf et al. (2013) convincingly argue that the elasticity of substitution should be greater when using expenditure measures because they embed inputs from the three sectors. However, they do not provide any justification for aggregate consumption being a Leontief aggregator of sectoral outputs.

We re-do the exercise of Herrendorf et al. (2013) using the nonhomothetic CES demand system rather than Stone-Geary, estimating

\[
\log \left( \frac{\omega_{at}}{\omega_{mt}} \right) = \zeta_{am} + (1 - \sigma) \log \left( \frac{p_{at}}{p_{mt}} \right) + (\epsilon_a - \epsilon_m) \log C_t + \nu_{amt}, \tag{24}
\]

\[
\log \left( \frac{\omega_{st}}{\omega_{mt}} \right) = \zeta_{sm} + (1 - \sigma) \log \left( \frac{p_{st}}{p_{mt}} \right) + (\epsilon_s - \epsilon_m) \log C_t + \nu_{smt}, \tag{25}
\]

where \(\omega_{it}\) denotes consumption expenditure or value added in sector \(i\) at time \(t\).\(^{50}\) Our estimates are reported in table 6. Our point estimate of the elasticity of substitution for the expenditure data is 0.88 which is very close to HRV’s estimate of 0.85. As in Herrendorf et al. (2013), we find that the elasticity of substitution is larger for expenditure data than value added. However, our estimated elasticity for value added measures is 0.57 with a standard deviation of 0.1. Thus, the preferences implied by our estimate differ significantly from the Leontief specification found using the Stone-Geary setting.

\(^{50}\)As we are using relative consumption shares rather than employment, there is no need to control for international trade in this regression because it is subsumed in consumption expenditure.
Table 6: Consumption Expenditure and Value Added for the U.S.

<table>
<thead>
<tr>
<th></th>
<th>σ</th>
<th>$\varepsilon_a - \varepsilon_m$</th>
<th>$\varepsilon_s - \varepsilon_m$</th>
<th>Obs</th>
<th>$R^2_{am}$</th>
<th>$R^2_{sm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value Added</td>
<td>0.57</td>
<td>-0.63</td>
<td>0.62</td>
<td>63</td>
<td>0.77</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.12)</td>
<td>(0.06)</td>
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<tr>
<td>Expenditure</td>
<td>0.88</td>
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<td>0.91</td>
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<tr>
<td></td>
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<td>(0.03)</td>
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</tbody>
</table>

The intuition for this difference in the estimate of the price elasticity is as follows. Expenditure shares in services measured in value-added terms raise at a faster rate than the relative price of services. The Stone-Geary demand system imposes that the income effects become less important as aggregate consumption grows. This implies that the estimation has to load the late increase in service expenditure to increases in the relative price of services. Thus, as the relative prices of services grow at a slower rate than value added, the estimation selects the minimal price elasticity to maximize the explanatory power of relative prices. In contrast, the nonhomothetic CES does not impose declining income effects. As a result, both income and price effects help account for the secular increase in expenditure shares in services and the estimation does not need to select Leontief demand system.

Our estimates of the income elasticities are very similar using measures of sectoral activity. In fact, the point estimates are identical for $\varepsilon_a - \varepsilon_m$, with an elasticity of $-0.63$. The estimates for $\varepsilon_s - \varepsilon_m$ are 0.62 for value added data and 0.55 for expenditure data and we cannot reject the null that they are statistically identical. Thus, our estimates imply that the role for nonhomotheticities is very similar regardless of whether utility is specified in terms of value added or expenditure.

**Forecasting U.S. Expenditure Shares** What do the estimated price and income elasticities imply for the evolution of the sectoral composition of the U.S. economy? Assuming that relative prices and aggregate consumption grow at the average rate of the postwar period, we forecast expenditure shares in 2025 and 2050 using the estimated price and income elasticities. The projected evolution of expenditure shares is depicted in figure D.1 in the online appendix. In the last year of our sample (2010) expenditure shares were 6% in agriculture, 20.5% in manufacturing and 74.2% in services. The projected shares in 2025 are 3.8% in agriculture, 18.1% manufacturing and 78.1% in services. In 2050, our projected shares are 2.1%, 14.5% and 83.4% in agriculture, manufacturing and services respectively. This exercise suggests that the process of structural transformation in the U.S. may continue in the next decades, with manufacturing and agriculture still accounting for a non-negligible part of the economy.

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51 We thank Paco Buera and Alex Monge-Naranjo for suggesting this exercise.
5 Equilibrium Dynamics and Asymptotics

To complete the presentation of our theoretical framework, in this section we characterize the dynamic and asymptotic properties of the economy introduced in Section 2. We choose a continuous time version of the model to simplify the derivations and fully characterize the equilibrium, showing conditions sufficient for its existence and uniqueness. We show that the economy asymptotically converges to a path of constant real consumption growth. In addition, we introduce a generalization of the baseline utility function that allows us to establish a precise connection to the empirical measures of the growth rate of real aggregate consumption, as for instance, defined in the Penn World Table (Feenstra et al., 2013).

Notation Henceforth, we denote by $\eta^f_x$ the elasticity of a (potentially multivariate) function $f$ with respect to its argument $x$, i.e., $\eta^f_x \equiv \partial \log f / \partial \log x$. When function $f$ is defined over a single variable $x$, we simply refer to the elasticity function as $\eta_f$. Moreover, we use bold face notation to indicate a collection of sectoral variables. For instance, $p(t)$ stands for the set of sectoral prices at time $t$, that is, $\{p_i(t)\}_{i=1}^I$.

5.1 Household Preferences and Demand

Consider a unit mass of households with identical preferences over a stream of real consumption per capita $[c(t)]_{t=0}^\infty$, defined as

$$U(0) \equiv \int_0^\infty e^{-(\rho-n)t} u(c(t)) \, dt,$$

(26)

where $u(\cdot)$ is the instantaneous utility function, $\rho > 0$ is the discount rate, and $n \geq 0$ denotes population growth. We make the standard assumption that $n < \rho$, and that the instantaneous utility function $u$ is asymptotically isoelastic, that is, $\lim_{c \to \infty} \eta_u \equiv cu'/u = 1 - \theta$ with $\theta > 0$. Furthermore, we henceforth focus attention to the empirically relevant case where the elasticity of substitution is not greater than unity, i.e., $\sigma \in (0, 1]$.

Per capita real consumption $c(t)$ aggregates consumption of a bundle $c(t) = \{c_i(t)\}_{i=1}^I$ of goods according to generalized nonhomothetic CES preferences defined implicitly through

$$\sum_{i=1}^I \left[ g(c(t))^{-\varepsilon_i} c_i(t) \right]^{\frac{\sigma-1}{\sigma}} = 1,$$

(27)

where each consumption good is characterized by an income elasticity parameter $\varepsilon_i > 0$ and $\sigma \in (0, 1)$ denotes the elasticity of substitution. Function $g(\cdot)$ is a positive-valued and monotonic function and will be characterized in what follows.\(^{52}\)

\(^{52}\)To find the basic isoelastic nonhomothetic CES aggregator presented in Section 2, one can simply replace $g(c) \rightarrow c$ and $\varepsilon_i \rightarrow \frac{\varepsilon_i - \sigma}{1 - \sigma}$.
When consumers face prices \( \mathbf{p} = \{p_i\}_{i=1}^I \), the expenditure function corresponding to the preferences above is be given by (see derivations in Appendix A)

\[
e(c; \mathbf{p}) \equiv \left( \sum_{i=1}^I [g(c)^{\varepsilon_i} p_i]^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.
\] (28)

Correspondingly, the consumption expenditure share of good \( i \) is given by

\[
\omega_i(c; \mathbf{p}) \equiv \frac{p_i c_i}{e(c; \mathbf{p})} = \left( \frac{g(c)^{\varepsilon_i} p_i}{e(c; \mathbf{p})} \right)^{1-\sigma}.
\] (29)

For any variable that varies across sectors, e.g., income elasticity parameters \( \varepsilon_i \), we can define expenditure-weighted averages. For instance, we define the economy-wide average income elasticity parameter as:

\[
\overline{\varepsilon}(c; \mathbf{p}) \equiv \sum_{i=1}^I \omega_i(c; \mathbf{p}) \varepsilon_i.
\] (30)

In the remainder of the paper, we will drop the dependence of expenditure-weighted average functions on the prices whenever the corresponding prices are clear from the context.

The real consumption aggregator defined by equation (27) characterizes the consumer’s instantaneous utility function up to a monotonic transformation implied by \( g(\cdot) \). In order for our concept of real aggregate consumption \( c(t) \) to correspond to empirical measures of real income, choice of \( g(\cdot) \) should express consumer utility in terms of equivalent expenditure in constant prices.\(^{53}\) Accordingly, we choose function \( g(\cdot) \) in such a way that aggregate consumption per capita \( c \) is expressed in terms of constant prices \( \mathbf{q} \), i.e., in real terms. We define \( g(\cdot) \) implicitly through:

\[
c \equiv \left( \sum_{i=1}^I [g(c)^{\varepsilon_i} q_i]^{1-\sigma} \right)^{\frac{1}{1-\sigma}},
\] (31)

that is, we let \( g \equiv e^{-1}(\cdot; \mathbf{q}) \). With this definition, Equation (27) defines \( c(t) \) as an aggregator of a bundle \( \{c_i(t)\}_{i=1}^I \) expressed in terms of the cost of an optimal bundle when consumers face given constant prices \( \mathbf{q} \).

The next proposition, characterizes the solution to the household problem.

**Proposition 2.** Consider the household’s problem of maximizing (26) where the aggregator

\(^{53}\)This point becomes critical only when we aim to characterize the asymptotic behavior of our economy. Measures of real consumption constructed based on chained Fisher indices provide a local approximation of the utility for any smooth utility function. Therefore, in so far as the local approximation holds, the choice of \( g(\cdot) \) does not bear on our empirical results in Section 3. We refer the readers to the online Appendix for more details on this point.
is defined by Equations (27) and (31), subject to the flow budget constraint

\[ \dot{a}(t) = w(t) + [r(t) - n]a(t) - e(c(t); p(t)), \tag{32} \]

and the No-Ponzi condition

\[ \lim_{t \to \infty} a(t) \exp \left( - \int_0^t (r(t') - n) \, dt' \right) \geq 0, \tag{33} \]

for some path of wage \( w(\cdot) \), interest rate \( r(\cdot) \), and sectoral prices \( p(\cdot) \). Then, the Euler Equation

\[ \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \frac{\overline{p}(t)}{\overline{p}(t)} - (1 - \sigma) \text{Cov}(\frac{\overline{c}(t)}{\overline{c}(t)}; \hat{\overline{p}}(t))}{\eta_{\epsilon_c}(c(t); t) - \eta_{\epsilon_u}(c(t))}, \tag{34} \]

along with the transversality condition

\[ \lim_{t \to \infty} a(t) e^{-(\rho-n)t} \frac{c(t)}{e(c(t); p(t))} \frac{\overline{\epsilon}(c; q)}{\overline{\epsilon}(c; p(t))} = 0, \tag{35} \]

characterize necessary conditions for any paths of consumption and assets to be the solution to the household problem. In the Euler Equation above \( \overline{p}/\overline{p} \) and \( \overline{\epsilon} \) are expenditure-weighted sectoral averages at price \( p(t) \), and the elasticity of marginal expenditure is given by

\[ \eta_{\epsilon_c}(c(t); t) = \left( \frac{\overline{\epsilon}(c; p(t))}{\overline{\epsilon}(c; q)} - 1 \right) \left[ 1 + (1 - \sigma) \left( \frac{\text{Var}(\epsilon; c, p(t))}{\overline{\epsilon}(c; p(t))^2} \right) \right] \]

\[ + (1 - \sigma) \left( \frac{\text{Var}(\epsilon; c, p(t))}{\overline{\epsilon}(c; p(t))^2} - \frac{\text{Var}(\epsilon; c, q)}{\overline{\epsilon}(c; q)^2} \right). \tag{36} \]

Furthermore, let \( \Delta \equiv \epsilon_{\text{max}}/\epsilon_{\text{min}} \) be the ratio of the largest to the smallest sectoral elasticity parameters corresponding to the household preference’s aggregator in Equation (27), and suppose the elasticity of intertemporal substitution is bounded above by \( 1/\theta \), that is, \( \eta_{\epsilon_u}(c) < -\theta \) for all \( c \). If the following inequality is satisfied

\[ \theta > (\Delta - 1) \left[ \frac{1}{\Delta} + \frac{1 - \sigma}{4} (\Delta - 1) \right], \tag{37} \]

then the household problem has a unique solution, fully characterized by the Euler equation and the transversality conditions above.

The proposition above establishes that under mild conditions on the concavity of instantaneous utility function \( u(\cdot) \), the household problem has a unique optimum that can be found by solving an Euler equation. First, note that if all income elasticity parameters are the same, then the real consumption elasticity of the marginal expenditure in Equation (36) becomes
zero and the Euler equation reduces to its familiar form of standard homothetic preferences with heterogenous technological growth growth rates across consumption sectors (see, e.g., Ngai and Pissarides, 2007).

When income elasticity parameters are heterogenous across sectors, we have two distinct modifications to the standard Euler equation. To unpack these two different modifications, note that \( \left( \eta^c_e - \eta_u' \right) \frac{\dot{c}}{c} \) is the rate of decline in the marginal utility of consumer spending (expenditure) at constant prices, which in turn depends on the concavity of the instantaneous utility function and the expenditure function. Accordingly, the numerator of Equation equation (34) describes the growth rate of the marginal utility of consumer spending, while the denominator expresses the ratio between growth rates of consumer spending and real aggregate consumption in terms of base prices \( q \).

First, due to nonhomotheticity, the expenditure function is a nonlinear function of real consumption. Therefore, the denominator in the right hand side of the Euler equation (34) includes an adjustment term that reflects the convexity of the expenditure function. The adjustment effectively increases the concavity of the instantaneous utility function by the degree of the convexity of the expenditure function. Equation (36) shows that this nonhomotheticity adjustment depends on the mean and the variance of the income elasticity parameters with distributions implied by expenditure shares, under current and base prices. The larger the mean and the variance of the income elasticity parameters under current prices relative to base prices, the larger is this adjustment. Intuitively, the components of consumer’s expenditure corresponding to goods with higher income elasticity have higher convexity in real aggregate consumption. When the income of a typical consumer grows, she spends a larger share of her income on more income elastic goods. As a result, the expenditure function as a whole becomes a more convex function of real aggregate consumption.

The last term on the numerator of Euler equation (34) accounts for the interaction of income elasticity and growth rates of sectoral prices. If the rates of growth in sectoral prices are positively correlated with income elasticity parameters, when (real) income grows consumers have to shift a larger share of their expenditure toward more expensive goods. This effectively reduces the growth rate of their real consumption.

We emphasize that Equations (34), (35), (36), and (37) are all invariant to common scaling of sectoral elasticity parameters \( \varepsilon_i \)’s. Therefore, our choice of cardinality for function \( g(\cdot) \) in Equation (31) highlights (and pins down) the one degree of freedom that we face in our choice of sectoral elasticity parameters.

5.2 Production

The production side of our model includes inter-sector heterogeneity in rates of technological progress, analyzed first by Ngai and Pissarides (2007), as well as inter-sector heterogeneity in
factor intensities, studied first in a two-sector setting by Acemoglu and Guerrieri (2008). We show that our growth model remains fully tractable when we incorporate both these supply side channels.

Households inelastically supply labor, \( L(t) \equiv L(0) e^{nt} \). Capital is accumulated using investment goods produced by sector \( i = 0 \),

\[
\dot{K}(t) = Y_0(t) - \delta K(t).
\]  

Labor and capital are combined by producers of consumption good sectors \( i \in \{1, \cdots, I\} \) to produce output using a Cobb-Douglas technology

\[
Y_i(t) = A_i(t) L_i(t)^{1-\alpha_i} K_i(t)^{\alpha_i}, \quad \text{for } i \in \{0, \cdots, I\},
\]  

where the production function in sector \( i \) is characterized with sector-specific capital intensity \( \alpha_i \in (0,1) \). The Hicks-neutral technological progress \( A_i(t) \) grows exogenously with the (potentially time varying) rate \( \gamma_i(t) \) that asymptotically becomes constant, i.e., rate \( \lim_{t \to \infty} \dot{A}_i(t) / A_i(t) = \gamma_i > 0 \) for all sectors \( i \).

Since we assume competitive labor and capital markets, the marginal revenue product of labor and capital have to equate their respective prices, that is,

\[
w(t) = \frac{(1-\alpha_i) p_i(t) Y_i(t)}{L_i(t)}, \quad R(t) = \frac{\alpha_i p_i(t) Y_i(t)}{K_i(t)}.
\]  

We define sectoral capital-labor ratios as

\[
\kappa_i(t) \equiv \frac{K_i(t)}{L_i(t)} = \frac{\alpha_i}{1-\alpha_i} \frac{w(t)}{R(t)}, \quad \text{for } i \in \{0, \cdots, I\}.
\]  

Equation (42) shows that sectoral capital-labor ratios are proportional to each other, and to the relative price of labor to capital.

As before, we normalize the price of the investment sector in each period to unity, \( p_0(t) \equiv 1 \). We find the prices of sectoral consumption goods by equalizing marginal revenue products of capital from Equation (41)

\[
p_i(t) = \frac{A_0(t)}{A_i(t)} \cdot \frac{w(t)}{R(t)} = \left( \frac{\alpha_0}{\alpha_i} \right)^{\alpha_i} \left( \frac{1-\alpha_0}{1-\alpha_i} \right)^{1-\alpha_i} A_0(t) \frac{(w(t))^{\alpha_0-\alpha_i}}{(R(t))^{\alpha_i}},
\]  

where in the second equality we have substituted for relative price of inputs from Equation
\( \kappa_0 \) denotes the capital-labor ratio in the investment sector. Equation (43) shows that consumption good prices depend only on sectoral TFPs and the capital-labor ratio in the investment sector. As expected, goods produced by sectors with lower TFPs are more expensive. The dependence of prices on capital-labor ratio in the investment sector in Equation (43) is a proxy for their dependence on the rental price of capital. Goods produced by sectors with higher capital intensities become more expensive as capital-labor ratios rise and the rental price of capital falls.

Equation (43) illustrates how both supply-side forces driving structural change appear through sectoral prices. A higher rate of technological progress in sector \( i \) (relative to the investment sector) is a force lowering the price in this sector, the mechanism featured in the model of Ngai and Pissarides (2007). As the economy accumulates capital, the capital-labor ratio grows proportionally in all sectors. A higher capital intensity in sector \( i \) (relative to the investment sector) is an alternative force lowering the price in this sector, one formalized in the model by Acemoglu and Guerrieri (2008).

5.3 Competitive Equilibrium and Equilibrium Dynamics

Since we assume a closed economy, market clearing implies for all \( t \geq 0 \)

\[
L(t) = \sum_{i=0}^{I} L_i(t),
\]

\[
K(t) = \sum_{i=0}^{I} K_i(t) = a(t)L(t),
\]

\[
p_i(t)Y_i(t) = \omega_i(t)e(t)L(t),
\]

where \( \omega_i(t) = \omega_i(c(t); p(t)) \) and \( e(t) = e(c(t); p(t)) \) are the expenditure share and expenditure functions as defined in Equations (28) and (29). Equation (46) connects the production allocations to the nonhomothetic CES demand system. In particular, this equation characterizes the total sectoral outputs that, together with prices of labor and capital, pin down equilibrium sectoral allocations of labor and capital from Equations (40) and (41). Moreover, the total value of output at time \( t \) is given by

\[
Y(t) \equiv \sum_{i=0}^{I} p_i(t)Y_i(t),
\]

\[
= Y_0(t) + L(t)e(t).
\]

An equilibrium path in our economy consists of the path \([c(\cdot), a(\cdot), \omega(\cdot), Y(\cdot), K(\cdot), L(\cdot)]_{t=0}^{\infty}\) of per capita real consumption and assets, sectoral shares in consumption expenditure, and sectoral outputs and allocations of capital and employment, along with the path \([r(\cdot), R(\cdot), w(\cdot), p(\cdot)]_{t=0}^{\infty}\).
of real interest rate, rental price of capital, wage, and sectoral output prices satisfying Equa-
tions (29), (34), (35), (38), (39), (44), (45), (46) and $r(t) = R(t) - \delta$ for all $t \geq 0$.

The next proposition characterizes the asymptotic properties of the competitive equilibrium of our economy.

**Proposition 3.** Let constant $\gamma^*$ be defined as

$$
\gamma^* \equiv \min_{i \in I/\{0\}} \frac{(1 - \alpha_0) \gamma_i + \alpha_i \gamma_0}{(1 - \alpha_0) \varepsilon_i / \varepsilon_{\max}},
$$

where $\varepsilon_{\max}$ is the maximum among all income elasticity parameters. Assume that condition (37) is satisfied (which ensures that the instantaneous utility function defined in Equations (26), (31), and (27) is concave in real consumption $c$), and that

$$
\rho > n + (1 - \theta) \gamma^*.
$$

Then, there exists a unique competitive equilibrium path for our economy, along which per capita real consumption asymptotically grows at a constant rate

$$
\lim_{t \to \infty} \frac{\dot{c}(t)}{c(t)} = \gamma^*.
$$

Let $I^*$ be the set of consumption sectors achieving the minimum in Equation (47). Asymptotically, the share of sectors belonging to this set in employment and consumption expenditure converges to a time-constant distribution. The employment and consumption expenditure shares of any sector $i$ outside the set $I^*$ vanishes at a rate

$$
\lim_{t \to \infty} \frac{\dot{l}_i(t)}{l_i(t)} = -\frac{\alpha_i}{1 - \alpha_0} \frac{\gamma_i - \varepsilon_i}{\varepsilon_{\max}} \gamma^*.
$$

Proposition 3 states the key asymptotic properties of our economy. First, as with standard growth models, the equilibrium is unique and the rate of growth of real consumption and real interest rate converge to constant values. More notably, the economy asymptotically features reallocation of labor across consumption sectors: while some sectors converge to comprising constant shares, others shrink at a constant rate. Crucially, the rate of growth of real consumption and the set of sectors that do not vanish asymptotically are determined through a combination two forces: 1) the supply-side substitution forces, as captured by the sectoral rates of technical growth and capital intensities in the numerator of Equation (47), and 2) the demand-side income forces, as captured by the preference elasticity parameter in the denominator of the same equation. This relation generalizes and encompasses the results of both Ngai and Pissarides (2007) and Acemoglu and Guerrieri (2008) and unifies them with our account of long-run demand nonhomotheticity.
Here, we offer some basic intuition for the results and leave a detailed proof of the proposition to Appendix B. Define the productivity-adjusted capital-labor ratio in the investment sector as

\[ \tilde{\kappa}_0(t) \equiv \frac{K_0(t)}{A_0(t)^{1-\alpha_0} L_0(t)}. \] (51)

This variable has a one-one relationship with real interest rate through \( r(t) = \alpha_0 \tilde{\kappa}_0(t)^{\alpha_0 - 1} - \delta \) and, as we will explain below, constitutes the main state variable of the economy. As we expect from a path of asymptotically constant growth, this variable has to converge to a constant. Now, substituting the normalized capital-labor ratio of the investment sector in Equation (43), we observe that the growth rates of consumption good prices take the form

\[ \frac{p_i(t)}{p_i(0)} = \left( \frac{\tilde{\kappa}_0(t)}{\tilde{\kappa}_0(0)} \right)^{\alpha_0 - \alpha_i} e^{-\left( \gamma_i - \frac{1-\alpha_i}{1-\alpha_0} \gamma_0 \right) t}. \] (52)

Therefore, if the rental price of capital remains constant, price of consumption good \( i \) falls at the rate \( (\gamma_i - \gamma_0) + \frac{\alpha_i - \alpha_0}{1-\alpha_0} \gamma_0 \), where the first terms in the parantheses captures technical growth in sector \( i \) and the second term captures the extent to which technical growth in the investment sector differentially impacts growth in sector \( i \) through differences in capital intensity.

Since households invest optimally, both investment and household expenditure comprise non-negligible values of the total value as the economy grows. Therefore, total consumption expenditure of households asymptotically grows at the same rate as the rate of growth of the investment sector. Combining these insights and Equations (52) and (29) yields

\[ \lim_{t \to \infty} \frac{\dot{\omega}_i(t)}{\omega_i(t)} = (1 - \sigma) \left[ \eta g \tilde{\omega}_i^- \gamma^* - \left( \gamma_i + \frac{\alpha_i}{1-\alpha_0} \gamma_0 \right) \right]. \] (53)

Since all shares are bounded above by one, the rate of growth of sectoral shares cannot remain asymptotically positive. Therefore, for the equilibrium to be well defined we need to ensure that these rates of growth do not asymptotically exceed 0. This simple rule in fact pins down the asymptotic rate of growth of real consumption per capita \( \gamma^* \) in Equation (47), which in general will be different from the rate of growth of per capita consumption expenditure \( \frac{\gamma_0}{1-\alpha_0} \).

While Proposition 3 provides a simple account of the asymptotic properties of our model, the dynamics of the equilibrium path in this economy may generally be more complex. However, the model still remains fully tractable and the dynamic equations can be written in terms of a state variable \( \tilde{\kappa}_0 \), investment-sector capital-labor ratio, and a control variable \( c \), per capita real consumption. Appendix C provides the full derivation of dynamic equations characterizing the competitive equilibrium everywhere along an path with an initial condition \((c(0), \tilde{\kappa}_0(0))\). A system of two linear equations in \( \frac{\dot{c}(t)}{c(t)} \) and \( \frac{\dot{\kappa}_0(t)}{\kappa_0(t)} \) determines rates of growth of the two variables for \((c(t), \tilde{\kappa}_0(t))\) at time \( t \). Our unified model includes heterogeneity across
sectors along three different dimensions, i.e., income elasticity of demand for sectoral outputs, capital intensity, and rate of technical growth, the interactions between all these sources of heterogeneity appear in the dynamic equations.

Here, for the sake of brevity, we present the dynamic equations for the special case where \( \alpha_i \equiv \alpha \) for all sectors \( i \). This case parallels the workhorse model analyzed by Buera and Kaboski (2009) and Herrendorf et al. (2013) including two competing forces: income non-homotheticity and heterogeneous rates of technological growth. When capital intensities are identical, capital-labor ratios equalize across all sectors in equilibrium and \( \tilde{\kappa}_0 \) equals the economy-wide capital-labor ratio. Dropping the subscript 0 to reflect this fact, the dynamics of equilibrium paths take the following form:

\[
\begin{align*}
\dot{c} &= \frac{\alpha \tilde{\kappa}^{\alpha-1} - (\delta + \rho) + \tilde{\gamma} (1 + (1 - \sigma) \rho_{\varepsilon, \gamma}) - \gamma_0}{-\eta \omega - 1 + (\tilde{\varepsilon} / \tilde{\sigma} - 1) \left( 1 + (1 - \sigma) \text{Var} (\tilde{\varepsilon}) \right) + (1 - \sigma) \left( \text{Var} (\tilde{\varepsilon}) - \text{Var}' (\tilde{\varepsilon}) \right)}, \\
\dot{\tilde{\kappa}} &= \tilde{\kappa}^{\alpha - 1} - \tilde{\varepsilon} \tilde{\kappa} - \left( n + \delta + \frac{\gamma_0}{1 - \alpha_0} \right) \tilde{\kappa},
\end{align*}
\]

where \( \tilde{\kappa} \) denotes the normalized capital-labor ratio (which is the same across all sectors), \( \tilde{\varepsilon} \), \( \text{Var}(\cdot) \), and \( \rho_{\cdot, \cdot} \) indicate mean, variance, and correlation coefficient of sectoral variables with distribution implied by expenditure shares under current prices, while \( \tilde{\varepsilon}' \) and \( \text{Var}'(\cdot) \) denote mean and variance of sectoral variables with distribution implied by expenditure shares under base prices.

6 Conclusion

This paper presents a tractable model of structural transformation that accommodates both long-run demand and supply drivers of structural change. Our main contribution is to introduce the nonhomothetic CES utility function to growth theory. These preferences generate nonhomothetic Engel curves at any level of development, which are in line with the empirical evidence that we have from both rich and developing countries. Moreover, for this class of preferences, price and income elasticities are independent (a feature of implicitly additive utility functions) and they can be used for an arbitrary number of sectors. We argue that these are desirable theoretical and empirical properties.

In contrast to generalized Stone-Geary utility functions, nonhomothetic CES utility functions do not assume that preferences become asymptotically homothetic as income grows. In contrast to PIGL preferences, recently employed in a growth model by Boppart (2013), nonhomothetic CES utility functions accommodate an arbitrary number of sectors with heterogeneous income elasticities. In contrast to both of these models, our demand system does not assume specific functional interrelations between elasticities of income and substitution. The latter property particularly makes our specification suitable for the exercise of separating
the contributions of income and prices to changes in relative sectoral demand. Relative to models with differential trends in relative prices and homothetic constant-elasticity preferences, our model has the advantage that can accommodate trends in both real and nominal measures.

From a quantitative stand-point, we show that our model succeeds in explaining sectoral reallocation patterns across countries, in spite of the parsimony of our approach (we only make use of three elasticity parameters). We estimate our model applied to three sectors (agriculture, manufacturing and services) using a panel of 25 countries for the postwar period. The sample covers countries with very different levels and trajectories of development. The model fit captures the evolution of these three sectors of the economy. In particular, it explains the hump-shaped evolution for the manufacturing sector in all cases where this pattern appears in the data.

To conclude, we believe that the proposed preferences provide a tractable departure from homothetic preferences. We think that they can be used in many applied general equilibrium settings that currently use homothetic constant elasticity of substitution as their workhorse model, such as international trade. These preferences can be also combined with productions functions without constant shares to study skilled-biased technological change or capital deepening.
References


A Nonhomothetic CES Preferences

In this section of the appendix, we provide an overview of the properties of the general family of Nonhomothetic CES preferences. We first introduce the general family of nonhomothetic CES preferences in Section A.1. We then specialize them to the case of isoelastic nonhomothetic CES functions in Section A.2 and contrast the latter with Stone-Geary and PIGL preferences in Section A.3.

These preferences have a number of distinctive features, extensively studied in an early literature on theoretical foundations of preferences and production functions. Sato (1975) derived a general family of CES functions as the solution to a partial differential equation that imposes the constancy of elasticity of substitution. This family includes standard homothetic CES functions as well as two classes of separable and nonseparable nonhomothetic functions. Hanoch (1975) showed that additivity of the direct or indirect utility (or production) function results in price and income effects that are nontrivially dependent on each other. He then introduced implicit additivity and derived in a family of functions where the income elasticity of demand is not fully dependent on the elasticity of substitution. Our nonhomothetic CES functions correspond to the nonseparable class of functions in the sense of Sato (1975), which also satisfy the condition of implicit additivity in the sense of Hanoch (1975).

Finally, Blackorby and Russell (1981) have proved an additional property that is unique to this class of functions. In general, different generalizations of the elasticity of substitution to cases involving more than two variables, e.g., the Allen-Uzawa definition or the Morishima definition, are distinct from each other. However, for the class of nonhomothetic CES functions they become identical and elasticity of substitution can be uniquely defined similar to the case of two-variable functions.

A.1 General Nonhomothetic CES Preferences

Consider preferences over a bundle $C = \{C_1, C_2, \cdots, C_I\}$ of goods defined through an implicit utility function:

$$
\sum_{i=1}^{I} \Omega_{i}^{\frac{1}{\sigma}} \left( \frac{C_i}{g_i(U)} \right)^{\frac{\sigma-1}{\sigma}} = 1, \tag{A.1}
$$

where functions $g_i$'s are differentiable in $U$ and $\sigma \neq 1$ and $\sigma > 0$. Standard CES preferences are a specific example of Equation (A.1) with $g_i(U) = U$ for all $i$'s. These preferences were first introduced, seemingly independently, by Sato (1975) and Hanoch (1975) who each characterize

54For the case of $\sigma = 1$, the preferences are simply defined according to $\sum_i \Omega_i \log \left( \frac{C_i}{g_i(U)} \right) = 1$. 

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different properties of these functions. Here, we state and briefly prove some of the relevant results to provide a self-contained exposition of our theory in this paper.

**Lemma 4.** If $\sigma > 0$ and functions $g_i(\cdot)$ are monotonically increasing for all $i$, the function $U(C)$ defined in Equation (A.1) is monotonically increasing and quasiconcave for all $C \gg 0$.

**Proof.** Establishing monotonicity is straightforward. To establish quasiconcavity, assume to the contrary that there exists two bundles of $C'$ and $C''$ and their corresponding utility values $U'$ and $U''$, such that $U \equiv U(\alpha C' + (1 - \alpha)C'')$ is strictly smaller than both $U'$ and $U''$. We then have:

$$
1 = \sum_i \Omega_i^{1/\sigma} \left( \alpha \frac{C_i'}{g_i(U')} + (1 - \alpha) \frac{C_i''}{g_i(U'')} \right)^{\frac{\sigma-1}{\sigma}},
$$

$$
> \sum_i \Omega_i^{1/\sigma} \left( \alpha \frac{C_i'}{g_i(U')} + (1 - \alpha) \frac{C_i''}{g_i(U'')} \right)^{\frac{\sigma-1}{\sigma}},
$$

$$
\geq \alpha \sum_i \Omega_i^{1/\sigma} \left( \frac{C_i'}{g_i(U')} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \alpha) \sum_i \Omega_i^{1/\sigma} \left( \frac{C_i''}{g_i(U'')} \right)^{\frac{\sigma-1}{\sigma}},
$$

where in the second inequality we have used monotonicity of the $g_i$’s and in the third we have used Jensen’s inequality and the assumption that $\sigma > 0$. Since the last line equals 1 from the definition of the nonhomothetic CES functions valued at $U'$ and $U''$, we arrive at a contradiction. ■

Henceforth, we assume the conditions in Lemma 4 are satisfied. The next lemma characterizes the demand for general nonhomothetic CES preferences and provides the solution to the expenditure minimization problem.

**Lemma 5.** Consider any bundle of goods that maximizes the utility function defined in Equation (A.1) subject to the budget constraint $\sum_i p_i C_i \leq E$. For each good $i$, the real consumption satisfies:

$$
C_i = \Omega_i \left( \frac{p_i}{E} \right)^{-\sigma} g_i(U)^{1-\sigma},
$$

and the share in consumption expenditure satisfies:

$$
\omega_i \equiv \frac{p_i C_i}{E} = \Omega_i \left( \frac{C_i}{g_i(U)} \right)^{\frac{\sigma-1}{\sigma}} = \Omega_i \left[ g_i(U) \left( \frac{p_i}{E} \right) \right]^{1-\sigma}.
$$

**Proof.** Let $\lambda$ and $\rho$ denote the Lagrange multipliers on the budget constraint and constraint (A.1), respectively:

$$
\mathcal{L} = U + \rho \left( 1 - \sum_i \Omega_i \left( \frac{C_i}{g_i(U)} \right)^{\frac{\sigma-1}{\sigma}} \right) + \lambda \left( E - \sum_i p_i C_i \right).
$$

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The FOCs with respect to $C_i$ yields:

$$\rho \frac{1 - \sigma}{\sigma} \frac{\omega_i}{C_i} = \lambda p_i,$$

where we have defined

$$\omega_i \equiv \Omega_i^{\frac{1}{\sigma}} \left( \frac{C_i}{g_i(U)} \right)^{\frac{\sigma - 1}{\sigma}}.$$

Equation (A.4) shows that expenditure $p_iC_i$ on good $i$ is proportional to $\omega_i$. Since the latter sums to one from constraint (A.1), it follows that $\omega_i$ is the expenditure share of good $i$, and we have:

$$E = \sum_{i=1}^{I} p_iC_i = \frac{1 - \sigma}{\sigma} \frac{\rho}{\lambda}.$$

We can now substitute the definition of $\omega_i$ from Equation (A.5) in expression (A.4) and use (A.6) to find (A.2) and (A.3).

Lemma 5 implies the following relationship, defining the expenditure (and implicitly the indirect utility function) for general Nonhomothetic CES preferences:

$$E = \left[ \sum_{i=1}^{I} \Omega_i \left( g_i(U) p_i \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}.$$

The expenditure function is continuous in prices $p_i$’s and $U$, and homogenous of degree 1, increasing, and concave in prices. The elasticity of the expenditure function with respect to utility is:

$$\eta_E^U \equiv \frac{U \partial E}{E \partial U} = \sum_i \omega_i \eta_{g_i}^U = \eta_{g_i}^{U},$$

which ensures that the expenditure function is increasing in utility if all $g_i$’s are monotonically increasing. It is straightforward to also show that the elasticity of the utility function (A.1) with respect to consumption of good $i$ is also given by:

$$\eta_{U}^{C_i} \equiv \frac{C_i \partial U}{U \partial C_i} = \frac{\omega_i}{\eta_{g_i}^{U}},$$

where $\omega_i$ is the ratio defined in Equation (A.5).

Examining sectoral demand from Equation (A.2) along indifference curves, shows the main properties of nonhomothetic CES preferences. As expected, the elasticity of substitution is constant:

$$\eta_{C_i/C_j} \equiv \frac{\partial \log (C_i/C_j)}{\partial \log (p_i/p_j)} = \sigma.$$
More interestingly, the elasticity of relative demand with respect to utility is in general different from unity:

\[ \eta_{C_i/C_j}^U = \frac{\partial \log (C_i/C_j)}{\partial \log U} = \frac{\partial \log (g_i/g_j)}{\partial \log U}. \]  

(A.10)

Since utility has a monotonic relationship with real income (and hence expenditure), it then follows that the expenditure elasticity of demand for different goods are different. More specifically, we can use (A.7) to find the expenditure elasticity of demand:

\[ \eta_{C_i}^E = \frac{\partial \log C_i}{\partial \log E} = \sigma + \frac{1}{1 - \sigma} \eta^U_{g_i}. \]  

(A.11)

The intuition for the normalization in expression (A.11) is that the elasticity \( \eta_{C_i}^E \) has to be invariant to all monotonic transformations of utility.

Preferences defined by Equation (A.1) belong to the general class of preferences with **Direct Implicit Additivity**. Hanoch (1975) shows that the latter family of preferences have the nice property that is illustrated by Equations (A.9) and (A.10): the separability of the income and substitution elasticities of the Hicksian demand. This is in contrast to the stronger requirement of **Explicit Additivity** commonly assumed in nonhomothetic preferences, whereby the utility is explicitly defined as a function \( U = F(\sum_i f_i(C_i)) \). In Section A.3 below, we will show examples of how substitution and income elasticities of Hicksian demand are not separable for preferences with explicit additivity in direct utility, e.g., generalized Stone-Geary preferences (Kongsamut et al., 2001), or indirect utility, e.g., PIGL preferences (Boppart, 2013).

Finally, let us investigate the convexity of the expenditure function in terms of utility. First, we express the second derivative of the expenditure function in terms of elasticities,

\[ \frac{\partial^2 E}{\partial U^2} = \frac{E}{U^2} \eta^U_{E} \left( \eta^U_{E} + \eta^U_{E} - 1 \right), \]  

(A.12)

where \( \eta^U_{E} \) is the second order elasticity of expenditure with respect to utility. We can compute this second order elasticity as follows:

\[
\eta^U_{E} = U \frac{\partial}{\partial U} \log \sum_i \eta_{g_i}(U) (g_i(U)p_i)^{1-\sigma} - (1 - \sigma) \frac{\partial \log E}{\partial \log U},
\]

\[
= \sum_i \eta_{g_i} \cdot \eta_{g_i} (g_i(U)p_i)^{1-\sigma} + (1 - \sigma) \sum_i \eta_{g_i}^2 (g_i(U)p_i)^{1-\sigma} - \frac{\sum_i \eta_{g_i} (g_i(U)p_i)^{1-\sigma}}{\sum_i \eta_{g_i} (g_i(U)p_i)^{1-\sigma}},
\]

\[
= \frac{\sum_i \eta_{g_i} \cdot \eta_{g_i} (g_i(U)p_i)^{1-\sigma}}{(\eta_{g_i}^2)^2} + (1 - \sigma) \text{Var} \left( \frac{\eta_{g_i}}{\eta_{g_i}}, \right), \]

(A.13)

where \( \bar{X}_i \) and \( \text{Var} (X_i) \) denote the expected value and variance of variable \( X_i \) across sectors with weights given by expenditure shares \( \omega_i \) for prices \( p \) and utility \( U \).
To make sense of (A.13), consider the choice of \( g_i(U) \equiv g(U)^{\epsilon_i} \) for some monotonically increasing function \( g(\cdot) \) (which corresponds to the aggregator introduced in Section 5). We have that \( \eta_{g_i} = \eta_g \varepsilon_i \) and \( \eta_{g_{\varepsilon_i}} = \eta_{g_{\varepsilon_i}} \), implying:

\[
\eta^U_{g_i} = \eta_g \varepsilon_i \left[ \eta_{g_{\varepsilon_i}} + (1 - \sigma) \text{Var} \left( \frac{\varepsilon_i}{\varepsilon_i} \right) \right].
\] (A.14)

**A.2 Isoelastic Nonhomothetic CES Preferences**

Now, consider the specific case used in our basic model in Section 2, where the isoelastic functions \( g_i \) are defined as:

\[
g_i(U) = U^{\epsilon_i - \sigma},
\] (A.15)

where \( \eta_{g_i} = (\epsilon_i - \sigma)/(1 - \sigma) \), and we retrieve standard CES preferences when \( \epsilon_i = 1 \) for all \( i \)’s.

To tie our exposition more closely to the discussions in Section 2, let us for now identify utility with \( C \), aggregate real income and define a corresponding aggregate price index \( P \equiv E/C \).

From Equations (A.2) and (A.3), we find demand to be:

\[
C_i = \left( \frac{p_i}{P} \right)^{-\sigma} C^{\epsilon_i},
\] (A.16)

\[
\omega_i = \frac{p_i C_i}{P C} = \left( \frac{p_i}{P} \right)^{1-\sigma} C^{\epsilon_i-1}.
\] (A.17)

The aggregate price index is

\[
P \equiv \frac{E}{C} = \left( \sum_{i=1}^{I} \omega_i C^{\epsilon_i-1} p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.
\] (A.18)

From (A.7), the real income elasticity of the expenditure function is:

\[
\eta^C_E \equiv \frac{C}{E} \frac{\partial E}{\partial C} = \frac{\bar{\epsilon} - \sigma}{1 - \sigma},
\] (A.19)

where \( \bar{\epsilon} = \sum_i \omega_i \epsilon_i \). Therefore, a sufficient condition for the function \( E (C; \{p_i\}_{i=1}^{I}) \) to be a one-to-one mapping for all positive prices is that all sectors have an income elasticity larger than the elasticity of substitution \( \epsilon_i > \sigma \) if \( \sigma < 1 \) (and \( \epsilon_i < \sigma \) if \( \sigma > 1 \)). This directly follows from Lemma 4.

The income elasticities of demand are given by Equations (A.10) and (A.11):

\[
\eta^C_{C_i/C_j} = \epsilon_i - \epsilon_j,
\] (A.20)

\[
\eta^E_{C_i} = \sigma + (1 - \sigma) \frac{\epsilon_i - \sigma}{\bar{\epsilon} - \sigma}.
\] (A.21)

Each good \( i \) is characterized by a parameter \( \epsilon_i \in \mathbb{R} \) that is a measure of its real income.
More generally, the relationship between utility $U$ and real aggregate consumption $C$ in Expression (A.15) can be defined by any monotonic function $G$ such that $U = G(C)$. In particular, let us define $G(\cdot)$ such that $C$ corresponds to consumption expenditure at constant prices $\{q_i\}_i$ such that

$$C^{1-\sigma} = \sum_{i=1}^{I} \Omega_i G(C)^{\epsilon_i - 1} q_i^{1-\sigma}. \quad (A.22)$$

Assuming $\sigma \in (0, 1)$, if $\epsilon_i > \sigma$ for all $i$, function $G(\cdot)$ defined through Equation (A.22) is monotonically increasing for all positive $C$. Therefore, we can approximate the relationship as:

$$\log G(C) \approx \log G(\hat{C}) + \frac{\partial \log G}{\partial \log C} \bigg|_{C=\hat{C}} \left( \log C - \log \hat{C} \right),$$

$$= \frac{1 - \sigma}{\bar{\epsilon} - \sigma} \log C + \text{const.}, \quad (A.23)$$

where $\bar{\epsilon}$ is the average elasticity parameter at constant price $q$ and real income $\hat{C}$.

### A.3 Comparison to Generalized Stone-Geary and PIGL Preferences

For comparison, now consider generalized Stone-Geary preferences that have been widely used in previous work on structural change (see, e.g., Kongsamut et al., 2001):

$$C = \left( \sum_{i=1}^{I} (C_i - \bar{C}_i) \frac{\sigma - 1}{\sigma - 1} \right)^{\frac{\sigma}{\sigma - 1}}, \quad (A.24)$$

where $\bar{C}_i$ are the usual coordinate shifters.\(^{55}\) The expenditure elasticity of demand for good $i$ is given by:

$$\eta^{E}_{C_i} = 1 - \frac{C_i}{\bar{C}_i}, \quad (A.25)$$

which is different from 1 as long as the shifter $\bar{C}_i \neq 0$. However, note that due to constancy of $\bar{C}_i$, this elasticity converges to unity at the same rate as the rate of growth of $C_i$. Therefore, nonhomotheticity is a short-run feature of Stone-Geary preferences: as the income grows Stone-Geary preferences asymptote to homothetic CES preferences.

Another important feature of the nonhomothetic CES preferences is the fact that elasticity of substitution $\sigma_{ij}$ between all goods $i$ and $j$ remains constant $\sigma$ and remains independent of

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\(^{55}\)In particular, standard 3-sector models of structural transformation generally assume $\bar{C}_a > 0$, $\bar{C}_s < 0$ and $\bar{C}_m = 0$. 46
expenditure (income) elasticities. In contrast, for Stone-Geary preferences we find:

\[
\sigma_{ij} = \sigma \cdot \frac{E}{E - \sum_k p_k c_k} \cdot \left(1 - \frac{c_i}{c_j}\right) \cdot \left(1 - \frac{c_j}{c_i}\right),
\]
(A.26)

\[
\eta^E_{c_i} = \frac{E}{E - \sum_k p_k c_k} \cdot \left(1 - \frac{c_i}{c_i}\right),
\]
(A.27)

where \( \eta_i \) is the nominal income elasticity of demand in sector \( i \) (Hanoch, 1975). It then follows that the elasticities of substitution between goods \( i \) and \( j \) always satisfies the following equality:

\[
\sigma_{ij} = \sigma \eta^E_{c_i} \eta^E_{c_j},
\]
(A.28)

creating a direct linkage between elasticities of substitution and expenditure for different sectors. As expected, when \( E \) goes to infinity we find that \( \sigma_{ij} \to \sigma \) and \( \eta^E_{c_i} \to 1 \) for all sectors.

An alternative specification for nonhomothetic preferences in the structural change literature, recently used by Boppart (2013), is the Price Independent Generalized Linear (PIGL) preferences. The canonical definition for these preferences involves a two-good system. In general, no closed-form representation for the utility function exists, but the indirect utility/expenditure function relationship can be specified as:

\[
C + \frac{\vartheta}{\vartheta - 1} \left[ \left( \frac{p_1}{p_2} \right)^\xi - 1 \right] = \frac{1}{\xi} \left[ \frac{E}{\xi} - 1 \right],
\]
(A.29)

where \( p = (p_1, p_2) \) is the pair of good prices, \( C \) is the aggregator (utility) and \( E \) is expenditure, \( 0 \leq \xi \leq \vartheta < 1 \), and \( \vartheta > 0 \).\(^{56}\) For these preferences, the expenditure elasticity of demand for good \( i \) is constant and less than unity: \( \eta^E_{c_i} = 1 - \xi < 1 \). Therefore, like nonhomothetic CES preferences and unlike Stone-Geary, PIGL preferences also feature nonhomotheticity at all levels of income. In contrast to nonhomothetic CES, however, there is no generalization of PIGL preferences to more than two good demand systems that preserves the independence of income elasticities across different goods.\(^{57}\)

Needless to say, since PIGL preferences are outside the CES family, the elasticity of substitution varies with income and prices. As Boppart (2013) shows, the elasticity of substitution

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\(^{56}\)PIGL preferences are not additive in the sense of Hanoch (1975).

\(^{57}\)As a reminder, from Engel aggregation we know that we can have up to \( I - 1 \) independent income elasticities in a demand system involving \( I \) goods. This is why we have one degree of freedom in specifying the \( I \) income elasticity parameters \( \bar{\epsilon} \) in nonhomothetic CES preferences defined in Section A.2.
between goods 1 and 2 are given by

\[ \sigma = 1 - \varrho - (\varrho - \xi) \left( \frac{E}{p_2} \right) \left[ \frac{\varrho}{(p_1/p_2)^{\varrho}} \right] \].

As a result, when PIGL preferences are embedded in a growth model, along an equilibrium path that involves growing income the elasticity of substitution will be monotonically increasing and converges toward \(1 - \varrho\). Therefore, the choice of PIGL preferences involves specific assumptions about the dynamics of substitution elasticities in a two-good model.

### B Proofs of Propositions and Lemmas

**Proof of Lemma 1.** The household problem can be written as that of finding

\[ \max_{\{C_t, K_t, \{C_it\}_i\}_t} \sum_t \left\{ \beta^t \frac{C_t^{1-\theta} - 1}{1 - \theta} + \lambda_t \left( E_t - \sum_{i=1}^I p_i C_{it} \right) + \rho_t (F_t - 1) \right\}, \]

where \(E_t = w_t + K_t(1 - \delta + R_t) - K_{t+1}\) and \(F_t\) is defined according to Equation (A.1) and (A.15). It is straightforward to see that the FOCs characterizing the optimal bundle \(\{C_{it}\}_i\) for any choice of \(E_t\) and \(C_t\) are the same as the ones in the static case and are given by:

\[ \rho_t \frac{1 - \sigma}{\sigma} \omega_{it} C_{it}^{1-\varrho} = \lambda_t p_{it}, \]

resulting in an expression similar to Equation (A.16):

\[ C_{it} = \left( \frac{p_{it}}{R_t} \right)^{-\sigma} C_t^{\varrho}. \]

Furthermore, aggregate price index \(P_t\) and total consumption expenditure \(E_t\) could be written according to expression (A.18). This shows the first part of the lemma, characterizing the intra-temporal problem.

Now, we can write down the intertemporal problem as:

\[ \max_{\{C_t\}_t} \sum_t \left\{ \beta^t \frac{C_t^{1-\theta} - 1}{1 - \theta} + \lambda_t \left( w_t + K_t(1 - \delta + R_t) - K_{t+1} - E(C_t) \right) \right\}, \]

where we have defined the expenditure function:

\[ E\left( C_t; \{p_{it}\}_{i=1}^I \right) = \left[ \sum_{i=1}^I C_{it}^{\varrho} \frac{1 - \sigma}{\sigma} p_{i}^{1-\varrho} \right]^{1/\varrho}. \]
The FOCs with respect to $K_t$ yield:

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\beta (1 - \delta + R_t)},$$

which will pin down all $\lambda$'s for any given $\lambda_0$.

The discussion above implies that for any initial marginal utility of wealth $\lambda_0$, Equation (B.6) gives $\lambda_t$, which then allows us to find $C_t$ that satisfies its respective FOC from:

$$C_t^{1-\theta} = \lambda_t E(C_t) \frac{\sum \epsilon_i \omega_{it} - \sigma}{1 - \sigma}.$$  \hspace{1cm} (B.7)

Combining the equation above with Equation (B.6) then yields the Euler equation (8) and completes the proof.

**Proof of Proposition 2.**

Using an argument similar to the one used for Lemma 1, we can decompose the problem into two intra-temporal and intertemporal problems. To avoid repetition, we focus on the latter, using the definition of the expenditure function in Equation (28) as the cost of real consumption $c(t)$ for the representative consumer in terms of the price of investment good at time $t$.

For a given path of wages $[w(t)]_{t=0}^\infty$, rental prices of capital $[r(t)]_{t=0}^\infty$, and sectoral good prices $[p(t)]_{t=0}^\infty$, the current-value Hamiltonian for the consumer problem (26) may be written as:

$$\hat{H}(t, c(t), a(t), \lambda(t)) \equiv u(c(t)) + \lambda(t) (w(t) + [r(t) - n] a(t) - e(c(t) ; p(t))).$$

Let us start with the necessary conditions. The FOCs for the Hamiltonian are as follows:

$$\frac{\partial \hat{H}}{\partial c} = 0 \Rightarrow u'(c) - \frac{\partial e}{\partial c} = 0,$$  \hspace{1cm} (B.8)

$$\frac{\partial \hat{H}}{\partial a} = (\rho - n) \lambda - \dot{\lambda} \Rightarrow \frac{\dot{\lambda}}{\lambda} = -(\rho - \rho).$$  \hspace{1cm} (B.9)

In addition, we impose that the solution satisfy the transversality condition:

$$\lim_{t \to \infty} e^{-(\rho-n)t} \lambda(t) a(t) = 0.$$  \hspace{1cm} (B.10)

Equations (B.8) and (B.9) together with the law of evolution of assets (35) and the transversality equation (B.10) characterize paths of per capita real aggregate consumption and asset holdings $[c(\cdot), a(\cdot)]$, and costate $\lambda(\cdot)$ that satisfy necessary conditions for optimality.

Next, we show conditions that ensure the solution above indeed corresponds to the unique solution to the household utility maximization problem. A standard argument (using (B.9)
and the No-Ponzi constraint) shows that for all feasible pairs \([c(\cdot), a(\cdot)]\), we have that 
\[\lim_{t \to \infty} \exp(- (\rho - n) t) \lambda(t) a(t) \geq 0.\] 
Therefore, we can establish that the pair characterized by Equations (B.8), (B.9), and (B.10) indeed correspond to the optimum if the Hamiltonian is concave in \(c\). Furthermore, since the Hamiltonian is separable in \((c, a)\) and linear \(a\), strict concavity in \(c\) implies the uniqueness of the optimum for the household problem. We will come back to the characterization of conditions ensuring the concavity of \(e\) at the end of the proof.

From Equation (B.8), we find that
\[
\lambda(t) = \frac{u'(c(t))}{\partial_e(c(t); p(t))} = \frac{u'(c(t))}{\frac{c}{\eta^e}} \cdot \frac{\eta^e}{c},
\] (B.11)
where \(\frac{\partial e}{\partial c}\) is the marginal (dollar) cost of consumption. We can compute the consumption elasticity of expenditure:
\[
\eta^e = \frac{\partial \log e}{\partial \log c} = \frac{c}{1 - \sigma} \frac{\partial}{\partial c} \log \sum_{i=1}^I (g^i p_i)^{1-\sigma},
\]
\[
= \eta_g \cdot \sum_{i=1}^I \varepsilon_i \omega_i,
\]
where the last equation can be explicitly expressed as \(\eta_g (c) \cdot \varepsilon(c; \bar{p})\) as a function of real aggregate consumption \(c\) and current prices \(p\).

To translate the equations above into the Euler format, we need to compute \(\frac{\dot{\lambda}}{\lambda}\), which corresponds to the growth in value of income at time \(t\). Using Equation (B.11), we can write the growth in utility value of income as the sum of the contribution of growth of real consumption, decline in the price index, and the decline in the average income elasticity parameter, that is,
\[
\frac{\dot{\lambda}}{\lambda} = \eta^e \cdot \frac{\dot{c}}{c} - \frac{(e/c)}{e/c} - \frac{\dot{\eta}^e}{\eta^e}.
\]
First, we simplify the growth of consumption expenditure:
\[
\frac{\dot{e}}{e} = \eta^e \cdot \frac{\dot{c}}{c} + \sum_i \frac{\dot{p}_i}{p_i} \omega_i,
\]
\[
= \eta_g \cdot \frac{\dot{c}}{c} + \frac{\dot{\bar{p}}}{\bar{p}} \cdot \frac{\bar{p}}{p_i},
\]
where we have used the fact that \(\eta^p_i = \omega_i\) from Lemma 6 (see below) and have defined \(\frac{\dot{\bar{p}}}{\bar{p}}\) to be the average rate of growth of prices across sectors, as weighted by their corresponding
consumption expenditures. Next, we compute the growth of the real consumption elasticity of expenditure:

\[ \frac{\dot{\eta}_e}{\eta_e} = \frac{\dot{\eta}_g}{\eta_g} + \frac{\dot{\bar{\varepsilon}}}{\bar{\varepsilon}}, \]

\[ = \eta_{tg} \cdot \frac{\dot{c}}{c} + \eta_e \cdot \frac{\dot{\bar{\varepsilon}}}{\bar{\varepsilon}} + \sum_i \eta_{p_i} \left( \frac{\dot{p}_i}{p_i} \right), \]

\[ = \eta_{tg} \cdot \frac{\dot{c}}{c} + (1 - \sigma) \eta_{g} \bar{\varepsilon} \cdot \text{Var} \left( \frac{\varepsilon}{\bar{\varepsilon}} \right) + (1 - \sigma) \sum_i \left( \frac{\varepsilon_i}{\bar{\varepsilon}} - 1 \right) \left( \frac{\dot{p}_i}{p_i} \right) \omega_i, \]

\[ = \eta_{tg} \cdot \frac{\dot{c}}{c} + (1 - \sigma) \eta_{g} \bar{\varepsilon} \cdot \text{Var} \left( \frac{\varepsilon}{\bar{\varepsilon}} \right) \cdot \frac{\dot{c}}{c} + (1 - \sigma) \text{Cov} \left( \frac{\varepsilon}{\bar{\varepsilon}}, \frac{\dot{p}}{p} \right), \]

where in the third equality, we have used the results of Lemma 6 substituting for the elasticities of average income elasticity parameter \( \bar{\varepsilon} \).

Combining all of the above, we find the Euler equation to be:

\[ \frac{\dot{c}}{c} = \frac{r - \rho - \frac{\dot{p}}{p} - (1 - \sigma) \text{Cov} \left( \frac{\varepsilon}{\bar{\varepsilon}}, \frac{\dot{p}}{p} \right)}{-\eta_{t'c} - 1 + \eta_{tg} + \eta_{g} \bar{\varepsilon} \left( 1 + (1 - \sigma) \text{Var} \left( \frac{\varepsilon}{\bar{\varepsilon}} \right) \right)}. \quad (B.12) \]

The Euler Equation (B.12) is expressed for a general function \( g(\cdot) \). Specializing this result to the specific function defined in Equation (31), we will now derive the Euler Equation for the real consumption stated in terms of constant prices \( q \). Since \( g \) is the inverse of the expenditure function at prices \( q \), from Equation (B.12), we have:

\[ 1 = \eta_g(c) \cdot \bar{\varepsilon}(c; q), \]

suggesting \( \eta_g = \frac{1}{\bar{\varepsilon}(c; q)} \). It follows that:

\[ \eta_e = \frac{\varepsilon(c; p)}{\bar{\varepsilon}(c; q)}, \]

which is positive if we have \( \varepsilon_i > 0 \) for all sectors. This ensures that the function \( e \) is monotonically increasing and one-to-one. This elasticity, which may in general be different from zero, characterizes the way income effects shifts expenditure across sectors with different prices. In particular, whenever the average elasticity parameter \( \bar{\varepsilon} \) is higher than the one at base prices, marginal cost of increasing real consumption exceeds the current aggregate price index.

Similarly, substituting for \( q \) in Equation (B.17) from Lemma 6 below, we find:

\[ \eta_{tg} = -\frac{\partial \log \varepsilon(c; q)}{\partial \log c} = -(1 - \sigma) \left( \frac{\varepsilon^2(c; q)}{\bar{\varepsilon}(c; q)^2} - 1 \right), \]

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To ensure the sufficiency of the FOCs and the uniqueness of the solution, we need to find conditions under which the Hamiltonian is strictly concave. The second order condition for $c$ is

$$u'' - \lambda \frac{\partial^2 e}{\partial c^2} = \frac{u}{c^2} \eta_u (\eta_u' - \eta_{\partial e/\partial c}),$$

where we have substituted for $\lambda$ from Equation (B.11). This expression has to always be negative. We need to only focus on the term within the parentheses. The first term on the right hand side, by assumption, is always greater than $\theta$. To compute the second term, note that

$$\eta^c_{e,c} \equiv \eta_{\partial e/\partial c} = \frac{c}{e \cdot \eta^e_{e,c}} \frac{\partial^2 e}{\partial c^2},$$

$$= \eta^c_{e} + \eta^c_{e,c} - 1,$$  \hspace{1cm} (B.13)

and $\eta^c_{e,c}$ is given by Equation (A.14) from Section A.

Finally, we can find $\eta_g$ in Equation (A.14) by using the definition of function $g(\cdot)$ in Equation (31). Since $e(c; q) = c$, we have that $\eta^c_{e} = 0$ at prices $q$ and therefore:

$$0 = \eta_g \bar{\varepsilon_i} \left[ \eta_g + (1 - \sigma) \text{Var}\left( \frac{\varepsilon_i(c; q)}{\varepsilon_i(c; q)} \right) \right],$$ \hspace{1cm} (B.14)

which implies $\eta_g = -(1 - \sigma) \text{Var}\left( \frac{\varepsilon_i(c; q)}{\varepsilon_i(c; q)} \right)$.

Combining all the results above, we find the second order condition to require:

$$\theta + \frac{\varepsilon(c; p)}{\varepsilon(c; q)} - 1 + (1 - \sigma) \left[ \frac{\varepsilon(c; p)}{\varepsilon(c; q)} \left( \frac{\text{Var}(\varepsilon; c; p)}{\varepsilon(c; q)^2} \right) - \frac{\text{Var}(\varepsilon; c; q)}{\varepsilon(c; q)^2} \right] > 0,$$

for all $c$. Remembering that the variance of any distribution on $\{\varepsilon_i\}_{i=1}^I$ is bounded above by $\frac{1}{4} (\varepsilon_{\max} - \varepsilon_{\min})^2$, it immediately follows that condition (37) is a sufficient condition for the FOC to always characterize an optimal solution.

**Lemma 6.** The real consumption and sectoral price elasticities of the expenditure function are given by:

$$\eta^c_e = \eta_g \bar{\varepsilon},$$  \hspace{1cm} (B.15)

$$\eta^p_i = \omega_i.$$  \hspace{1cm} (B.16)

Furthermore, define the average $\bar{x}$ of some sectoral parameters $x_i$ across the consumption
sector weighted by expenditure shares:

\[ \bar{x} \equiv \sum_{i=1}^{I} x_i \omega_i. \]

The elasticities of this function are given by:

\[
\eta^\epsilon_{\bar{x}} = \frac{1 - \sigma}{\bar{x}} \eta_g \text{Cov} (\varepsilon, x), \tag{B.17}
\]

\[
\eta^p_{\bar{x}} = (1 - \sigma) \left( \frac{x_i}{\bar{x}} - 1 \right) \omega_i. \tag{B.18}
\]

**Proof.** We can compute the consumption elasticity of expenditure:

\[
\eta^\epsilon_c = \frac{\partial \log e}{\partial \log c} = \frac{c}{1 - \sigma} \frac{\partial}{\partial c} \log \sum_{i=1}^{I} (g^\epsilon_i p_i)^{1 - \sigma},
\]

\[= \eta_g \cdot \sum_{i=1}^{I} \varepsilon_i \omega_i,
\]

\[= \eta_g \bar{\varepsilon}.
\]

where the last equation can be explicitly expressed as \( \eta_g (c) \cdot \bar{\varepsilon} (c; \mathbf{p}) \) as a function of real aggregate consumption \( c \). Similarly, we can compute the price elasticity of the expenditure function:

\[
\eta^p_i = \frac{\partial \log e}{\partial \log p_i} = \frac{p_i}{1 - \sigma} \frac{\partial}{\partial p_i} \log \sum_{j=1}^{I} (g^\epsilon_j p_j)^{1 - \sigma},
\]

\[= \left( \frac{g^\epsilon_i p_i}{e} \right)^{1 - \sigma},
\]

\[= \omega_i.
\]

Next, we use the expressions above to compute the elasticities of the sectoral shares in consumption expenditure. From Equation (29) we have:

\[
\eta^\epsilon_{\omega_i} = (1 - \sigma) (\eta_g \varepsilon_i - \eta^\epsilon_e) = (1 - \sigma) \eta_g (\varepsilon_i - \bar{\varepsilon}),
\]

\[
\eta^p_{\omega_i} = (1 - \sigma) (\delta_{ij} - \eta^p_{\epsilon_i}) = (1 - \sigma) (\delta_{ij} - \omega_j),
\]

where \( \delta_{ij} \) stands for the kronecker delta function.

Finally, we use the elasticities of the expenditures shares to compute elasticities of a general
We find:

\[
\eta^c_x = \frac{1}{\bar{x}} \sum_j x_j \omega_j \eta^c_{\omega_j}, \quad \eta^g_x = \frac{1}{\bar{x}} \eta_g \sum_j x_j \omega_j (\varepsilon_j - \bar{\varepsilon}), \quad \eta^{p_i}_x = \frac{1}{\bar{x}} \sum_j x_j \omega_j (\delta_{ij} - \omega_i), \quad \eta^{p_i}_x = (1 - \sigma) \left( \frac{x_i}{\bar{x}} - 1 \right) \omega_i.
\]

For instance, for \( x_i \equiv \varepsilon_i \), we find:

\[
\eta^c_{\varepsilon} = (1 - \sigma) \eta_g Cov \left( \frac{\varepsilon_i}{\bar{\varepsilon}}, \varepsilon_i \right) = (1 - \sigma) \eta_g Var \left( \frac{\varepsilon}{\bar{\varepsilon}} \right).
\]

**Proof of Proposition 3**

Our strategy for the proof of this proposition is as follows. To establish the existence and uniqueness of the competitive equilibrium, we invoke the second Welfare Theorem. We formulate the social planner’s problem, whose potential solutions have to correspond to different competitive equilibria in our economy. We solve the social planner’s problem and show that it has a unique solution, and further establish a direct correspondence between this solution and the competitive equilibrium, which thus has to also be unique.

Let \( \hat{u} (c_1, \cdots, c_I) \equiv u (c) \), where \( c \) is defined through Equation (27), denote the instantaneous utility of the representative household over a bundle of \( c = (c_i)_{i=1}^I \) per capita consumption of \( I \) different goods. The social planner’s problem can be stated as the following maximization problem:

\[
\max_{\{k_i(t), l_i(t)\}_{t=0}^T} \int_0^\infty e^{-(\rho-n)t} \hat{u} (c_1(t), \cdots, c_I(t)),
\]

where

\[
c_i = A_i k^{\alpha_i} l_1^{1-\alpha_i}, \quad 1 \leq i \leq I, \quad (B.20)\\
\dot{k} = A_0 k^{\alpha_0} l_1^{1-\alpha_0} - (\delta + n) k, \quad (B.21)
\]

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subject to constraints:

\[
\sum_{i=0}^{I} l_i = 1, \quad (B.22)
\]

\[
\sum_{i=0}^{I} k_i = k, \quad (B.23)
\]

and \( k(0) = K(0)/L(0) \) and \( k(t) \geq 0 \) for all \( t \).

The corresponding present value Hamiltonian is given by:

\[
H = \hat{u} + \mu \left( A_0 L k_0^{\alpha_0} l_0^{1-\alpha_0} - (\delta + n) k \right),
\]

where we substitute for per capita consumption, i.e., Equations (B.20) and (B.23), in the expression for \( \hat{u} \) making the latter a function of vectors of sectoral per capita stocks of capital and employment shares \((k, l)\).\(^{58}\)

We can show that the function \( M(k) = \max_{(k, l)} \hat{H} \) under the constraints suggested by Equations (B.22) and (B.23) is a strictly concave function of \( k \), if \( \hat{u} \) is a strictly concave function of its arguments. To see why, let us define functions \( F(k, l) \equiv \hat{u}(k, l) + \mu A_0 L k_0^{\alpha_0} l_0^{1-\alpha_0} \) and \( \hat{F}(k) = \max_l F(k, l) \). First, it is straightforward to see that \( F \) is jointly strictly concave in \((k, l)\) for \( \mu \geq 0 \), which implies that \( \hat{F} \) is a strictly concave function of vector \( k \). The strict concavity of the latter then implies that \( M(k) = \max_{\sum_{i=k}^{I} k_i = k} \hat{F}(k) \) is a strictly concave function of \( k \).

Now let us find a candidate solution for the social planner’s problem that satisfies the following conditions:

\[
- \frac{\partial H}{\partial k} = (\delta + n) \mu - \zeta = \hat{\mu} - (\rho - n) \mu, \quad (B.24)
\]

\[
\frac{\partial H}{\partial k_i} = \alpha_i \frac{\partial \hat{u}}{\partial c_i} A_i \left( \frac{k_i}{l_i} \right)^{\alpha_i-1} = \zeta, \quad 1 \leq i \leq I, \quad (B.25)
\]

\[
\frac{\partial H}{\partial l_i} = (1 - \alpha_i) \frac{\partial \hat{u}}{\partial c_i} A_i \left( \frac{k_i}{l_i} \right)^{\alpha_i} = \xi, \quad 1 \leq i \leq I, \quad (B.26)
\]

\[
\frac{\partial H}{\partial k_0} = \alpha_0 \mu A_0 \left( \frac{k_0}{l_0} \right)^{\alpha_0-1} = \zeta, \quad (B.27)
\]

\[
\frac{\partial H}{\partial l_0} = (1 - \alpha_0) \mu A_0 \left( \frac{k_0}{l_0} \right)^{\alpha_0} = \xi, \quad (B.28)
\]

\(^{58}\)Let us remember that according to our vector notation \( k \equiv (k_0, \ldots, k_I) \) denotes the vector of sectoral per capita stocks of capital, which is distinct from \( k \), the economy-wide total per capita stock of capital.
and the transversality condition

\[
\lim_{t \to \infty} e^{-(\rho-n)t} \mu(t) k(t) = 0, \tag{B.29}
\]

where \(\xi\) and \(\varsigma\) are Lagrange multipliers corresponding to the two constraints Equations (B.20) and (B.23), respectively.

From Equations (B.24) and (B.27) we find:

\[
\mu(t) = \mu(0) \exp \left(- \int_0^t \left( \alpha_0 \tilde{\kappa}_0(t') \alpha_0^{-1} - (\rho + \delta) \right) dt' \right), \tag{B.30}
\]

where we have used the productivity adjusted definition of capital-labor ratio:

\[
\tilde{\kappa}_0(t) \equiv A_0(t)^{-1/(1-\alpha_0)} \frac{k_0(t)}{l_0(t)}. \tag{B.31}
\]

Similarly, we define the economy-wide aggregate capital-labor ratio \(\bar{k}(t) \equiv A_0(t)^{-1/(1-\alpha_0)} k(t)/l(t)\) and rewrite the transversality condition as:

\[
\lim_{t \to \infty} \bar{k}(t) \mu(0) \exp \left(- \int_0^t \left( \alpha_0 \tilde{\kappa}_0(t') \alpha_0^{-1} - \left( \delta + n + \frac{\gamma_0}{1-\alpha_0} \right) \right) dt' \right) = 0, \tag{B.32}
\]

where we have used the fact, which we will show later, that asymptotically \(k(t) \to \bar{k}^* e^{\frac{\gamma_0}{1-\alpha_0} t}\).

Note that with this transformation Equation (B.21) can be written as:

\[
\dot{\bar{k}}(t) = l_0 \tilde{\kappa}_0(t) \alpha_0 - \left( n + \delta + \frac{\gamma_0}{1-\alpha_0} \right) \bar{k}(t). \tag{B.33}
\]

Henceforth, we use the notation that \(\bar{x}(t)\) denotes variable \(x(t)\) adjusted by productivity in the investment sector, i.e., \(\bar{x}(t) \equiv A_0(t)^{-1/(1-\alpha_0)} x(t)\). Dividing (B.26) by (B.25), we find:

\[
\frac{1-\alpha_i}{\alpha_i} \tilde{\kappa}_i(t) = \frac{\tilde{\xi}(t)}{\xi(t)}, \tag{B.34}
\]

suggesting that capital-labor ratios in all sectors are proportional to each other. This relation echoes Equation (42), suggesting, as we show below, that \(\xi\) and \(\varsigma\) correspond to the wage and rental price of capital in a competitive equilibrium.

Starting from any initial value \(\mu(0)\), Equation (B.30) along with conditions (B.25) to (B.28) define a unique path for the allocations. The argument follows from the strict concavity of function \(F\) defined earlier, and the intuition is as follows. Equation (B.30) determines \(\mu(t)\) and therefore function \(F\) at time \(t\). The optimal allocation at this point it time is simply given by maximizing function \(F\) under constraints (B.22) and (B.22), which has to be unique.

Note that any candidate path that satisfies the conditions above has to further satisfy the
following two asymptotic conditions:

\[
\begin{aligned}
\lim_{t \to \infty} \tilde{\kappa}_0(t) &= \tilde{\kappa}^*_0 > 0, \\
\lim_{t \to \infty} l_0(t) &= l^*_0, \quad 0 < l^*_0 < 1,
\end{aligned}
\] (B.35) (B.36)

implying that the asymptotic capital-labor ratio and employment in the investment sector are constant and interior. If, on the contrary, we asymptotically have \(\tilde{\kappa}_0 \to 0\), we can show that \(\xi \to 0\) and Equation (B.26) has to be violated.\(^{59}\) If \(\tilde{\kappa}_0 \to \infty\), Equation (B.33) implies that the transversality condition B.32 has to be violated. Therefore, condition (B.35), has to hold.

Now from Equation (B.34) we learn that all sectoral capital-labor ratios asymptote to nonzero constants and therefore \(\lim_{t \to \infty} \tilde{k}(t) = \bar{k}^* = \sum_{i=1}^{I} l_i^* \tilde{\kappa}_i^* > 0\). Hence, from Equation (B.33) we know that \(l_0^* > 0\), since otherwise \(\tilde{k} \to 0\). Finally, assuming \(l_0^* = 1\) would suggest that \(\bar{k}^* = \tilde{\kappa}_0^*\) and \((\tilde{\kappa}_0^*)^{\alpha_0 - 1} = n + \delta + \frac{\gamma_0}{1 - \alpha_0}\). Substituting this in Equation (B.32) would violate the transversality condition. Therefore, any candidate path satisfying the conditions above will asymptotically be an interior candidate solution, in the sense that labor-capital ratios grow at the same rate as the rate of technological progress in the investment sector and there is an interior split of employment between the investment and the consumption sector. This implies, as we will see shortly, that the per capita consumption expenditure in the corresponding competitive equilibrium also grows at the same rate as the rate of growth of technology in the investment sector.

Next, we show that the growth of real consumption per capita along any candidate path satisfying the conditions above has to be asymptotically constant. To see this, note that combining Equations (B.20) and (B.34), we find that \(c_i = A_i^{\frac{\alpha_i}{1 - \alpha_0}} A_0 l_i \tilde{\kappa}_0^{\alpha_i}\) for \(i \in \{1, \cdots, I\}\). Equation (B.35) then implies:

\[
\lim_{t \to \infty} \frac{\dot{c}_i(t)}{c_i(t)} = \gamma_i + \frac{\alpha_i}{1 - \alpha_0} \gamma_0 + \dot{\gamma}_i^C,
\] (B.37)

where we defined the asymptotic rate of growth of the share of employment in sector \(i\) as:\(^{60}\)

\[
\dot{\gamma}_i^C \equiv \lim_{t \to \infty} \frac{\dot{l}_i(t)}{l_i(t)} \leq 0.
\] (B.38)

\(^{59}\)Assume \(\tilde{\kappa}_0\) converges to zero exponentially at a constant rate. From Equation (B.30), \(\mu\) has to converge to zero at the rate of \(-\infty\), which implies the same has to be the case for \(\xi\). From Equation (B.26), we need to have that \(\partial \hat{u} / \partial c_i\) converges to zero at the rate of \(-\infty\), which can hold only if \(c_i\)'s all grow at an ever increasing rate. This contradicts the initial assumption that capital-labor ratios converge to zero.

\(^{60}\)Note that Equation (B.36) establishes the total employment share in the consumption sector asymptotically converges to a constant \(l_C \to 1 - l_0^*\). However, within the aggregate consumption sector, some sectors could continue to shrink asymptotically which can result in Equation (B.38) taking nonzero values.
Now, from Equation (27), we have that for all \( t \geq 0 \):

\[
\sum_{i=1}^{I} \nu_i \left( \gamma_i + \frac{\alpha_i}{1-\alpha_0} \gamma_0 + g_i - \eta g \varepsilon_i \frac{\dot{c}}{c} \right) = 0,
\]

where we have defined the effective share of sector \( i \) in consumption as

\[ \nu_i \equiv \left( g(c)\varepsilon_i p_i/c \right)^{\sigma-1}/\sigma. \]

The rate of growth of per capita real consumption is then given by

\[
\frac{\dot{c}}{c} = \frac{1}{\eta g \varepsilon} \left( \overline{\gamma} + \frac{\overline{\alpha}}{1-\alpha_0} \gamma_0 + \frac{\gamma^C}{\overline{\gamma}} \right), \tag{B.39}
\]

where averages are taken with respect to the distribution implied by \( \{\nu_i\}_{i=1}^{I} \). Equation (B.26) suggests that, up to a constant, consumption share \( \nu_i \)'s are the same as employment shares \( l_i \)'s. To see this, rewrite Equation (B.26) as:

\[
\xi = (1 - \alpha_i) u'(c) \frac{\partial c}{\partial c_i} l_i = (1 - \alpha_i) u'(c) \frac{c}{\eta g \varepsilon} l_i,
\]

where we have used \( \eta_i = \nu_i/\eta g \varepsilon \) from Equation (A.8).

Define set \( \mathcal{I}^* \subset \{1, \cdots, I\} \) as the set of consumption sectors with nonzero asymptotic employment shares, i.e., \( \mathcal{I}^* = \{i | \hat{\gamma}_i^C = 0\} \). Consider some sector \( i \notin \mathcal{I}^* \), for which \( \hat{\gamma}_i^C < 0 \) implying that the asymptotic employment share of this sector is zero, i.e., \( \lim_{t \to \infty} l_i = 0 \). Since consumption shares \( \nu_i \)'s have to grow proportionally to employment shares as we showed above, for any such sector \( i \) we have \( \lim_{t \to \infty} \nu_i = 0 \). It then follows that \( \lim_{t \to \infty} \overline{\gamma}^C = \sum_{i \in \mathcal{I}^*} \hat{\gamma}_i^C = 0 \), and taking the limit of expression (B.39), we find the asymptotic rate of growth of per capital consumption as:

\[
\gamma^* \equiv \lim_{t \to \infty} \frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta g \varepsilon^*} \left( \overline{\gamma}^* + \frac{\alpha^*}{1-\alpha_0} \gamma_0 \right),
\]

where averages are taken with respect to the distribution implied by \( \{\nu_i^*\}_{i=1}^{I} \) the limit of distribution \( \{\nu_i\}_{i=1}^{I} \), with support \( \mathcal{I}^* \).

In order to characterize the set \( \mathcal{I}^* \), we need to compute the asymptotic rate of growth of \( \nu_i \) for each sector \( i \). Substituting Equation (B.34) in Equation (B.26), we find:

\[
u_i = \frac{c}{\eta g \varepsilon} \frac{\partial c}{\partial c_i}, \tag{B.40}
\]

As a reminder, share \( \nu_i \) equals the share of sector \( i \) in consumption expenditure \( \omega_i \equiv (g(c)^{\varepsilon_i} p_i/c)^{1-\sigma} \) for the corresponding prices in the competitive equilibrium. We introduce a different variable here solely to respect the conceptual distinction between the formulation of the social planner’s problem and the competitive equilibrium where the prices are implicit in the former (and the current formulation).
where the right hand side captures the social cost of producing good \( i \) in marginal utility terms at time \( t \). From Equations (B.27) and (B.28), we have:

\[
\zeta = \mu \alpha_0 \bar{\kappa}_0^{\alpha_0 - 1}, \tag{B.41}
\]

\[
\xi = \mu A_0^{1\alpha_0} (1 - \alpha_0) \bar{\kappa}_0^{\alpha_0}. \tag{B.42}
\]

Substituting in the expression above we find:

\[
\frac{u'(c) c \nu_i}{\mu c_i \eta g \overline{\varepsilon}} = \left( \frac{\alpha_0}{\alpha_i} \right)^{\alpha_i} \left( \frac{1 - \alpha_0}{1 - \alpha_i} \right)^{1 - \alpha_i} A_0^{1 - \alpha_i} \frac{A_0^{1 - \alpha_i}}{A_i} \bar{\kappa}_0^{\alpha_0 - \alpha_i}, \tag{B.43}
\]

where we have again used \( \eta_c^{\varepsilon_i} = \nu_i / \eta g \overline{\varepsilon} \).

We proceed by establishing that the asymptotic rate of growth of the term \( \frac{u'(c) c \nu_i}{\mu c_i \eta g \overline{\varepsilon}} \) is \( \gamma_0 \). To see this, let us rewrite Equation (B.26) as:

\[
\frac{c u'(c) c \nu_i}{\eta g \overline{\varepsilon}} = \frac{l_i}{1 - \alpha_i} \xi,
\]

which we can then sum over \( i \in \{1, \ldots, I\} \) to find:

\[
\frac{c u'(c) c \nu_i}{\eta g \overline{\varepsilon}} = \xi \sum_{i=1}^{I} \frac{l_i}{1 - \alpha_i}.
\]

Since \( \sum_{i=1}^{I} l_i \) converges to a nonzero value \( 1 - l_0' \), the rate of growth of the expression on the left hand side is the same as \( \xi \), which we know, from Equation (B.42), grows at the same rate as \( \mu A_0^{1/(1 - \alpha_0)} \). We use this fact and the fact that \( \frac{\nu_i}{c_i} = v_i^{1/(1 - \sigma)} g(c)^{\overline{\varepsilon}_i} \) to conclude from Equation (B.43) that:

\[
\lim_{t \to \infty} \frac{\dot{v}_i}{v_i} = (1 - \sigma) \left( \eta g \overline{\varepsilon}_i \gamma^* - \gamma_i - \frac{\alpha_i}{1 - \alpha_i} \gamma_0 \right). \tag{B.44}
\]

Crucially, Equation (B.44) implies that for any sector \( i^* \in I^* \), whose employment share does not asymptotically vanish, we should have:

\[
\gamma^* = \frac{1}{\eta g \overline{\varepsilon}_i} \left( \gamma_i^* + \frac{\alpha_i}{1 - \alpha_i} \gamma_0 \right).
\]

Now, consider the set \( \bar{I} \), defined as follows:

\[
\bar{I} \equiv \arg\min_{i \in \mathcal{I}/(0)} \frac{(1 - \alpha_0) \gamma_i + \alpha_i \gamma_0}{(1 - \alpha_0) \overline{\varepsilon}_i}.
\]
Consider the rate of growth of employment share in a consumption sector \( \hat{i} \in \hat{I} \):

\[
\lim_{t \to \infty} \frac{\dot{v}_i}{v_i} = (1 - \sigma) \left( \frac{\eta_g \varepsilon_i \gamma^* - \gamma_i - \frac{\alpha_i}{1 - \alpha_0} \gamma_0}{\varepsilon_i} \right),
\]

\[
= (1 - \sigma) \frac{\varepsilon_i}{\varepsilon_i^*} \left( \gamma_i^* + \frac{\alpha_i^*}{1 - \alpha_0} \gamma_0 \right) - \frac{1}{\varepsilon_i} \left( \gamma_i^* + \frac{\alpha_i^*}{1 - \alpha_0} \gamma_0 \right).
\]

The expression shows that \( \hat{I} = I^* \). If \( \hat{i} / \notin I^* \), this expression has to be strictly negative, violating the assumption that \( \hat{i} \in \hat{I} \). If \( i^* / \notin \hat{I} \), will have to be negative violating the assumption that \( i^* \in I^* \). It then follows that:

\[
\gamma^* = \min_{i \in I / \{0\}} \left( 1 - \alpha_0 \right) \gamma_i + \alpha_i \gamma_0.
\]

Finally, we need to check that the transversality condition is indeed satisfied. Since we know that the asymptotic rate of growth of \( \frac{c u_i (c)}{\mu \eta_g \varepsilon} \) is \( \frac{\gamma_0}{1 - \alpha_0} \), we find that:

\[
\lim_{t \to \infty} \frac{\dot{\mu}(t)}{\mu(t)} = (1 - \theta) \gamma^* - \frac{\gamma_0}{1 - \alpha_0},
\]

where we have used the fact that \( \lim_{t \to \infty} \pi(t) = \pi^* \) and \( \lim_{t \to \infty} \eta_g (c(t)) = \eta_g \) are both constants. Therefore, in order to satisfy condition B.32, we need to ensure that

\[
\rho > n + (1 - \theta) \gamma^*.
\]

From Equations (B.24) and (B.27), \( \mu(t) \geq 0 \) for all \( t \). Therefore, from strict concavity of \( F \) it follows that these equations together give a unique path of \( \{k(\cdot), l(\cdot), k(\cdot), \mu(\cdot)\}_{t=0}^{\infty} \). Due to the strict concavity of \( M \), we conclude that the resulting path corresponds to the unique solution to the social planner’s problem (see Theorem 7.14 in Acemoglu, 2008). This completes the proof of the proposition.

For completeness, we state the correspondence between the variables above and the variables characterizing the competitive equilibrium. Comparing expressions (B.41) and (B.42) with expressions (40) and (41) in the derivations of the competitive equilibrium, we find that \( \varsigma \equiv \mu R \) and \( \xi \equiv \mu w \). Substituting in the expression (B.40) yields:

\[
\frac{u'(c)}{\mu} \frac{\partial c}{\partial c_i} = \frac{1}{A_i} \left( \frac{R}{\alpha_i} \right) \left( \frac{w}{1 - \alpha_i} \right)^{1 - \alpha_i},
\]

where the right hand side corresponds to the expression for unit price of goods in sector \( i \) in Equation (43). We use \( \eta_i \equiv \nu_i / \eta g \varepsilon \) to rewrite this expression as

\[
\frac{c u' (c)}{\mu} \frac{\nu_i}{\eta g \varepsilon} = p_i c_i,
\]

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which, once we sum over \( i \in \{1, \ldots, I\} \), implies
\[
\frac{cu'(c)}{\mu \eta g} \equiv e.
\]

In light of this connection, the key step of the proof above establishing the rate of growth of the expression on the left hand side has a straightforward interpretation: per capita consumption expenditure grows at the same rate as the output of the investment sector.

\[\blacksquare\]

**C Equilibrium Dynamic Equations**

In this section, we characterize the dynamics of this economy along the equilibrium path. Define investment sector productivity-adjusted aggregate and investment variables:
\[
\tilde{k}(t) \equiv \frac{K(t)}{A_0^{\frac{1-\alpha_0}{\alpha_0}}(t) L(t)}, \quad \tilde{y}_0(t) \equiv \frac{Y_0(t)}{A_0^{\frac{1-\alpha_0}{\alpha_0}}(t) L(t)}, \quad \tilde{e}(t) \equiv \frac{e(t)}{A_0^{\frac{1-\alpha_0}{\alpha_0}}(t)}, \quad \tilde{\kappa}_0(t) \equiv \frac{\kappa_0(t)}{A_0^{\frac{1-\alpha_0}{\alpha_0}}(t)},
\]
and denote the share of labor employed in the investment sector by \( l_0(t) \equiv L_0(t)/L \). With this normalization, the rental price of capital and the capital-labor ratio in the investment sector have the following one-to-one relationship \( R(t) = \alpha_0\tilde{\kappa}_0(t)^{\alpha_0-1} \) and we can use them interchangeably to characterize the path of the aggregate economy.

Per capita consumption and productivity-adjusted capital-labor ratio in the investment sector \( (c(t), \tilde{\kappa}_0(t)) \) fully characterize the state of the economy at time \( t \). First, note that productivity-adjusted per capita consumption expenditure \( \tilde{e}(t) \) is a function of these two variables and time:
\[
\tilde{e}(t) = \tilde{e}(c(t), \tilde{\kappa}_0(t), t) = \left( \sum_{i=1}^{I} [\varphi_i(t) g(c(t))^{\bar{\varepsilon}_i} \tilde{\kappa}_0(t)^{\alpha_0-\alpha_i}]^{1-\sigma} \right)^{1/(1-\sigma)}, \quad (C.2)
\]
where we have substituted from Equations (43) and (C.1) in the definition of Equation (28) and have defined a (exogenously given) time-dependent function
\[
\varphi_i(t) \equiv \left( \frac{\alpha_0}{\alpha_i} \right)^{\alpha_i} \left( \frac{1-\alpha_0}{1-\alpha_i} \right)^{1-\alpha_i} A_0(t)^{-\frac{\alpha_i}{\alpha_0}} A_i(t)^{-1}. \]

The direct dependence on time is due to the impact of time-varying sectoral technologies on prices. Similarly, we can write the share of sector \( i \) in consumption expenditure as a function of \( (c(t), \tilde{\kappa}_0(t)) \) and time as \( \omega_i(t) = \omega_i(c(t), \tilde{\kappa}_0(t), t) \). Averages of the income elasticity parameter and capital share in the consumption sector of the economy \( \bar{\varepsilon} \) and \( \bar{\alpha} \) then also become functions of the two state variables (and time). We emphasize in the special
case where capital intensities are identical across all consumption sectors, the expenditure function in Equation (28) becomes independent of capital-labor ratios and solely depends on real consumption per capita \(c\) and time.

In addition to the expenditure and sectoral shares, the duplet \((c(t), \tilde{\kappa}_0(t))\) also pins down total investment and total per-capita stock of capital at time \(t\) along any equilibrium path. To see that, we first compute the employment share of the investment sector, dropping the dependence on time to simplify notation:

\[
l_0 \equiv \frac{L_0}{L} = \frac{1}{1 + \frac{L_C}{L_0}},
\]

where in the second line we have used the equality \(\frac{L_C(t)}{L_0(t)} = \frac{1 - \tilde{\alpha}}{1 - \alpha_0} \tilde{\kappa}_0\). Combining Equation (C.3) with \(\tilde{y}_0 = l_0 \tilde{\kappa}_0\), we can write both the normalized output and the employment share of the investment sector as:

\[
\begin{align*}
\tilde{y}_0 &= \tilde{\kappa}_0 - \frac{1 - \tilde{\alpha}}{1 - \alpha_0} \tilde{e}, \\
l_0 &= 1 - \frac{1 - \tilde{\alpha}}{1 - \alpha_0} \tilde{e},
\end{align*}
\]

Therefore, since average capital share \(\tilde{\alpha}\) and normalized expenditure \(\tilde{e}\) are both functions of \((c(t), \tilde{\kappa}_0(t))\) and time, so are the employment share and normalized output of the investment sector. Finally, the following lemma establishes that the total per-capita stock of capital is also a function of the same pair of variables.

**Lemma 7.** Along any equilibrium path and for all times \(t\), the aggregate productivity-adjusted per-capita stock of capital \(\tilde{k}\) and the productivity-adjusted capital-labor ratio in the investment sector \(\tilde{\kappa}_0\) satisfy the following equation:

\[
\tilde{k} = \tilde{\kappa}_0 \left[ 1 + \frac{\tilde{e}}{\tilde{\kappa}_0} \left( \frac{\tilde{\alpha} - \alpha_0}{\alpha_0 (1 - \alpha_0)} \right) \right],
\]

where \(\tilde{e}\) and \(\tilde{\alpha}\) are functions of \((c, \tilde{\kappa}_0)\) and time as defined by Equation (C.2). Moreover, for any level of per capita real consumption \(c > 0\) at time \(t\), Equation (C.6) defines a monotonically increasing and one-to-one mapping between \(\tilde{k}\) and \(\tilde{\kappa}_0\).

**Proof.** Along any equilibrium path, the output of all consumption goods are strictly positive and therefore \(\kappa_i > 0\) for all \(i \geq 1\). From Equation (41), we know that \(R \geq \alpha_0 A_0 \kappa_0^{\alpha_0 - 1}\) and therefore along any equilibrium path \(\kappa_0 > 0\). Hence, the allocations of labor and capital to all sectors are always interior.
Aggregate capital to labor ratio in the economy may be written as:

\[
k = \frac{K}{L} = \frac{L_0}{L} \kappa_0 + \frac{L_C}{L} \kappa_C, \]

\[
= l_0 \kappa_0 + (1 - l_0) \kappa_0 \frac{\kappa_C}{\kappa_0},
\]

\[
= \kappa_0 \left[ l_0 + (1 - l_0) \frac{\tilde{\alpha}/(1 - \tilde{\alpha})}{\alpha_0/(1 - \alpha_0)} \right],
\]

\[
= \kappa_0 \left[ 1 + \frac{\tilde{e}}{\kappa_0} \left( \frac{\alpha}{\alpha_0} - \frac{1 - \tilde{\alpha}}{1 - \alpha_0} \right) \right],
\]

(C.7)

where in the second equality, we have defined \( L_C \) and \( \kappa_C \) as the total employment and capital-labor ratio in the consumption sector, and in the third equality, we have used the expressions for capital-to-labor ratios in Equations (42) as well as

\[
\kappa_C = \frac{\tilde{\alpha}}{1 - \tilde{\alpha}} \frac{w}{R}.
\]

In the last equality, we have used the expression for the share of employment in the investment sector from Equation (C.5). Adjusting both sides of Equation (C.7) with respect to the productivity in the investment sector yields the desired result.

We will now show that the function defined by Equation (C.7) is one-to-one and monotonically increasing, mapping values of \( \tilde{k} \) to \( \kappa_0 \) everywhere along any equilibrium path. To show this, it is sufficient to establish that the derivative of this function with respect to \( \kappa_0 \) is everywhere strictly positive.

\[
\frac{\partial \tilde{k}}{\partial \kappa_0} = 1 + \left( \frac{\tilde{\alpha} - \alpha_0}{\alpha_0 (1 - \alpha_0)} \right) \frac{\partial}{\partial \kappa_0} \left( \frac{\tilde{r}^{1 - \alpha_0}}{\kappa_0} \right) + \left( \frac{\tilde{r}^{1 - \alpha_0}}{\kappa_0} \right) \frac{\partial}{\partial \kappa_0} \left( \frac{\tilde{r} - \alpha_0}{\alpha_0 (1 - \alpha_0)} \right),
\]

\[
= 1 + \left( \frac{\tilde{\alpha} - \alpha_0}{\alpha_0 (1 - \alpha_0)} \right) \tilde{r}^{1 - \alpha_0} \frac{\partial}{\partial \log \tilde{r}} \left( \frac{1}{\kappa_0} \right) + \frac{\tilde{r}^{1 - \alpha_0}}{\alpha_0 (1 - \alpha_0)} \frac{\partial}{\partial \log \kappa_0} \tilde{r} - \alpha_0,
\]

\[
= 1 + \left( \frac{\tilde{\alpha} - \alpha_0}{\alpha_0 (1 - \alpha_0)} \right) \frac{\tilde{r}^{1 - \alpha_0}}{\alpha_0 (1 - \alpha_0)} \left( \frac{\tilde{\alpha} - \alpha_0}{\alpha_0 (1 - \alpha_0)} - \sum_i \eta_i \eta_p \frac{\tilde{r}_0}{\alpha_0} \right) + \tilde{r}^{1 - \alpha_0} \frac{\partial}{\partial \log \alpha_0} \left( 1 + \frac{\alpha_1}{\alpha - \alpha_0} \right) \left( \frac{\alpha_0}{\alpha} \right) - \left( \frac{\alpha_1}{\alpha} \right),
\]

\[
= 1 + \left( \frac{\tilde{\alpha} - \alpha_0}{\alpha_0 (1 - \alpha_0)} \right) \left( \frac{\tilde{\alpha} - \alpha_0}{\alpha_0 (1 - \alpha_0)} - \sum_i \eta_i \eta_p \frac{\tilde{r}_0}{\alpha_0} \right) + \tilde{r}^{1 - \alpha_0} \frac{\partial}{\partial \log \alpha_0} \left( 1 + \frac{\alpha_1}{\alpha - \alpha_0} \right) \left( \frac{\alpha_0}{\alpha} \right) - \left( \frac{\alpha_1}{\alpha} \right),
\]

where in the third equality we have invoked the results of Lemma 6. Recalling the expression for the employment share of the investment sector \( l_0 \) from Equation C.5, we can now rewrite
this as follows:

\[
\frac{\partial \tilde{k}}{\partial \tilde{\kappa}_0} = 1 + (1 - l_0) \left( \frac{\bar{\alpha} - \alpha_0}{\alpha_0} - \frac{(1 - \sigma) \text{Var}(\alpha)}{\alpha_0 (1 - \bar{\alpha})} \right),
\]

\[
= l_0 + \frac{1 - l_0}{\alpha_0 (1 - \bar{\alpha})} \left[ \bar{\alpha} (1 - \bar{\alpha}) - (1 - \sigma) \text{Var}(\alpha) \right].
\]

Finally, note that the expression within the square bracket on the right hand side is always positive. This is because:

\[
\bar{\alpha} (1 - \bar{\alpha}) - (1 - \sigma) \text{Var}(\alpha) \geq \bar{\alpha} (1 - \bar{\alpha}) - \text{Var}(\alpha),
\]

\[
= \bar{\alpha} - \bar{\alpha}^2 > 0.
\]

where the inequality in the second line follows from the fact that for all sectors \(i\), we have \(0 < \alpha_i < 1\). This completes the proof that the mapping of \(\tilde{k}\) to \(\tilde{\kappa}_0\) is monotonic and one-to-one.

Equation (C.6) shows that whenever average capital share of the consumption sector exceeds that of the investment sector, the economy-wide capital-labor ratio \(k\) is greater than the capital-labor ratio in the investment sector \(\kappa_0\). Furthermore, the lemma ensures us that the relationship that the two ratios is one-to-one. This point is critical since it allows us to use the investment sector capital-labor ratio \(\tilde{\kappa}_0\) as the state variable fully characterizing the path of capital accumulation in the economy.

The next proposition characterizes the dynamics of competitive equilibria in our economy in terms of the two state variables \((\tilde{c}, \tilde{\kappa}_0)\). Before introducing the dynamic equations, let us introduce a function a function \(\chi\) of the state variables:

\[
\chi(c, \tilde{\kappa}_0; t) \equiv \frac{\tilde{\kappa}_0}{\alpha_0} \frac{\tilde{c} \tilde{\kappa}_0 (c, \tilde{\kappa}_0; t)}{\alpha_0 (1 - \alpha_0)}.
\]

This function has the property that \(\chi = \frac{\tilde{c} \tilde{\kappa}_0}{\alpha_0 (1 - \alpha_0)} = \frac{\tilde{k} \tilde{\kappa}_0 - 1}{\bar{\alpha} - \alpha_0}\) and will greatly simplify the exposition of the forthcoming lemma.

**Proposition 8.** The following system of two equations characterizes the dynamics of state
variables \((c, \tilde{\kappa}_0)\) in any competitive equilibria of our economy:

\[
\begin{align*}
-\eta \bar{u} &+ \frac{\eta g}{\bar{\varepsilon}} \left(1 + (1 - \sigma) \operatorname{Var} \left(\frac{\varepsilon}{\bar{\varepsilon}}\right)\right) \frac{\dot{c}}{c} \\
+ \left[\alpha_0 - \bar{\alpha} (1 + (1 - \sigma) \rho_{\varepsilon,\alpha})\right] \frac{\dot{\kappa}_0}{\bar{\kappa}_0} = \alpha_0 \kappa_0^{\alpha_0 - 1} - \delta + \bar{\gamma} (1 + (1 - \sigma) \rho_{\varepsilon,\gamma}) \\
- \frac{1 - \bar{\alpha} (1 + (1 - \sigma) \rho_{\varepsilon,\alpha})}{1 - \alpha_0} \gamma_0,
\end{align*}
\]

\((C.9)\)

\[
\eta g \bar{\varepsilon} \left[\bar{\alpha} (1 + (1 - \sigma) \rho_{\varepsilon,\alpha}) - \alpha_0\right] \frac{\dot{c}}{c} + \frac{1 + (1 - \bar{\alpha})(\bar{\alpha} - \alpha_0) \chi - (1 - \sigma) \operatorname{Var}(\alpha) \chi}{\bar{\kappa}_0} \frac{\dot{\kappa}_0}{\bar{\kappa}_0} = (1 - \alpha_0 (1 - \bar{\alpha}) \chi) \kappa_0^{\alpha_0 - 1} \\
- (1 + (\bar{\alpha} - \alpha_0) \chi) \left(n + \delta + \frac{\gamma_0}{1 - \alpha_0}\right) \\
+ \left[\bar{\alpha}(1 + (1 - \sigma) \rho_{\alpha,\gamma}) - \alpha_0\right] \chi \bar{\gamma},
\]

\((C.10)\)

where \(\bar{x}\) and \(\operatorname{Var}(x)\) denote the average and variance of a sector-specific set of parameters \(x\) and \(\rho_{x,x'}\) denotes the correlation coefficient between this parameter and another set of parameters \(x'\), all according to the distribution implied by sectoral expenditures shares at time \(t\), and \(\chi\) is defined by Equation \((C.8)\).

If the condition \((37)\) is satisfied (the instantaneous utility function defined in Equations \((26)\), \((31)\), and \((27)\) is concave in real consumption \(c\)), then the system above uniquely determines \((\dot{c}/c, \dot{\kappa}_0/\kappa_0)\) at time \(t\).

Proof. First, let us express the Euler Equation \((B.12)\) in terms of the variables \((c, \tilde{\kappa}_0, t)\) by substituting for the growth of sectoral prices based on the production side of our economy. From Equation \((43)\) we can write sectoral prices as

\[p_i = \tilde{\phi}_i A_i^{-1} \Lambda_0^{1 - \alpha_i} \kappa_0^{\alpha_0 - \alpha_i},\]

where \(\phi_i\) is a constant sector-specific parameter. This implies that the rate of growth of sectoral prices is given by:

\[\frac{\dot{p}_i}{p_i} = \frac{1 - \alpha_i}{1 - \alpha_0} \gamma_0 - \gamma_i + \frac{\dot{\kappa}_0}{\kappa_0} (\alpha_0 - \alpha_i).\]

This allows us to compute the average of growth rates of sectoral prices and their covariance
with income elasticity parameters under the distribution implied by expenditure shares:

\[
\frac{\dot{p}_i}{p_i} = \frac{1 - \alpha}{1 - \alpha_0} \gamma_0 - \gamma + (\alpha_0 - \alpha) \frac{\dot{k}_0}{k_0},
\]

\[
Cov\left(\varepsilon_i, \frac{\dot{p}_i}{p_i}\right) = -Cov(\varepsilon_i, \gamma_i) - \left(\frac{\gamma_0}{1 - \alpha_0} + \frac{\dot{k}_0}{k_0}\right) Cov(\varepsilon_i, \alpha_i).
\]

Substituting this relation and the fact that \(r = R - \delta = \alpha_0 \kappa_0^{-1} - \delta\) in the Euler equation, yields:

\[
-\eta\epsilon - 1 + \eta y + \eta \hat{\varepsilon} \left(1 + (1 - \sigma) Var\left(\frac{\hat{\varepsilon}}{\varepsilon}\right)\right) \frac{\dot{c}}{c} + [\alpha_0 - \alpha (1 + (1 - \sigma) \rho_{\varepsilon,\alpha})] \frac{\dot{k}_0}{k_0} = \alpha_0 \kappa_0^{-1} - (\delta + \rho) + \gamma_0 \left(1 + (1 - \sigma) \rho_{\varepsilon,\gamma}\right) \frac{1 - \alpha}{1 - \alpha_0} \gamma_0,
\]

where \(\rho_{\varepsilon,\gamma}\) and \(\rho_{\varepsilon,\alpha}\) denote correlation coefficients between the income elasticity parameters and the technological rates of growth and capital shares at the sectoral levels, both under the distributions implied by expenditure shares.

The equation governing the evolution of aggregate capital stock can be written as follows

\[
\dot{k} = \left(\frac{\dot{K}}{K} - n - \frac{\gamma_0}{1 - \alpha_0}\right) \bar{k},
\]

\[
= \frac{Y_0}{K} - \left(n + \delta + \frac{\gamma_0}{1 - \alpha_0}\right) \bar{k},
\]

\[
= \tilde{y}_0 - \left(n + \delta + \frac{\gamma_0}{1 - \alpha_0}\right) \bar{k},
\]

\[
= \tilde{k}_0 \left(1 - \alpha_0 (1 - \alpha) \chi\right) - \left(n + \delta + \frac{\gamma_0}{1 - \alpha_0}\right) \bar{k},
\]

(C.11)

where we have used \(\dot{K} = Y_0 - \delta K\) in the second equality and Equations (C.4) and (C.8) in the fourth equality. Next, we need to transform this equation into one described in terms of the per-capita stock of capital in the investment sector \(\bar{k}_0\). This will complete the characterization of the dynamics of the pair \((c, \bar{k}_0)\).

Lemma 7 established that along any equilibrium path a one-to-one mapping exists that relates a level of (productivity-adjusted) per capita stock of capital \(\bar{k}\) to a corresponding level of (productivity-adjusted) capital-per-worker \(\bar{k}_0\) in the investment sector. Taking the
derivative of this function (from Equation (C.6)) yields:

\[
\dot{k} = \dot{k}_0 + (1 - \alpha_0) \dot{k}_0^{-\alpha_0} \frac{\alpha - \alpha_0}{\alpha_0 (1 - \alpha_0)} \dot{\epsilon} + \dot{k}_0^{1-\alpha_0} \frac{\alpha - \alpha_0}{\alpha_0 (1 - \alpha_0)} \dot{\epsilon} + \frac{1}{\alpha_0 (1 - \alpha_0)} \dot{\alpha},
\]

\[
\begin{aligned}
&= \left[ \dot{k}_0 + (1 - \alpha_0) \dot{k}_0^{-\alpha_0} \frac{\alpha - \alpha_0}{\alpha_0 (1 - \alpha_0)} \right] \frac{\dot{k}_0}{\dot{k}_0} + \frac{\dot{k}_0^{1-\alpha_0} \dot{\epsilon}}{\alpha_0 (1 - \alpha_0)} \left[ (\alpha_0 - \alpha_0) \dot{\epsilon} + \alpha_0 (1 - \alpha_0) \dot{\alpha} \right].
\end{aligned}
\]  

(C.12)

Therefore, we need to compute the growth rates of expenditure \(\dot{\epsilon}/\dot{\epsilon}\) and average (consumption-sector) capital intensities \(\dot{\alpha}/\dot{\alpha}\) in terms of the growth rates of real consumption \(\dot{c}/c\) and investment-sector capital-to-labor ratio \(\dot{k}_0/\dot{k}_0\).

Now, we can use the expressions for the elasticity of function \(\varpi\) with respect to real consumption and price, from Lemma 6, to find:

\[
\frac{\dot{\alpha}}{\dot{\alpha}} = \frac{\alpha_c \cdot \dot{c}}{\dot{c}} + \sum_i \eta_{\alpha} \frac{\dot{p}_i}{p_i},
\]

\[
= \frac{1}{\varpi} \eta_0 \mathrm{Cov} (\varv, \alpha) \cdot \dot{c} + (1 - \sigma) \sum_i \left( \frac{\alpha_i}{\alpha} - 1 \right) \omega_i \left[ \frac{1 - \alpha_i}{1 - \alpha_0} \frac{A_0}{A_0 - \dot{A}_i} + (\alpha_0 - \alpha_i) \frac{\dot{k}_0}{\dot{k}_0} \right],
\]

\[
= (1 - \sigma) \eta_0 \epsilon_{\varv \alpha} \cdot \dot{c} - \frac{1}{\varpi} \left[ \mathrm{Cov} (\alpha_\gamma) + \left( \frac{\gamma_0}{1 - \alpha_0} + \frac{\dot{k}_0}{\dot{k}_0} \right) \mathrm{Var} (\alpha) \right].
\]

Similarly, we can write the growth rate of consumption expenditure as:

\[
\frac{\dot{\epsilon}}{\dot{c}} = \frac{\eta_c \cdot \dot{c}}{\dot{c}} + \sum_i \eta_c \frac{\dot{p}_i}{p_i} - \frac{1}{1 - \alpha_0} \frac{\dot{A}_0}{A_0},
\]

\[
= \eta_0 \epsilon \cdot \dot{c} + \sum_i \omega_i \left[ \frac{1 - \alpha_i}{1 - \alpha_0} \frac{A_0}{A_0 - \dot{A}_i} + (\alpha_0 - \alpha_i) \frac{\dot{k}_0}{\dot{k}_0} \right] - \frac{1}{1 - \alpha_0} \frac{\dot{A}_0}{A_0},
\]

\[
= \eta_0 \epsilon \cdot \dot{c} - \chi + (\alpha_0 - \varpi) \frac{\dot{k}_0}{\dot{k}_0} - \frac{\alpha_0 \gamma_0}{1 - \alpha_0}.
\]

We can write Equation (C.12) as:

\[
\frac{\dot{k}}{\dot{k}_0} = \left[ 1 + (1 - \alpha_0) (\alpha - \alpha_0) \chi \right] \frac{\dot{k}_0}{\dot{k}_0} + \chi \left[ (\alpha - \alpha_0) \frac{\dot{\epsilon}}{\dot{c}} + \frac{\dot{\alpha}}{\dot{\alpha}} \right],
\]

\[
= \left[ 1 + (1 - \alpha) (\alpha - \alpha_0) \chi - (1 - \sigma) \mathrm{Var} (\alpha) \right] \frac{\dot{k}_0}{\dot{k}_0} + \eta_0 \epsilon \left[ \alpha (1 + (1 - \sigma) \rho_{\varv \alpha}) - \alpha_0 \right] \frac{\dot{\epsilon}}{\dot{c}}\]

\[
- \left[ \alpha (1 + (1 - \sigma) \rho_{\alpha \gamma}) - \alpha_0 \right] \chi \gamma - \left[ \alpha (\alpha - \alpha_0) + (1 - \sigma) \mathrm{Var} (\alpha) \right] \chi \gamma_0.
\]
Finally, substituting this expression into Equation (C.11) give us:

\[
\left[ 1 + (1 - \bar{\alpha})(\bar{\alpha} - \alpha_0) \chi - (1 - \sigma) Var(\alpha) \right] \frac{\dot{\kappa}_0}{\bar{\kappa}_0} \\
+ \eta g \bar{\alpha} \left[ (1 + (1 - \sigma)\rho_{\xi,\alpha}) - \alpha_0 \right] \chi \frac{\dot{c}}{c} = (1 - \alpha_0 (1 - \bar{\alpha}) \chi) \bar{\kappa}_0^{\alpha_0 - 1} \\
- (1 + (\bar{\alpha} - \alpha_0) \chi) \left( \rho + \delta + \frac{\gamma_0}{1 - \alpha_0} \right) \\
+ [\bar{\alpha} (\bar{\alpha} - \alpha_0) + (1 - \sigma) Var(\alpha)] \chi \gamma_0 \\
+ [\bar{\alpha} (1 + (1 - \sigma)\rho_{\alpha,\gamma}) - \alpha_0] \chi \gamma.
\]

where we have used the fact that \( l_0 = 1 - l_C = 1 - \alpha_0 (1 - \bar{\alpha}) \chi \) and \( \tilde{\kappa}/\bar{\kappa}_0 = 1 + (\bar{\alpha} - \alpha_0) \chi \).

To ensure that the system of Equations above indeed has a solution, we need to establish that the determinant of the following matrix is nonzero:

\[
M = \begin{bmatrix}
-\eta g \bar{\alpha} \left[ (1 + (1 - \sigma)\rho_{\xi,\alpha}) - \alpha_0 \right] \chi & \alpha_0 - \bar{\alpha} (1 + (1 - \sigma)\rho_{\xi,\alpha}) \\
\eta g \bar{\alpha} \left[ (1 + (1 - \sigma)\rho_{\xi,\alpha}) - \alpha_0 \right] \chi & 1 + (1 - \bar{\alpha}) (\bar{\alpha} - \alpha_0) \chi - (1 - \sigma) Var(\alpha) \chi
\end{bmatrix}.
\]

A necessary condition for the Euler equation to have a unique solution is that the expression \( M_{1,1} \) is strictly positive. Since \( M_{1,2} \cdot M_{2,1} \leq 0 \) and \( M_{2,2} = \frac{\partial k}{\partial \kappa_0} > 0 \) (from Lemma 7), this is sufficient to ensure that the determinant of matrix \( M \) is indeed positive.

Starting from initial level of per capita stock of capital \( k(0) \), Equation (C.6) and a choice of \( c(0) \), give the corresponding allocation of capital to the investment sector \( \kappa_0 \). For all \( t \geq 0 \), Equations (C.9) and (C.10) describe the dynamics of the economy in terms of \( (c, \tilde{\kappa}_0) \).

The two state variables \( (c, \tilde{\kappa}_0) \) at time \( t \) are sufficient to fully specify the economy. As we discussed, both \( \tilde{\xi} \) and \( \tilde{\alpha} \) are functions of \( c \) and \( \tilde{\kappa}_0 \), with the dependence on the latter going through the dependence of functions (29) on prices, as specified by Equation (43). Knowing the two state variables at any given time \( t \), capital and labor employed in each consumption good sector \( i \) may be written as:

\[
K_i = \left( \frac{\alpha_i}{\alpha_0} \omega_i (c, \tilde{\kappa}_0) \cdot \tilde{\varepsilon} (c, \tilde{\kappa}_0) \right) \cdot LA_0 \frac{1}{1 - \alpha_0} \nu_0 ,
\]
\[
L_i = \left( \frac{1 - \alpha_i}{1 - \alpha_0} \omega_i (c, \tilde{\kappa}_0) \cdot \tilde{\varepsilon} (c, \tilde{\kappa}_0) \right) \cdot L ,
\]

which completes the characterization of the economy.
Table 7: Estimation by Quartiles and Sub-periods  

<table>
<thead>
<tr>
<th></th>
<th>Quartiles</th>
<th>Pre 93</th>
<th>Post 93</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.63</td>
<td>0.76</td>
<td>0.75</td>
</tr>
<tr>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.01)</td>
</tr>
<tr>
<td>Food</td>
<td>-0.44</td>
<td>-0.31</td>
<td>-0.42</td>
</tr>
<tr>
<td>(0.02)</td>
<td>(0.07)</td>
<td>(0.09)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>Housing</td>
<td>-0.20</td>
<td>-0.44</td>
<td>-0.38</td>
</tr>
<tr>
<td>(0.02)</td>
<td>(0.06)</td>
<td>(0.07)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>Services</td>
<td>0.47</td>
<td>0.74</td>
<td>0.78</td>
</tr>
<tr>
<td>(0.03)</td>
<td>(0.15)</td>
<td>(0.18)</td>
<td>(0.04)</td>
</tr>
</tbody>
</table>

Note: Std. Errors clustered at the household level. Elasticity estimates are relative to non-durables consumption. (1) Corresponds to the first quartile, (2), to the second, etc. All estimates contain household and time-sector fixed effects. Data from Heathcote et al. (2010).
Figure 4: Regression Fit for Japan using common world parameters \( \{\sigma, \epsilon_a, \epsilon_m, \epsilon_s\} \)

(a) Regression Fit using all regressors

(b) Relative Prices

(c) Consumption

(d) Net Exports

(e) Partial fit: Prices only

(f) Partial fit: Consumption only

(g) Partial fit: Net Exports only
Figure 5: Baseline Country Fit
Figure 6: Baseline Country Fit (ct’d)
Table 8: 10-Sector Regression

<table>
<thead>
<tr>
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<th>World</th>
<th>OECD</th>
<th>Non-OECD</th>
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</thead>
<tbody>
<tr>
<td>Price Elasticity $\sigma$</td>
<td>0.82</td>
<td>0.84</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.04)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>Income Elasticity</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>(relative to Manufacturing)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Mining</td>
<td>-1.43</td>
<td>-1.14</td>
<td>-1.78</td>
</tr>
<tr>
<td></td>
<td>(0.17)</td>
<td>(0.19)</td>
<td>(0.28)</td>
</tr>
<tr>
<td>Agriculture</td>
<td>-0.96</td>
<td>-0.82</td>
<td>-1.15</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.17)</td>
<td>(0.15)</td>
</tr>
<tr>
<td>Public Utilities</td>
<td>-0.02</td>
<td>0.12</td>
<td>-0.20</td>
</tr>
<tr>
<td></td>
<td>(0.08)</td>
<td>(0.07)</td>
<td>(0.14)</td>
</tr>
<tr>
<td>Transp., Storage, Communications</td>
<td>0.10</td>
<td>0.32</td>
<td>-0.17</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(0.11)</td>
<td>(0.15)</td>
</tr>
<tr>
<td>Government Services</td>
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<td>0.50</td>
<td>-0.36</td>
</tr>
<tr>
<td></td>
<td>(0.21)</td>
<td>(0.20)</td>
<td>(0.26)</td>
</tr>
<tr>
<td>Construction</td>
<td>0.18</td>
<td>0.26</td>
<td>0.12</td>
</tr>
<tr>
<td></td>
<td>(0.09)</td>
<td>(0.08)</td>
<td>(0.13)</td>
</tr>
<tr>
<td>Wholesale and Retail</td>
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<td>0.56</td>
<td>0.12</td>
</tr>
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<td></td>
<td>(0.13)</td>
<td>(0.17)</td>
<td>(0.20)</td>
</tr>
<tr>
<td>Community, Social and Personal Serv.</td>
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<td>0.83</td>
<td>-0.03</td>
</tr>
<tr>
<td></td>
<td>(0.14)</td>
<td>(0.26)</td>
<td>(0.07)</td>
</tr>
<tr>
<td>Finance, Insurance, Real State</td>
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<td>1.19</td>
<td>0.64</td>
</tr>
<tr>
<td></td>
<td>(0.15)</td>
<td>(0.25)</td>
<td>(0.17)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.95</td>
<td>0.91</td>
<td>0.93</td>
</tr>
</tbody>
</table>

Note: All sectoral elasticities computed relative to Manufacturing. Standard errors clustered at the country level. All regressions include a sector-country fixed effect. Source: GDDC 10-Sector database (Vries et al., 2014).