Turbulence Modeling via the Fractional Laplacian

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Herein, we derive the fractional Laplacian operator as a means to represent the ensemble-averaged friction force arising in a turbulent flow:

$$\rho \frac{D\bar{u}}{Dt} = -\nabla \bar{p} + \mu \nabla^2 \bar{u} + \rho C_\alpha \int \int \int \frac{\bar{u}(t,x') - \bar{u}(t,x)}{|x' - x|^\alpha + 3} \, dx' \tag{0.1}$$

where $\bar{u}(t,x)$ is the ensemble-averaged velocity and $C_\alpha$ is a turbulent mixing coefficient (with units (length)$^\alpha$/time). The derivation is grounded in Boltzmann kinetic theory, which presumes an equilibrium probability distribution of particle speeds, $f^{eq}_\alpha(t,x,\mathbf{u})$. While historically $f^{eq}_\alpha$ has been assumed to be the Maxwell-Boltzmann (normal) distribution, we show that any member of the family of Lévy $\alpha$-stable distributions is a suitable alternative, with parameter $\alpha$ selecting the distribution. If $\alpha = 2$, then $f^{eq}_\alpha$ is the Maxwell-Boltzmann distribution, with large particle speeds very unlikely, and equation (0.1) reverts to the Navier-Stokes equations (with $C_\alpha = 0$). If $0 < \alpha < 2$, then $f^{eq}_\alpha$ is a Lévy $\alpha$-stable distribution, with “heavy tails” that permit large velocity fluctuations, as in turbulence. For shear turbulence, the choice $\alpha = 1$ (Cauchy distribution for $f^{eq}_\alpha$) leads to the logarithmic velocity profile known as the Law of the Wall. Since equation (0.1) is restricted to unbounded flows, we also consider unidirectional shear flow and derive additional operators that account for boundary conditions. We then present examples of 1D Couette flow and 2D boundary layer flow. Turbulent transport of passive scalars is also discussed within this kinetic theory framework, and we derive the fractional Laplacian as a means to represent the ensemble-averaged dispersion in a turbulent flow. This work lays out a new framework for turbulence modeling that will hopefully lead to new methods for analysis of turbulent flows.

Key words:

1. Introduction

1.1. Literature Review

While it is widely accepted that the Navier-Stokes equations describe turbulent fluid flows, it would take lifetimes to perform direct numerical simulation of Navier-Stokes for high-Reynolds-number flows of engineering and environmental interest. Thus, a wide array of turbulence models have been proposed (Tennekes & Lumley 1972; Pope 2000; Wilcox 2006; McDonough 2007). However, any path that follows Osborne Reynolds’ (1895) method of averaging the governing equations, and then solving, leads to the closure

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small steps yield Brownian motion and molecular diffusion island clusters of small steps consistent with incoherent fluid motion

Figure 1: Fluid particle trajectories with steps drawn from a Maxwell-Boltzmann distribution (left) and Lévy $\alpha = 3/2$ distribution (right). Both trajectories have 7000 steps. Figure from (Chechkin et al. 2008), with annotations.

problem, with more unknowns than equations. Our approach is reverse of Reynolds’; starting with the Boltzmann equation from kinetic theory, we first solve and then ensemble average. The result of our analysis is equation (0.1), which governs the ensemble-averaged flowfield and uses the fractional Laplacian to represent turbulent friction. It should be emphasized that this operator is not chosen ad hoc but rather is derived from the first principles of Boltzmann kinetic theory and the statistics of turbulent transport. Our premise is that turbulence is a non-local phenomenon, statistically bringing together distant fluid particles and causing exchanges of mass and momentum from distant parts of the flowfield. Thus, this literature review focuses on three key ideas: (i) the description of anomalous diffusion using Lévy $\alpha$-stable distributions; (ii) non-local turbulence closure models, including fractional derivatives; and (iii) the kinetic theory framework for turbulence modeling.

Turbulence exhibits anomalous diffusion, which is diffusion that occurs over distances $\xi$ that scale by time to a power other than one half, i.e. not $\xi \sim t^{1/2}$ (Zaslavsky 2002). For example, air pollution dispersed from a point source will disperse in a patch that grows linearly with time, $\xi \sim t$ (Kämpf & Cox 2016). This so called super-diffusion can be accurately modeled using a fractional Laplacian for the diffusion operator (in lieu of the regular Laplacian, which results in the usual $\xi \sim t^{1/2}$ Fickian diffusion) (Cushman-Roisin 2008, 2013). In another famous example, Richardson (1926) observed that the root-mean-square distance separating fluid particles initially near one another in a turbulent atmosphere scales as $\xi \sim t^{3/2}$. Shlesinger et al. (1987) formally showed that Richardson’s law could be derived assuming the turbulence followed the $\alpha = 3/2$ Lévy distribution. In their derivation, they assumed that the trajectory of a fluid particle follows a Lévy walk, characterized by occasional large steps due to coherent fluid motion (see Figure 1). Lévy walks are random walks where the particle moves with constant velocity for random periods of time, instantly choosing another random velocity at each turning point (collision with another particle within the context of Boltzmann kinetics).

Lévy $\alpha$-stable distributions (for $0 < \alpha < 2$) are probability distributions that describe fluctuating processes characterized by large bursts or outliers, such as turbulence (Lévy 1937; Chechkin et al. 2008; Nolan 2017; Shintani & Umeno 2017). For example, wind speeds have been shown to be Lévy-distributed with $1.5 \lesssim \alpha \lesssim 1.72$ (Boettcher et al. 2003; Metzler et al. 2009; Blackledge et al. 2011). Lévy-distributed velocity fluctuations with $\alpha \approx 1$ have been observed in studies such as: turbulent pipe flow behind a grid (Tong & Goldburg 1988; Onuki 1988); flowfield induced by a large number of point vortices (Min et al. 1996); and Couette flow between parallel rotating disks (Mordant et al. 2001).

A number of non-local turbulence models have been explored in the literature, including the fractional Laplacian. Without derivation, W. Chen (2006) speculated that inertial-range turbulence could be modeled by the fractional Laplacian with $\alpha = 2/3$. Recently,
Churbanov & Vabishchevich (2016) used the fractional Laplacian to model turbulent flow in a rectangular duct, with \( \alpha = 1/4 \) (chosen \textit{ad hoc}) showing reasonable agreement with experiments. Similarly, Xu et al. (2017) analyzed plane Poiseuille flow with encouraging results. Modification of Navier-Stokes with a fractional time derivative has also been proposed without derivation (El-Shahed & Salem 2004; Kumar et al. 2015). However it is important to note that the literature neither offers a rigorous derivation of the fractional Laplacian as a turbulence model, nor offers a theory as to how to choose the fractional order \( \alpha \).

A number of non-local diffusion operators have also been proposed. Based on phenomenological arguments, Schumer et al. (2003, 2009) model advection-dispersion using the fractional Laplacian. The forms of other diffusion operators stem from the fact that the Fourier transform of the regular Laplacian \( \mu \nabla^2 u \) is \( -\mu|k|^2 \hat{u} \), where \( k \) is the wavenumber.† Thus, Berkowicz & Prahm (1980) proposed \( -\mu(k)|k|^2 \hat{u} \), where \( \mu(k) \) is an \textit{ad hoc} function of wavenumber \( k \). Following suit, several workers proposed various forms for \( \mu(k) \), differing essentially in the manner by which they construct \( \mu(k) \) and the number of tunable parameters that it contains (Fiedler 1984; Stull 1984, 1993; Nakayama & Bandou 1995). One model of note is that of Cushman-Roisin & Jenkins (2006): \[
F(z) = \frac{\rho A}{\pi} \int_0^\infty \frac{|u(z') - u(z)(u(z') - u(z))}{(z' - z)^2} dz'.
\]
For wall-bounded shear flows, the solution of \( F(z) = 0 \) is the logarithmic profile \( u(z) = \ln(z) \), in agreement with the \textit{Law of the Wall}.

Efforts have been made to model turbulence using \textit{Boltzmann kinetic theory}. Several authors have pursued numerical solution of the \textit{Boltzmann equation} with different collision operators (right hand sides), including: non-local collision operator (Hayot & Wagner 1996); fractional time derivative (Baule & Friedrich 2006); orthogonal projector (Degond & Lemou 2002); or an integral forcing term (Tsugé & Sagara 1976; Srinivasan 1966). In order to describe Reynolds stresses using kinetic theory, Girimaji (2007) decomposes the Boltzmann equation into filtered and unresolved parts, and he shows direct correspondence between the resulting kinetic model and the Reynolds stresses.

Another noteworthy series of papers (published in this journal) is that of H. Chen and colleagues, who explored the Boltzmann equation with Maxwell-Boltzmann \( f^{eq} \) but long collision time \( \tau \). H. Chen et al. (2004) extended the \textit{Chapman-Enskog expansion} \( f = f^{eq} + \epsilon f^{(1)} + \ldots \) to include the second order term \( \epsilon^2 f^{(2)} \) and found that using a finite collision time \( \tau \) leads to memory effects and nonlinear constitutive relations consistent with turbulence. H. Chen et al. (2007, 2010) provided the exact solution of the Boltzmann equation (4.10) and used it to evaluate the stress tensor (4.9). Considering Maxwell-Boltzmann \( f_{\alpha}^{eq} \) and finite \( \tau \), they determined an integro-differential equation that reduces to the Navier-Stokes equations in the limit \( \tau \to 0 \). H. Chen et al. (2013) applied their model (finite \( \tau \), Gaussian \( f_{\alpha}^{eq} \)) to Couette flow and found that (i) for small \( \tau \), their theory replicated a linear velocity profile consistent with Navier-Stokes, as expected; but (ii) for large \( \tau \), the velocity profile was still mostly linear but exhibited slip along the walls (not in agreement with experimental observations (Robertson & Johnson 1970) that reveal an S-shape velocity profile).

1.2. Key Ideas and Assumptions

Several elements distinguish this work from the literature: While the fractional Laplacian has been used previously, we provide the first rigorous derivation of the fractional Laplacian for the representation of turbulence. Moreover, we provide the first theory rigorously justifying the choice of parameter \( \alpha \): The appropriate \( \alpha \) is the one that corresponds to the scaling \( \xi \sim t^{1/\alpha} \) of the observed macro-scale transport. Our derivation

† The Fourier transform of the \textit{fractional Laplacian} is \(-|k|^\alpha \hat{u} \).
is rooted in the Boltzmann equation and Lévy statistics, two ideas that have been used to describe turbulence for some time but never combined as they are herein. Finally, in contrast to H. Chen and colleagues who considered a large $\tau$ and Maxwell-Boltzmann $f_\alpha^{eq}$, this paper considers small $\tau$ and other Lévy $\alpha$-stable $f_\alpha^{eq}$.

The key ideas that lead to equation (0.1) are as follows: Our foundation is Boltzmann kinetic theory, which has been shown to lead to the Navier-Stokes equations if the distribution of molecular speeds $f_\alpha^{eq}$ is assumed to be a Maxwell-Boltzmann (normal) distribution. However, the Maxwell-Boltzmann $f_\alpha^{eq}$ is uniquely tied to $\xi \sim t^{1/2}$ diffusion. In order to impose the desired $\xi \sim t^{1/\alpha}$ diffusion scaling, we model $f_\alpha^{eq}$ using the broader family of Lévy $\alpha$-stable distributions (of which the Maxwell-Boltzmann distribution is the $\alpha = 2$ member). The key algebra “trick” in evaluating the stress field is to separately consider “small” and “large” particle displacements; the small displacements produce the usual viscous stress, whereas the large displacements produce a turbulent stress that leads to the fractional Laplacian.

### 1.3. Organization of this Article

In §2 we develop a number of key ideas by considering turbulent transport of passive scalars. Section §3 provides a bridge between turbulent transport (§2) and Boltzmann kinetics (§4). In §4, we use Boltzmann kinetics to derive the fractional Laplacian as a model for the ensemble-averaged friction force arising in an unbounded turbulent flow. In Section §5, we consider boundary conditions through solution of the Boltzmann equation for the simple case of unidirectional shear flow. Numerical examples provided in §6 (for 1D Couette flow and 2D boundary layer flow) show excellent agreement with laboratory data. Conclusions are offered in §7, and appendices provide supplemental information.

### 2. Turbulent Transport

In this section, we consider the transport of passive scalars in a turbulent flow (Epps & Cushman-Roisin 2017). Through this warm-up problem, we introduce several ideas, including the probability and phase-space fundamentals needed for the more complex analysis of momentum transport in §4. We derive the requirement that the probability distribution describing particle velocities must be stable and then show that this stability constraint is satisfied by the Lévy $\alpha$-stable distributions. Further, we develop the hypothesis that these probability distributions are self-similar and that a similarity variable can be defined that links the macro-scale transport with the micro-scale particle motions. Finally, we conclude that turbulent diffusion can be described by a fractional Laplacian operator, paving the way for the momentum analysis in §4.

Our premise is to consider transport of a conserved quantity $c(t,x,u)$ being advected by a random flow field, $u$. Let $c(t,x,u)$ be a passive scalar quantity, such as the concentration of some species (of particles with velocity $u$). The 3D transport equation is

$$\frac{\partial c}{\partial t} + u \cdot \nabla c = 0$$

with zero right hand side to indicate no changes to $c$ due to sources/sinks. Random velocity $u = [u,v,w]$ is a 3D random variable governed by probability density function (pdf) $p_u(u)$, which denotes the joint pdf of $p_{u(u,v,w)}$. At this point, we require two conditions on $p_u(u)$:

- **Normalization:**

$$\iiint_{-\infty}^{\infty} p_u(u)d\mathbf{u} = 1 .$$

(2.2)
• **Isotropy:** That is, \( p_u(u) \) is a function of the magnitude of the velocity deviation \( |u - \bar{u}| \), where \( \bar{u}(t, x) \) is the mean flow speed.

The observable average, \( \bar{c}(t, x) \), is the weighted average of the \( c(t, x; u) \) corresponding to all possible velocities \( u \):

\[
\bar{c}(t, x) = \int \int_{-\infty}^{\infty} c(t, x; u)p_u(u)\, du .
\]  

(2.3)

We would like to determine the ensemble-averaged behavior of (2.1) using the probability/phase-space framework. Our general strategy is the reverse of Osborne Reynolds (1895): While he averaged the governing equations and then solved, we solve equation (2.1) and then ensemble average. For any single realization of random velocity \( u \) over time interval \( \delta t \), the solution of (2.1) is

\[
c(t + \delta t, x; u) = c(t, x - u\delta t; u) ,
\]  

(2.4)

which merely states that \( c \) is advected by the flow.

The ensemble-averaged behavior of (2.1) can be deduced as follows. First, construct the ensemble-averaged time derivative consistent with the rules of calculus:

\[
\frac{\partial \bar{c}}{\partial t} = \lim_{\delta t \to 0} \frac{\bar{c}(t + \delta t, x) - \bar{c}(t, x)}{\delta t} .
\]  

(2.5)

Using (2.3) and (2.4) to evaluate (2.5), we can write

\[
\frac{\partial \bar{c}}{\partial t} = \lim_{\delta t \to 0} \int \int_{-\infty}^{\infty} \frac{c(t, x - u\delta t; u) - c(t, x; u)}{\delta t}p_u(u)\, du .
\]  

(2.6)

To proceed, we need to constrain the probability distribution \( p_u(u) \).

### 2.1. Stability Constraint

Note that as \( \delta t \to 0 \) in (2.6), the probability \( p_u(u) \) must remain non-trivial. This requires that \( p_u(u) \) must be a **stable distribution**. A probability distribution is **stable** if the shape of the distribution is preserved under addition. That is, if \( X_1 \) and \( X_2 \) are two independent random variables drawn from a stable distribution, then their sum \( X_1 + X_2 \) also follows that distribution (up to scale and shift) (Nolan 2017).

In order to better expose this requirement, we switch from velocity to the corresponding displacement:

\[
\xi = u\, \delta t .
\]  

(2.7)

The displacement is governed by a pdf \( p_\xi(\xi; \delta t) \), where \( p_\xi(\xi; \delta t)\, d\xi = p_u(u)\, du \) and \( d\xi = d\xi_1\, d\xi_2\, d\xi_3 = (du\, \delta t)(dv\, \delta t)(dw\, \delta t) = (\delta t)^3 du \). Thus,

\[
p_\xi(\xi; \delta t) = p_u(u = \xi / \delta t) / (\delta t)^3 .
\]  

(2.8)

We simultaneously require that as \( \delta t \to 0 \), \( p_\xi(\xi; \delta t) \) becomes a Dirac function \( \delta(\xi - \xi^*) \), while \( p_\xi(\xi; \delta t) / \delta t \) remains finite. Together, these requirements produce streaming to the mean displacement \( \xi^* \) and finite diffusion about that location. In order for these constraints to be simultaneously met, \( p_\xi(\xi; \delta t) \) must be a **stable distribution**.

Realizing that the shift \( \xi^* = \bar{u} \delta t \) represents the mean motion, we simplify the exposition by introducing the displacement deviation \( \tilde{\xi} = \xi - \xi^* \) and corresponding probability distribution, \( p_\tilde{\xi}(\tilde{\xi}; \delta t) = p_\xi(\xi = \xi^* + \tilde{\xi}; \delta t) \). Then, distribution \( p_\tilde{\xi}(\tilde{\xi}; \delta t) \) must be a stable distribution with zero shift.

The simplest way of ensuring that \( p_\tilde{\xi}(\tilde{\xi}; \delta t) \) be a stable distribution is to require that the displacement deviation \( \tilde{\xi} \) in time interval \( \delta t \) follows the same pdf as the displacement
Table 1: Physical parameters and similarity variables associated with molecular diffusion, shear turbulence, and inertial range turbulence.

<table>
<thead>
<tr>
<th>physical parameter</th>
<th>units</th>
<th>similarity variable</th>
<th>exponent</th>
<th>distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>kinematic viscosity, (\nu)</td>
<td>(L^2/T)</td>
<td>(\eta = \xi/(\nu \delta t)^{\frac{1}{2}})</td>
<td>(\alpha = 2)</td>
<td>Maxwell-Boltzmann</td>
</tr>
<tr>
<td>friction velocity, (u_r)</td>
<td>(L/T)</td>
<td>(\eta = \xi/(u_r \delta t))</td>
<td>(\alpha = 1)</td>
<td>Cauchy</td>
</tr>
<tr>
<td>energy dissipation rate, (\epsilon)</td>
<td>(L^3/T)</td>
<td>(\eta = \xi/(\epsilon^{\frac{1}{3}} \delta t)^{\frac{2}{3}})</td>
<td>(\alpha = \frac{2}{3})</td>
<td>other Lévy</td>
</tr>
<tr>
<td>general, (q)</td>
<td>(L^\alpha/T)</td>
<td>(\eta = \xi/(q \delta t)^{1/\alpha})</td>
<td>(\alpha)</td>
<td>other Lévy</td>
</tr>
</tbody>
</table>

In half of that time interval. In other words, the probability of jump \(\tilde{\xi}\) in time \(\delta t\) is given by the convolution of the probabilities of all possible intermediate jumps \(\xi\) and then complementary jumps \(\xi - \tilde{\xi}\) made in successive half intervals \(\delta t/2\).†

\[
p_{\xi}(\xi; \delta t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{\xi}(\xi'; \delta t/2)p_{\xi}(\xi - \xi'; \delta t/2) \, d\xi'.
\] (2.9)

If the time interval can be halved, it can be halved once more, and so forth \(ad \, infinitum\) to reach the limit \(\delta t \to 0\).

For applicability across systems of different sizes and flow fields, there must exist a general formulation in which the structure of the pdf remains the same irrespective of the spatial size and strength of the flowfield. This necessitates the existence of a dimensionless variable with a canonical pdf. This in turn necessitates the reliance on a dimensional quantity that can be used for scaling. Table 1 lists three such possible scalings. In general, we invoke the existence of a dimensional quantity, \(q\), with dimensions \((\text{length})^{\alpha}/(\text{time})\), from which we can form the scaled displacement (similarity variable) \(\tilde{\eta}\) from the dimensional displacement \(\xi\):

\[
\tilde{\eta} \equiv \frac{\xi}{(q \delta t)^{1/\alpha}}.
\] (2.10)

We assert that (2.10) holds in general, with parameter \(\alpha\) describing the temporal scaling of the turbulent transport. This hypothesis is well justified by experimental evidence for \(\alpha = 2, 1, \text{ and } 2/3\). For laminar diffusion \(\alpha = 2\), momentum transport scales as \(\xi \sim (\nu \delta t)^{\frac{1}{2}}\). For wall bounded turbulent flows \(\alpha = 1\), turbulent transport scales as \(\xi \sim u_r \delta t\), where \(u_r \equiv \sqrt{\tau_w/\rho}\) is the friction velocity.‡ For turbulent dispersion, Richardson’s (1926) observations showed \(\xi \sim L^{3/2}\) scaling, corresponding to \(\alpha = 2/3\); from the Kolmogorov energy cascade of inertial turbulence, the pertinent dimensional quantity is the energy dissipation rate \(\epsilon\), giving \(\xi \sim (\epsilon^{1/3} \delta t)^{3/2}\) consistent with Richardson’s observations.

In order to solve equation (2.9), we must recast it in terms of a universal probability density function \(p_{\eta}(\eta)\) that governs the scaled displacement \(\tilde{\eta}\). So we define \(p_{\eta}(\eta)\) such that \(p_{\eta}(\eta) \, d\eta = p_{\xi}(\xi; \delta t) \, d\xi\) with \(d\eta = d\eta_1 \, d\eta_2 \, d\eta_3 = d\xi_1 \, d\xi_2 \, d\xi_3/(q \delta t)^{3/\alpha} = d\xi/(q \delta t)^{3/\alpha}\). Thus \(d\xi = (q \delta t)^{3/\alpha} d\eta\), and

\[
p_{\xi}(\xi; \delta t) = p_{\eta}(\eta = \xi/(q \delta t)^{1/\alpha})/(q \delta t)^{3/\alpha}.
\] (2.11)

† Properly stated, the stability constraint (2.9) should hold for any initial jump displacement \(\tilde{\xi}\) in time lapse \(\delta t\), followed by complementary \(\xi - \tilde{\xi}\) in \(\delta t - \delta t\). It is straightforward to show that solutions to (2.9) also satisfy this more general requirement. See Appendix A.1.

‡ The relevant physical constant is the wall stress, \(\tau_w\), which is repackaged as a friction velocity, \(u_r \equiv \sqrt{\tau_w/\rho}\) in order to have the correct units: \((\text{length})^{1}/(\text{time})\).
Turbulence Modeling via the Fractional Laplacian

To convert $p_{\xi}(\xi', \delta t/2)$, we need to determine the scaled parameter that corresponds to a jump of $\xi'$ in half time interval $\delta t/2$; this value is $\xi'/(q \delta t/2)^{1/\alpha} = 2^{1/\alpha} \tilde{\xi}' / (q \delta t)^{1/\alpha} = 2^{1/\alpha} \tilde{\eta}'$. Thus, $p_{\xi}(\xi', \delta t/2) = p_{\tilde{\eta}}(2^{1/\alpha} \tilde{\eta}') / (q \delta t/2)^{3/\alpha}$. Using these conversions, equation (2.9) can be written as

$$ \frac{p_{\tilde{\eta}}(\tilde{\eta})}{(q \delta t)^{3/\alpha}} = \iiint_{-\infty}^{\infty} \frac{p_{\tilde{\eta}}(2^{1/\alpha} \tilde{\eta}')}{(q \delta t/2)^{3/\alpha}} \frac{p_{\tilde{\eta}}(2^{1/\alpha}(\tilde{\eta} - \tilde{\eta}'))}{(q \delta t/2)^{3/\alpha}} \, d\tilde{\eta}'(q \delta t)^{3/\alpha}, \quad (2.12) $$

which simplifies to

$$ p_{\tilde{\eta}}(\tilde{\eta}) = \iiint_{-\infty}^{\infty} 2^{6/\alpha} p_{\tilde{\eta}}(2^{1/alpha} \tilde{\eta}') p_{\tilde{\eta}}(2^{1/alpha}(\tilde{\eta} - \tilde{\eta}')) \, d\tilde{\eta}' . \quad (2.13) $$

Note that with this similarity form, the dimensional quantities $q$ and $\delta t$ have been eliminated. We can put (2.13) into a more friendly form by setting $\tilde{\eta} = 2^{1/\alpha} \tilde{\eta}'$ such that $d\tilde{\eta}' = 2^{3/\alpha} d\tilde{\eta}$. Upon doing so and immediately replacing the checks with tildes, equation (2.13) simplifies to the stability constraint equation:

$$ 2^{-3/\alpha} p_{\tilde{\eta}}(2^{-1/\alpha} \tilde{\eta}) = \iiint_{-\infty}^{\infty} p_{\tilde{\eta}}(\tilde{\eta}') p_{\tilde{\eta}}(\tilde{\eta} - \tilde{\eta}') \, d\tilde{\eta}' . \quad (2.14) $$

Equation (2.14) provides a constraint on which probability distributions are admissible. Solutions to (2.14) may be used to model laminar through turbulent flows.

2.2. Lévy $\alpha$-stable Distributions

In this subsection, we show that the solution to the stability constraint equation (2.14) is the Lévy $\alpha$-stable distribution. The derivation is straightforward: take the Fourier transform of (2.14), solve in Fourier space, and inverse Fourier transform the solution.

Herein, we adopt the unitary form of the Fourier transform (in 3-dimensional space):

- **Transform:** $\mathcal{F}\{p_{\tilde{\eta}}(\tilde{\eta})\}(k) \equiv \widehat{p}_{\tilde{\eta}}(k) \equiv \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} p_{\tilde{\eta}}(\tilde{\eta}) e^{-ik \cdot \tilde{\eta}} \, d\tilde{\eta} ; \quad (2.15) $
- **Inverse transform:** $\mathcal{F}^{-1}\{\widehat{p}_{\tilde{\eta}}(k)\}(\tilde{\eta}) \equiv p_{\tilde{\eta}}(\tilde{\eta}) \equiv \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} \widehat{p}_{\tilde{\eta}}(k) e^{ik \cdot \tilde{\eta}} \, dk . \quad (2.16) $

In order to evaluate the Fourier transform of (2.14), we will use the scale property: $\mathcal{F}\{p_{\tilde{\eta}}(\alpha \tilde{\eta})\}(k) = \frac{1}{\alpha^{3/2}} \widehat{p}_{\tilde{\eta}}(k/\alpha)$ and the convolution theorem: $\mathcal{F}\{(f \otimes g)(\tilde{\eta})\}(k) = (2\pi)^{3/2} \tilde{f}(k) \tilde{g}(k)$. Thus, the Fourier transform of (2.14) is

$$ \widehat{p}_{\tilde{\eta}}(2^{1/\alpha} k) = (2\pi)^{3/2} (\widehat{p}_{\tilde{\eta}}(k))^{2} . \quad (2.17) $$

The solution to (2.17) is

$$ \widehat{p}_{\tilde{\eta}}(k) = \frac{1}{(2\pi)^{3/2}} e^{-|\gamma| k^\alpha} , \quad (2.18) $$

where $\gamma$ is a free parameter. The inverse Fourier transform of (2.18) is

$$ p_{\tilde{\eta}}(\tilde{\eta}) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} e^{-|\gamma| k^\alpha} e^{ik \cdot \tilde{\eta}} \, dk , \quad (2.19) $$

Then, upon making the substitution $k \rightarrow -k$ and using the negative signs in $-dk$ to flip the integration limits, we have the final result:

$$ p_{\tilde{\eta}}(\tilde{\eta}) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} e^{-|\gamma| k^\alpha} e^{-i k \cdot \tilde{\eta}} \, dk , \quad (2.20) $$

which is the multivariate Lévy $\alpha$-stable distribution with scale $\gamma$ but no skew or shift. For a primer on Lévy $\alpha$-stable distributions, see Appendix B.
Ensemble-averaged transport behavior: "Maxwell-Boltzmann" distribution ($\alpha = 2$)

2.3. Ensemble-Averaged Transport Behavior

Returning to the problem of turbulent transport, the ensemble-averaged transport, equation (2.6), can be put into similarity form upon substitution $p_u(u)\, du = p_\eta(\eta)\, d\eta$ and $u\, \delta t = \xi = (q\, \delta t)^{1/\alpha} \eta$:

$$\frac{\partial \bar{c}}{\partial t} = \lim_{\delta t \to 0} \iiint_{-\infty}^{\infty} \frac{c(t, x - (q\, \delta t)^{1/\alpha} \eta; \eta) - c(t, x; \eta)}{\delta t} p_\eta(\eta)\, d\eta.$$  

(2.21)

Upon specification of $q$, $\alpha$, and $p_\eta(\eta) = p_\eta(\eta = \eta - \eta)$, we can evaluate (2.21).

2.3.1. Ensemble-averaged transport behavior: "Maxwell-Boltzmann" distribution ($\alpha = 2$)

Upon inserting $\alpha = 2$ into (2.20) and carrying out the integration, one finds a normal distribution with standard deviation $\sqrt{2}\, \gamma$

$$p_\eta(\eta) = \frac{1}{(2\pi)^{3/2}(\sqrt{2}\, \gamma)^3} \exp\left(-\frac{|\eta|^2}{2(\sqrt{2}\, \gamma)^2}\right).$$

(2.22)

For proof, see Appendix A.2.

Due to the exponential decay of (2.22), only small displacements will be important in (2.21). Thus, we expand in Taylor series (switching to index notation for clarity)

$$c(t, x_i - \eta_i \sqrt{\nu\, \delta t}; \eta_i) = c(t, x_i; \eta_i) - \eta_j \sqrt{\nu\, \delta t} \frac{\partial c}{\partial x_j} + \frac{1}{2} \eta_j \eta_k (\nu\, \delta t) \frac{\partial^2 c}{\partial x_j \partial x_k} + O(\delta t^{3/2}).$$

(2.23)

Inserting (2.23) and (2.22) into (2.21), we have:

$$\frac{\partial \bar{c}}{\partial t} = \lim_{\delta t \to 0} \iiint_{-\infty}^{\infty} -\eta_j \sqrt{\nu\, \delta t} \frac{\partial c}{\partial x_j} + \frac{1}{2} \eta_j \eta_k (\nu\, \delta t) \frac{\partial^2 c}{\partial x_j \partial x_k} + O(\delta t^{3/2}) \frac{e^{-|\eta - \eta|^2}}{(2\pi)^{3/2}(\sqrt{2}\, \gamma)^3} \, d\eta,$$

(2.24)

which evaluates to

$$\frac{\partial \bar{c}}{\partial t} = \lim_{\delta t \to 0} \left[ -\frac{\eta_j \sqrt{\nu\, \delta t}}{\delta t} \frac{\partial \bar{c}}{\partial x_j} + \frac{1}{2} \nu (\bar{\eta}_j \bar{\eta}_k + (\sqrt{2}\, \gamma)^2 \delta_{jk}) \frac{\partial^2 \bar{c}}{\partial x_j \partial x_k} + O(\delta t^{1/2}) \right].$$

(2.25)

where $\delta_{ij}$ is the Kronecker delta. Note that the mean displacement is equivalently given by $\bar{\xi} = \bar{\eta} \sqrt{\nu\, \delta t} = \bar{u} \, \delta t$, where $\bar{u}$ is the mean velocity. Thus, for fixed mean velocity, $\lim_{\delta t \to 0} \frac{\eta_j \sqrt{\nu\, \delta t}}{\delta t} = \lim_{\delta t \to 0} \frac{\bar{a}_j}{\delta t} = \bar{u}_j$, while $\lim_{\delta t \to 0} \bar{\eta}_j = \lim_{\delta t \to 0} \frac{\bar{a}_j \delta t}{\sqrt{\nu\, \delta t}} = 0$. Since $\gamma$ is a free parameter (that scales the diffusion rate), we can choose $\gamma = 1$ so that we can interpret $\nu$ as the diffusivity. Thus, (2.25) simplifies the final result

$$\frac{\partial \bar{c}}{\partial t} + \bar{u} \cdot \nabla \bar{c} = \nu \nabla^2 \bar{c},$$

(2.26)

which is the classical transport equation with Fickian diffusion. Equation (2.26) represents the ensemble-averaged behavior of (2.1) when the probability distribution is Maxwell-Boltzmann. It echoes the celebrated result of Einstein (1905), who showed that a random walk following the Maxwell-Boltzmann distribution is equivalent to Fickian diffusion.
2.3.2. Ensemble-averaged transport behavior: Cauchy distribution \((\alpha = 1)\)

The Cauchy distribution,

\[
p_{\tilde{\eta}}(\tilde{\eta}) = \frac{1}{\pi^2 \gamma^3} \left( \frac{1}{|\tilde{\eta}|^2 + 1} \right)^2
\]

(2.27)
corresponds to \(\alpha = 1\) and \(q = u_\tau\) (Table 1). Due to the algebraic decay of (2.27), both small and large displacements are important in (2.21). Therefore, we find it convenient to recast the transport equation (2.6) in displacement form:

\[
\frac{\partial \bar{c}}{\partial t} = \lim_{\delta t \to 0} \int_{-\infty}^{\infty} \frac{c(t,x-\xi) - c(t,x)}{\delta t} p_{\xi}(\xi;\delta t) \, d\xi.
\]

(2.28)

To proceed, define the total displacement as the average plus a deviation \(\xi = \bar{\xi} + \tilde{\xi}\), such that \(d\xi = d\tilde{\xi}\). Since \(\tilde{\xi} = \tilde{u} \delta t\) and \(\tilde{u}\) is finite, then \(\tilde{\xi}\) is small in the limit of \(\delta t\) tending to zero. Therefore, we may expand \(c(t,x-\xi;\xi)\) in a Taylor series in \(\tilde{\xi}\):

\[
c(t,x-\xi;\xi) = c(t,x-\bar{\xi};\xi) - \tilde{\xi} \cdot \nabla c(t,x-\bar{\xi};\xi) + O(|\tilde{\xi}|^2).
\]

(2.29)

Then (2.28) becomes

\[
\frac{\partial \bar{c}}{\partial t} = \lim_{\delta t \to 0} \int_{-\infty}^{\infty} \frac{c(t,x-\bar{\xi};\xi) - \tilde{\xi} \cdot \nabla c(t,x-\bar{\xi};\xi) + O(|\tilde{\xi}|^2)}{\delta t} c(t,x;\xi) p_{\xi}(\xi;\delta t) \, d\xi.
\]

(2.30)

Equation (2.11) prescribes

\[
p_{\xi}(\xi;\delta t) = \frac{p_{\tilde{\eta}}(\tilde{\eta} = -\bar{\xi}/(u_\tau \delta t))}{(u_\tau \delta t)^3} = \frac{1}{(\gamma u_\tau \delta t)^3} \frac{1}{\pi^2} \left( \frac{|\tilde{\xi}|^2}{(\gamma u_\tau \delta t)^2 + 1} \right)^2 = \frac{1}{\pi^2} \left( \frac{|\tilde{\xi}|^2 + (\gamma u_\tau \delta t)^2}{(\gamma u_\tau \delta t)^2 + 1} \right)^2.
\]

(2.31)

Inserting (2.31) into (2.30) yields

\[
\frac{\partial \bar{c}}{\partial t} = \frac{\gamma u_\tau}{\pi^2} \int_{-\infty}^{\infty} \frac{c(t,x-\bar{\xi};\xi) - c(t,x;\xi)}{|\tilde{\xi}|^4} \, d\tilde{\xi} - \lim_{\delta t \to 0} \int_{-\infty}^{\infty} \tilde{u} \cdot \nabla c(t,x-\bar{\xi};\xi) \frac{1}{\pi^2} \frac{\gamma u_\tau \delta t}{|\tilde{\xi}|^2 + (\gamma u_\tau \delta t)^2} \, d\tilde{\xi}.
\]

(2.32)

In the gradient term, the limit \(\delta t \to 0\) makes the fraction in the integrand behave as a delta function, picking off the value \(\tilde{u} \cdot \nabla c(t,x;\xi)\). To see this, realize that as \(\delta t \to 0\), the integrand becomes zero for all \(\tilde{\xi} \neq 0\). Therefore, \(\tilde{\xi}\) can be assumed to be small, and a Taylor series used \(\nabla c(t,x-\bar{\xi};\xi) = \nabla c(t,x;\xi) - \tilde{\xi} \cdot \nabla (\nabla c(t,x;\xi)) + O(|\tilde{\xi}|^2)\). The \(\tilde{\xi}\) term in this Taylor series integrates to zero by symmetry. The remaining integral evaluates to unity:

\[
\int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{\gamma u_\tau \delta t}{|\tilde{\xi}|^2 + (\gamma u_\tau \delta t)^2} \, d\tilde{\xi} = 1,
\]

(2.33)

and all that remains is \(\tilde{u} \cdot \nabla c(t,x;\xi)\). Thus, the transport equation reduces to

\[
\frac{\partial \bar{c}}{\partial t} + \tilde{u} \cdot \nabla c(t,x;\xi) = \frac{\gamma u_\tau}{\pi^2} \int_{-\infty}^{\infty} \frac{c(t,x-\bar{\xi};\xi) - c(t,x;\xi)}{|\tilde{\xi}|^4} \, d\tilde{\xi}.
\]

(2.34)

We can now take an ensemble average of both sides using (2.3) in displacement form:

\[
\bar{c}(t,x) = \int_{-\infty}^{\infty} c(t,x;\xi) p_{\xi}(\xi;\delta t) \, d\xi,
\]

(2.35)
and we can make the substitution $x' = x - \xi$ to obtain the final result

$$\frac{\partial \bar{c}(t,x)}{\partial t} + \bar{u} \cdot \nabla \bar{c}(t,x) = \frac{\gamma u_r}{\pi^2} \iint_{-\infty}^{\infty} \bar{c}(t,x') - \bar{c}(t,x) \frac{d\xi'}{|x' - x|^\alpha}.$$  \hspace{1cm} (2.36)

Equation (2.36) predicts turbulent dispersion according to a fractional Laplacian. This is analogous to the fractional Laplacian appearing in the momentum equation (in §4).

2.3.3. Ensemble-averaged transport behavior: Any $0 < \alpha < 2$

We now generalize equation (2.36) to any value of $\alpha$ in the range $0 < \alpha < 2$. Recall the development in §2.3.2 that lead to equation (2.30), which is rearranged here

$$\frac{\partial \bar{c}}{\partial t} = \lim_{\delta t \to 0} \iint_{-\infty}^{\infty} \frac{c(t,x - \xi, t) - c(t,x, \xi)}{\delta t} p\xi(\bar{\eta}, \delta t) d\xi - \lim_{\delta t \to 0} \iint_{-\infty}^{\infty} \bar{u} \cdot \nabla c(t,x - \xi, t) p\xi(\bar{\eta}, \delta t) d\xi,$$

where

$$p\xi(\bar{\eta}, \delta t) = \frac{p\bar{\eta}(\bar{\eta} = \xi/(q \delta t)^{1/\alpha})}{(q \delta t)^{3/\alpha}},$$  \hspace{1cm} (2.37)

and $\xi = \xi - \bar{\xi}$. As in §2.3.2, the limit $\delta t \to 0$ makes $\xi/(q \delta t)^{1/\alpha}$ very large for any finite $\bar{\xi}$, which enables simplification of both terms in (2.37). For the second term of (2.37), we note that as in §2.3.2, $p\xi(\bar{\eta}, \delta t)$ behaves as a delta function, so the second term simplifies to $\bar{u} \cdot \nabla c(t,x,\xi)$. For the first term of (2.37), we may limit ourselves to retaining only the tails of $p\xi(\bar{\eta}, \delta t)$, which at 3D, for an arbitrary value of $\alpha$ and with elasticity $\gamma$ are given by (Nolan 2006) $p\bar{\eta}(\bar{\eta}) \simeq \frac{\alpha C_\alpha}{\gamma |\bar{\eta}/\gamma|^{\alpha+3}}$, such that

$$p\xi(\bar{\xi}, \delta t) \simeq \frac{1}{(q \delta t)^{3/\alpha}} \frac{\gamma^\alpha C_\alpha}{|\bar{\xi}/(q \delta t)^{1/\alpha}|^{\alpha+3}} = \frac{\gamma^\alpha q \delta t C_\alpha}{|\bar{\xi}|^{\alpha+3}}$$  \hspace{1cm} (2.39)

The first term in (2.37) can thus be expressed as

$$\lim_{\delta t \to 0} \iint_{-\infty}^{\infty} \frac{c(t,x - \xi, t) - c(t,x, \xi)}{\delta t} p\xi(\bar{\eta}, \delta t) d\xi = \frac{\gamma^\alpha q C_\alpha}{\pi^2} \iint_{-\infty}^{\infty} \frac{c(t,x - \xi, t) - c(t,x, \xi)}{|\xi|^{\alpha+3}} d\xi.$$  \hspace{1cm} (2.40)

This straightforward and non-trivial limit is the result of the stability property of the distribution $p\bar{\eta}(\bar{\eta})$, and this verifies the necessity of a stable distribution.

Upon inserting (2.40) into (2.37), taking an ensemble average of both sides using (2.35), and making the substitution $x' = x - \bar{\xi}$, we obtain the final result

$$\frac{\partial \bar{c}(t,x)}{\partial t} + \bar{u} \cdot \nabla \bar{c}(t,x) = \gamma^\alpha q C_\alpha \iint_{-\infty}^{\infty} \frac{\bar{c}(t,x') - \bar{c}(t,x)}{|x' - x|^{\alpha+3}} d\xi'.$$  \hspace{1cm} (2.41)

Equation (2.41) predicts turbulent dispersion according to a fractional Laplacian, as a generalization of (2.36) to fractional order $0 < \alpha < 2$.

In light of (2.41), a nondimensional number measuring the relative magnitudes of advection to dispersion can be formed: Given a characteristic concentration $C$, velocity $V$, and length $L$, the scales of the advection and dispersion terms in (2.41) are

$$\text{advective transport:} \hspace{1cm} \bar{u} \cdot \nabla \bar{c}(t,x) \sim VC/L,$$

$$\text{diffusive transport:} \hspace{1cm} \gamma^\alpha q C_\alpha \iint_{-\infty}^{\infty} \frac{\bar{c}(t,x') - \bar{c}(t,x)}{|x' - x|^{\alpha+3}} d\xi' \sim qC/L^\alpha.$$  \hspace{1cm} (2.42)
Therefore, the nondimensional number is

\[
\frac{\text{advective transport}}{\text{diffusive transport}} = \frac{VL^{\alpha-1}}{q}.
\]  

(2.43)

For \(\alpha = 2\), this ratio reverts to the usual Pécelt number \(Pe = VL/\nu\), and for \(\alpha = 1\), this ratio is \(V/u_\tau\).

In this section, we have shown that the canonical diffusion operator \(\nu \nabla^2 \bar{c}\) is uniquely tied to similarity variable \(\eta = \bar{X}/\sqrt{\nu \delta t}\), which has an \(O(\sqrt{\nu \delta t})\) scaling on the diffusion distance. Similarly, we showed that the case \(\alpha = 1\) is uniquely tied to similarity variable \(\eta = \bar{X}/(u_\tau \delta t)\). It is well known that in shear turbulent flows, mixing occurs over distances that scale by \(O(u_\tau \delta t)\), where \(u_\tau\) is the friction velocity (or some other appropriate characteristic eddy velocity) (Cushman-Roisin & Jenkins 2006; Kämpf & Cox 2016). Thus, the similarity parameter \(\eta = \bar{X}/(u_\tau \delta t)\) and Cauchy distribution are appropriate for modeling shear turbulent flows.

### 3. Equilibrium Distributions \(f_{\alpha}^{eq}\)

In §2, we derived that the isotropic Lévy \(\alpha\)-stable distribution \(p_\eta(\eta)\) from (2.20) can be used to model laminar through turbulent flows. Moreover, we related the probability distribution of particle velocities \(p_\eta(\bar{u})\) to this similarity form \(p_\eta(\bar{\eta})\) as follows:

\[
p_\eta(\bar{u}) = (\delta t)^3 p_{\xi}(\bar{\xi} = \bar{u} \delta t; \delta t) = \frac{(\delta t)^3}{(q \delta t)^{3/\alpha}} p_\eta(\bar{\eta} = \frac{\bar{u} \delta t}{(q \delta t)^{1/\alpha}}).
\]  

(3.1)

We can simplify the form of \(p_\eta(\bar{u})\) by defining a scale parameter \(U\) so as to absorb the \((q \delta t)^{1/\alpha}/\delta t\) explicitly shown in (3.1) and the (arbitrary) scale parameter \(\gamma\) hidden within \(p_\eta(\bar{\eta})\). Using the relaxation time \(\tau\) (to be defined in §4) in place of the infinitesimal timestep \(\delta t\), we can define:

\[
U \equiv \gamma (q \tau)^{1/\alpha}/\tau.
\]  

(3.2)

Equation (3.2) states that the microscale displacement \(U \tau\) gives rise to a net macro-scale transport of distance \(\gamma (q \tau)^{1/\alpha}\). Table 2 lists the \(U\) for the cases of interest. \(\dagger\)

Note that the free parameter \(\gamma\) is the constant of proportionality between the microscale displacement \(U \tau\) and the macro-scale distance \((q \tau)^{1/\alpha}\). Further, note that the value of \(\gamma\) may be different when considering transport of momentum versus transport of a passive scalar; indeed the Prandtl number (momentum diffusivity / thermal diffusivity) and the Schmidt number (momentum diffusivity / mass diffusivity) are typically not unity.

In §4, we will model the flowfield using Boltzmann kinetic theory, which presumes that the probability distribution of particle speeds is only ever slightly perturbed from a so called equilibrium distribution \(f_{\alpha}^{eq}(t, x, u)\). Bridging the derivation in §2 with what is to come in §4, we can write

\[
f_{\alpha}^{eq}(t, x, u) = \rho \ p_\eta(\bar{u} = u - \bar{u}(t, x)).
\]  

(3.3)

Combining (3.1)–(3.3), we have

\[
f_{\alpha}^{eq}(t, x, u) = \frac{\rho}{U^3} p_\nu \left( v = \frac{u - \bar{u}(t, x)}{U} \right).
\]  

(3.4)

\(\dagger\) In this context, \(\nu\) is interpreted as a mass/thermal diffusivity, as in the Pécelt number.

\(\dagger\) Note that \(U\), \(\gamma\), and \(q\) all change with \(\alpha\). We will use subscripts to indicate specific values \((U_1, U_2, \text{etc.})\) only when necessary.

\(\ddagger\) See footnote to Table 3.
In Table 2, we simply omit this factor of $p$. Note that if one evaluates the general Lévy $\alpha$-stable distribution with $U_d$ notation $\nu$ standard normal variable exponent in the fractional Laplacian, as shown in Figure 2.

Table 3 lists the formulae for these distributions, and Figure 2 shows the univariate turbulence, and inertial range turbulence. $\alpha$ where $\alpha$ other Lévy $\alpha$ exponent distribution physical parameter, $q$ scale parameter, $U$

<table>
<thead>
<tr>
<th>exponent</th>
<th>distribution</th>
<th>physical parameter, $q$</th>
<th>scale parameter, $U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 2$</td>
<td>Maxwell-Boltzmann</td>
<td>kinematic viscosity, $\nu$</td>
<td>$U_2 \equiv \sqrt{\nu/\tau}$ ($\gamma_2 \equiv 1/\sqrt{2}$) $^\dagger$</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>Cauchy</td>
<td>friction velocity, $u_\tau$</td>
<td>$U_1 = \gamma_1 u_\tau$</td>
</tr>
<tr>
<td>$\alpha = \frac{3}{2}$</td>
<td>other Lévy</td>
<td>energy dissipation rate, $c^\frac{1}{2}$</td>
<td>$U_\frac{3}{2} = \gamma_\frac{3}{2} \sqrt{\tau}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>other Lévy</td>
<td>general, $q$</td>
<td>$U_\alpha \equiv \gamma_\alpha (q \tau)^{1/\alpha} / \tau$</td>
</tr>
</tbody>
</table>

Table 3: Isotropic Lévy $\alpha$-stable distributions, $p_\nu(v)$, in 1, 2, 3, and $n$ dimensions for the standard normal variable $\nu \equiv (u - \bar{u})/U$. In the general Lévy $\alpha$-stable case, we use the notation $dk = dk_1 dk_2 \ldots dk_n$. In the Cauchy case, we define $c \equiv (n + 1)/2$.

<table>
<thead>
<tr>
<th>Dim.</th>
<th>Maxwell-Boltzmann†</th>
<th>Cauchy</th>
<th>general Lévy-α-stable</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p(v) \equiv \frac{1}{(2\pi)^{v/2}} e^{-v/2}$</td>
<td>$p(v) = \frac{1}{\pi} \frac{1}{v^2 + 1}$</td>
<td>$p(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-</td>
</tr>
<tr>
<td>2</td>
<td>$p(v) \equiv \frac{1}{(2\pi)^{v/2}} e^{-v/2}$</td>
<td>$p(v) = \frac{1}{\pi} \frac{1}{(v^2 + 1)^{1/2}}$</td>
<td>$p(v) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-</td>
</tr>
<tr>
<td>3</td>
<td>$p(v) \equiv \frac{1}{(2\pi)^{v/2}} e^{-v/2}$</td>
<td>$p(v) = \frac{1}{\pi} \frac{1}{(v^2 + 1)^{1/2}}$</td>
<td>$p(v) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-</td>
</tr>
<tr>
<td>$n$</td>
<td>$p(v) \equiv \frac{1}{(2\pi)^{v/2}} e^{-v/2}$</td>
<td>$p(v) = \frac{1}{\pi} \frac{1}{c^n}$</td>
<td>$p(v) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^n} e^{-</td>
</tr>
</tbody>
</table>

where $p_\nu(v)$ is the standard Lévy $\alpha$-stable distribution (with zero skew and unit scale). Table 3 lists the formulae for these distributions, and Figure 2 shows the univariate distributions for $\alpha = 2, 1, \text{ and } 2/3$, as well as the tails of the distributions for several $\alpha$. In §4, we will show that the exponent in the tail of the distribution is related to the exponent in the fractional Laplacian, as shown in Figure 2.

In §4, we will find it convenient to use the following alternate notation:

$$ f_{\alpha}^\nu(t, x, u) \equiv \frac{\rho}{U^3} F(\Delta(t, x, u)) , \quad \Delta(t, x, u) \equiv \frac{|u - \bar{u}(t, x)|^2}{U^2} ,$$

† The Maxwell-Boltzmann distribution is traditionally defined as $f_{\alpha}^\nu \equiv \rho \frac{1}{(2\pi)^{v/2}} e^{-|u - \bar{u}|^2 / 2U^2}$ with $U_2 \equiv \sqrt{\nu/\tau}$. This distribution can be written in the form of (3.4) as $f_{\alpha}^\nu \equiv \frac{\rho}{U^3} p(v = \frac{u - \bar{u}}{c_\alpha^{1/2}})$, with $p(v) \equiv \frac{1}{(2\pi)^{v/2}} e^{-|v|^2 / 2}$ as given in Table 3 and $U_2 = \sqrt{\nu/\tau}$ as given in Table 2. Note that if one evaluates the general Lévy $\alpha$-stable distribution with $\alpha = 2$, one finds $p(v) = \frac{1}{(2\pi)^{v/2}} \int_{-\infty}^{\infty} e^{-|k|^{\alpha} - ikv} dk = \frac{1}{(2\pi)^{v/2}} \frac{1}{(v^2 + 1)^{1/2}} e^{-|v/\sqrt{2}|^2 / 2}$. This result can be made copacetic with the Maxwell-Boltzmann distribution by writing $f_{\alpha}^\nu = \frac{\rho}{U^3} p(v = \frac{u - \bar{u}}{c_\alpha^{1/2}})$, with $U$ properly defined by the general formula (3.2) as $U = \gamma_2 \sqrt{\nu/\tau}$, and by taking the value $\gamma_2 = 1/\sqrt{2}$. In Table 2, we simply omit this factor of $\gamma_2 = 1/\sqrt{2}$ when defining $U_2$. 
Figure 2: (a) Univariate Lévy $\alpha$-stable distributions, of which Maxwell-Boltzmann ($\alpha = 2$) and Cauchy ($\alpha = 1$) are special cases. (b) Tails of these distributions for several $\alpha$. (c) Log-slope of the tail of the distributions, comparing numerical results from the center figure with the theoretical asymptotic value $1 + \alpha$. These log-slopes are identical to the exponent of the fractional Laplacian (in one dimension).

where $F(\Delta(t,x,u)) \equiv p_u(\nu = \sqrt{\Delta})$. Two important examples are as follows:

Maxwell-Boltzmann ($\alpha = 2$):  
\[ F(\Delta) = \frac{1}{(2\pi)^{3/2}} e^{-\Delta/2}; \]  
Cauchy ($\alpha = 1$):  
\[ F(\Delta) = \frac{1}{\pi^2 (\Delta + 1)^2}. \]  

A concern arises whether higher moments are finite when the equilibrium function decays algebraically instead of exponentially. For the Cauchy distribution, for example, the integrals for the second and higher moments diverge, raising objections about the value of the ensuing analysis. In any physical situations, however, the spatial domain is always finite, and fluctuating velocities remain within bound. Thus, no integration ever needs to be carried to infinity, and all moments remain finite.
4. Derivation of the Fractional Laplacian as a Turbulence Model

4.1. Boltzmann Kinetics

Our premise is Boltzmann kinetic theory, wherein the flowfield is described by the mass probability density function \( f(t,x,u) \) with time \( t \), position \( x = \{x, y, z\} \), and particle velocity \( u = \{u, v, w\} \) as independent variables. By definition, \( f(t,x,u) \, du \, dx \) is the mass of fluid particles at time \( t \) located within volume \( dx \) surrounding \( x \) that have velocities within the range \( du \) surrounding velocity \( u \). The ensemble-averaged hydrodynamic quantities are derived from \( f \) via integrals over all possible particle velocities (Chen 2011):

\[
\rho(t,x) = \iiint_{-\infty}^{\infty} f(t,x,u) \, du
\]

(4.1)

velocity \( \bar{u}_i(t,x) = \frac{1}{\rho} \iiint_{-\infty}^{\infty} u_i \, f(t,x,u) \, du \)

(4.2)

specific internal energy \( \bar{u}(t,x) = \frac{3}{2} \bar{T}^2 = \frac{1}{\rho} \iiint_{-\infty}^{\infty} \frac{1}{2} |u - \bar{u}(t,x)|^2 \, f(t,x,u) \, du \)

(4.3)

where \( \bar{u}_i \) is the \( i^{th} \) component of the ensemble-averaged Eulerian flow velocity, and \( U \) is the particle agitation speed, which in the original Boltzmann kinetics is understood as the mean speed of a particle between collisions. It is important to emphasize that we interpret \( \bar{u}_i \) as the ensemble-averaged velocity, with the integral over all possible particle velocities \( u_i \) corresponding to an ensemble average and taking the place of Reynolds’ time or ensemble average. Note that the mass distribution \( f(t,x,u) \) is a function of seven independent variables (time, 3D space, and 3D velocity space). \( \dagger \) \( \ddagger \)

The evolution of the mass distribution \( f(t,x,u) \) is governed by the Boltzmann equation, with the classical BGK (Bhatnagar et al. 1954) formulation for the collision term on the right hand side (Succi 2001):

\[
\frac{\partial f}{\partial t} + u \cdot \nabla f = \frac{1}{\tau} (f^{eq}_\alpha - f),
\]

(4.4)

where \( f^{eq}_\alpha(t,x,u) \) is an equilibrium distribution to which \( f \) relaxes, and \( \tau \) is the relaxation time, which in original Boltzmann kinetics can be understood as the mean time between successive collisions of a particle (Succi 2001).

The equilibrium distribution \( f^{eq}_\alpha \) (3.5) has a prescribed structure such that it shares certain moments with the actual distribution \( f \): The free parameters defining \( f^{eq}_\alpha \) are such that the hydrodynamic variables (\( \rho, \bar{u}, \bar{u} \)) are recovered when inserting \( f^{eq}_\alpha \) (3.5) into (4.1)–(4.3). This structure is required, because \( f^{eq}_\alpha \) is an allowable value for \( f \). This structure also ensures that the collision term (right hand side) of the Boltzmann equation (4.4) conserves mass, momentum and energy.

4.2. Hydrodynamic Equations

The ensemble-averaged hydrodynamic equations for mass and momentum are recovered by multiplying the Boltzmann equation by 1 or \( u \) and integrating over the velocity space. The only “trick” is that \( u \) is an independent variable, so it can “hop into” the derivatives, and the order of the derivatives and velocity integrations can be switched. For example, \( \iiint (u \cdot \nabla f) \, du = \nabla \cdot (\iiint uf \, du) = \nabla \cdot (\rho \bar{u}). \) In this way, \( \iiint (\text{Boltzmann eqn. (4.4)}) \, du \)

\( \dagger \) Using the fact that \( |u - \bar{u}|^2 = |u|^2 - 2\bar{u} \cdot u + |\bar{u}|^2 \), it is easy to show that the specific total energy \( \bar{e} = \frac{1}{\rho} \iiint \frac{1}{2} |v|^2 \, dv = \bar{u} + \frac{1}{2} |\bar{u}|^2 \) is the sum of the specific internal energy \( \bar{u} \) and the specific kinetic energy \( \frac{1}{2} |\bar{u}|^2 \).

\( \ddagger \) Unless otherwise noted, we use bold variables to indicate vectors and use Einstein’s convention of summing over repeated indices. Also, we will hereafter use the shorthand \( \iiint (\ldots) \, du \) to imply the definite integral \( \iiint_{-\infty}^{\infty} (\ldots) \, du \, dv \, dw \).
yields the continuity equation
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{u}) = 0, \]
and \( \iint u \, (Boltzmann \ eqn. \ (4.4)) \, du \) yields the momentum equation as follows:
\[ \iint \left\{ \frac{\partial (u f)}{\partial t} + \nabla \cdot (uu^\top f) \right\} \, du = \iint u \frac{1}{2} (f - f_{\alpha}^{eq}) \, du. \] (4.6)
The right hand side of (4.6) is zero, since both \( f \) and \( f_{\alpha}^{eq} \) obey (4.2). The unsteady term evaluates to \( \frac{\partial (\rho \bar{u})}{\partial t} \) also by virtue of (4.2). The advective term can be evaluated by first noting that \( (u - \bar{u})(u - \bar{u})^\top = uu^\top - \bar{u}u^\top - \bar{u}\bar{u}^\top + \bar{u}\bar{u}^\top \). Then,
\[ \iint \nabla \cdot (uu^\top f) \, du = \nabla \cdot \left[ \iint (u - \bar{u})(u - \bar{u})^\top f \, du + \iint (uu^\top - \bar{u}u^\top - \bar{u}\bar{u}^\top) f \, du \right]. \] (4.7)
Thus, (4.6) becomes \( \frac{\partial (\rho \bar{u})}{\partial t} + \nabla \cdot (\rho \bar{u}\bar{u}^\top) = \nabla \cdot \sigma \), which can be simplified using (4.5) and the definition \( \frac{D\bar{u}}{Dt} \equiv \frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} \) to yield the ensemble-averaged momentum equation:
\[ \rho \frac{D\bar{u}}{Dt} = \nabla \cdot \sigma \] (4.8)
in which the ensemble-averaged stress tensor \( \sigma \) is defined as
\[ \sigma_{ij}(t, x) \equiv -\iint (u_i - \bar{u}_i(t, x))(u_j - \bar{u}_j(t, x)) \, f(t, x, u) \, du. \] (4.9)
Note the similarity between (4.9) and a Reynolds stress, with the velocity departures from their respective means playing the role of Reynolds’ velocity fluctuations and the integral over \( f \, du \) playing the role of Reynolds’ ensemble average. However, the stress (4.9) is not a Reynolds stress: Expression (4.9) is a definition within the framework of kinetic theory, and (4.9) is rooted in the Boltzmann equation (rather than Navier-Stokes).

It is widely accepted that turbulence arises from the nonlinear advective term in the Navier-Stokes equations, \( \nabla \cdot \nabla \bar{v} \). Indeed, with a Reynolds decomposition \( \bar{v} = \bar{v} + \tilde{v} \), the Reynolds stresses arise as follows \( \bar{v} \cdot \nabla \bar{v} = (\bar{v} + \tilde{v}) \cdot \nabla (\bar{v} + \tilde{v}) = \bar{v} \cdot \nabla \bar{v} + \bar{\tilde{v}} \cdot \nabla \bar{\tilde{v}} \). Within the kinetic theory framework, both the advection term \( \bar{u} \cdot \nabla \bar{u} \) and the force term \( \nabla \cdot \sigma \) arise from the advection term in the Boltzmann equation, \( u \cdot \nabla f \). Since the Boltzmann equation is the parent of the Navier-Stokes equations, and since the turbulent force arises from the advection term as desired, we expect that the force \( \nabla \cdot \sigma \) is a reasonable alternative to the Reynolds force \( \bar{\tilde{v}} \cdot \nabla \bar{\tilde{v}} \).

Moving on, our goal is to obtain closed-form expressions for the stress \( \sigma \) and force \( \nabla \cdot \sigma \) that depend only on the ensemble-averaged macroscopic quantities \( \rho, \bar{u} \), and \( q \). We seek to evaluate the stress tensor for the Maxwell-Boltzmann, Cauchy, and general Lévy \( \alpha \)-stable \( f_{\alpha}^{eq} \) distributions.

Note that in traditional studies of Boltzmann kinetics, the Chapman-Enskog perturbation expansion has been shown to recover the Navier-Stokes equations (from the Boltzmann equation); this derivation assumes \( \tau \) is short and \( f_{\alpha}^{eq} \) is Maxwell-Boltzmann (Chapman & Cowling 1991). Herein, we pursue a different approach: We first determine the analytic solution the Boltzmann equation, expressing \( f \) in terms of \( f_{\alpha}^{eq} \) in (4.10) below, and then we use this \( f \) to evaluate the stress, as defined in (4.9). If we then assume \( f_{\alpha}^{eq} \) is Maxwell-Boltzmann, we recover the Navier-Stokes equations, as expected. However, if we assume \( f_{\alpha}^{eq} \) is another Lévy \( \alpha \)-stable distribution, then we find that the momentum equation is augmented by the fractional Laplacian.
4.3. Mass Probability Distribution

Equation (4.4) is linear in its unknown variable \( f(t,x,u) \) and possesses the following analytical solution for the mass probability distribution:

\[
f(t,x,u) = \frac{1}{\tau} \int_{-\infty}^{t} f^e_{\alpha}(t',x-(t-t')u,u) e^{-t-t' / \tau} dt'.
\] (4.10)

Equation (4.10) prescribes \( f(t,x,u) \) as the weighted sum of the particles with velocity \( u \), with the weight being an exponential attenuation to account for the relaxation between their earlier location, \( x' \equiv x - (t-t')u \), and their current location, \( x \). †

To simplify the algebra, we find it convenient to define the dimensionless flight time \( s \equiv (t-t')/\tau \) such that \( ds = -dt'/\tau \), \( x' = x - s\tau u \), and \( dx' = (dx')(dy')(dz') = (-s\tau du)(-s\tau dv)(-s\tau dw) = (-s\tau)^3 du \). With this notation, (4.10) takes the more compact form:

\[
f(t,x,u) = \int_{0}^{\infty} f^e_{\alpha}(t-s\tau,x-s\tau u,u) e^{-s} ds.
\] (4.11)

Unless otherwise noted, we will hereafter use the shorthand \( \int (\ldots)ds \) in lieu of \( \int_{0}^{\infty} (\ldots)ds \).

4.4. Scales, Definitions, and Assumptions

We define the following quantities:

- \( L \) characteristic length scale of the macro-scale flow (e.g. channel width).
- \( V \) characteristic velocity scale of the macro-scale flow (e.g. boundary speed);
- \( U \) agitation speed of the particles (standard deviation of the velocity fluctuations);
- \( \tau \) relaxation time (mean time between particle collisions);
- \( \ell \) demarcation length scale that separates “small” from “large” particle displacements;
- \( \lambda \) mean free path (average distance traveled between particle collisions):

\[
\lambda \equiv U_2 \tau.
\] (4.12)

\( Kn \) Knudsen number, defined as

\[
Kn \equiv \frac{\lambda}{L} = \frac{U_2 \tau}{L}.
\] (4.13)

Using these quantities, we make the following assumptions:

**Assumption #1**: Herein, we restrict our attention to incompressible, isothermal flows. That is, we assume that the density \( \rho \) and agitation speed \( U \) are both constant, from which follows \( \frac{\partial u_i}{\partial x_i} = 0 \) (implied summation over \( i = 1, 2, 3 \)) from (4.5).

**Assumption #2**: We further restrict our attention in §4 to the case of a constant \( \alpha \). Conceivably, \( \alpha \) could be a function of time and/or space, perhaps as a function of the local velocity gradient. For example, it would be expected that \( \alpha = 2 \) in the laminar sublayer near a wall but then decrease (say to \( \alpha = 1 \)) as one moves away from the wall. In §5, we consider the simple case of piecewise-constant \( \alpha \), with \( \alpha = 2 \) to model laminar sublayers and \( \alpha = 1 \) to model shear turbulence away from walls.

**Assumption #3**: We need only consider dimensionless time delays of order one:

\[
s = O(1).
\] (4.14)

† Note that (4.10) is restricted to the case of an unbounded domain (\( x \in \mathbb{R}^3 \)). If the domain were finite, then (4.10) would need to include an additional term (the homogeneous solution of (4.4)) in order to enforce boundary conditions. See §5.
The reason for this is readily apparent from examination of (4.11). The \( e^{-s} \) attenuation factor renders any value much larger than unity inconsequential.

**Assumption #4**: Assume that the characteristic flow speed is less than or on the order of the agitation speed:

\[
V \leq \mathcal{O}(U_2) .
\]  

(4.15)

Moreover, since \(|\bar{u}| \sim \mathcal{O}(V)\) by definition, equation (4.15) implies \(|\bar{u}| \leq \mathcal{O}(U_2)\). Note that with (4.12) and (4.13) assumption (4.15) implies \(V/U_2 = ReKn \leq 1\), which implies moderate speeds, consistent with the incompressibility assumption above.

**Assumption #5**: We assume that the following ordering of scales exists:

\[
\lambda \ll \ell \ll L .
\]  

(4.16)

The definition of the intermediate scale \(\ell\) is flow-dependent. For shear flows (\(\alpha = 1\)), \(\ell\) would be the thickness of the laminar sublayer near a wall:

\[
\ell_1 \sim \frac{\nu}{u\tau} ,
\]  

(4.17)

because viscosity dominates for distances less than this \(\ell\) from a wall. In a turbulent boundary layer, \(u\tau/V \sim 1/\ln(Re)\), so \(\ell_1/L \sim \ln(Re)/Re \ll 1\). For inertial turbulence (\(\alpha = 2/3\)), the appropriate definition is the Kolmogorov microscale

\[
\ell_2 \sim \nu \frac{3}{4} \epsilon^{-\frac{1}{4}} ,
\]  

(4.18)

which is the size of the smallest eddies in the turbulent cascade, \(\ell_2/L \approx Re^{-3/4} \ll 1\) (see Appendix C). Both of these demarcations makes sense, because they place the cut between small and large displacements precisely where turbulence ends and viscosity takes over.

While (4.17) or (4.18) justifies the second inequality in (4.16), the first inequality \(\lambda \ll \ell\) is the classical assumption of a continuum (\(Kn \ll 1\)), even at the smallest eddy scale.

**Assumption #6**: We assume that the relaxation time \(\tau\) is much shorter than the time scale of the flow:

\[
\tau \ll \frac{L}{V} , \quad \text{or equivalently} \quad \frac{\partial}{\partial t} \sim \frac{V}{L} \ll \frac{1}{\tau} .
\]  

(4.19)

This relation follows from (4.15) and (4.16), which prescribe \(V\tau/L < \mathcal{O}(U_2\tau/L) = \mathcal{O}(\lambda/L) \ll 1\). Phenomenologically, assumption (4.19) is consistent with the fact that the evolution of flow as a whole occurs as a result of a great many collisions among particles.

This assumption also justifies the use of the BGK collision model \(\frac{1}{\tau}(f_{eq}^{\alpha} - f)\), which essentially is a finite difference approximation to the rate of change of \(f\) that is brought about by particle collisions.

**Example**: For example numbers, consider dry air at standard temperature \(T = 15^\circ\text{C}\) and pressure \(p = 101.3\) kPa. At these conditions, laboratory measurements give the following values: density \(\rho = 1.23\) kg/m\(^3\), dynamic viscosity \(\mu = 1.79 \times 10^{-5}\) kg/m.s, kinematic viscosity \(\nu = \mu/\rho = 1.46 \times 10^{-5}\) m\(^2\)/s, and mean free path \(\lambda = 68 \times 10^{-9}\) m.

The agitation speed is \(U_2 = \sqrt{3k_BT/\rho} = 502\) m/s, where \(k_B = 1.38 \times 10^{-23}\) J/K is Boltzmann’s constant, and \(m = (29\text{ g/mol})/(6.02 \times 10^{23}\text{ molecules/mol}) = 4.82 \times 10^{-26}\) kg/particle is the molecular weight. Using these numbers, we can infer the collision time \(\tau = \lambda/U_2 = 1.35 \times 10^{-10}\) s. Equation (4.37) predicts the viscosity is \(\mu = \rho U_2^2\tau = 1.23\)
kg/m³ * (502 m/s)² * 1.35 × 10⁻¹⁰ s = 4.18 × 10⁻⁵ kg/m.s, which is on the order of the measured laboratory value. Equation (4.21) predicts \( p = \rho U_2^2 \) = 3.1 × 10⁶ Pa, which also is of the correct order of magnitude. Assuming \( V = 1 \text{ m/s} \) and \( L = 1 \text{ m} \), we have \( Re = VL/\nu = 6.9 \times 10^4 \), \( Kn = \lambda/L = 6.8 \times 10^{-8} \), \( Re Kn = 4.7 \times 10^{-3} \ll 1 \). Therefore, these numbers show that the above assumptions are reasonable and mutually acceptable.

### 4.5. Evaluation of the Stress Tensor

Our goal is to obtain closed-form expressions for the stress \( \sigma \) and force \( \nabla \cdot \sigma \) in terms of the hydrodynamic variables \( (\rho, \bar{u}, q) \) for \( f^e_\alpha \) being each of the Maxwell-Boltzmann, Cauchy, and general Lévy \( \alpha \)-stable distributions. Inserting (4.11) into (4.9), we have for the stress:

\[
\sigma_{ij}(t,x) = -\iiint (u_i - \bar{u}_i(t,x))(u_j - \bar{u}_j(t,x)) f^e_\alpha(t-s\tau,x-s\tau \mathbf{u}, \mathbf{u}) \, e^{-s} \, ds \, d\mathbf{u}. \tag{4.20}
\]

The integrals in this expression would be straightforward to evaluate were it not for the temporal and spatial shifts in \( f^e_\alpha(t-s\tau,x-s\tau \mathbf{u}, \mathbf{u}) \). Indeed, with no shift, the time integral would decouple, and the stress would be:†

\[
-\iiint_{-\infty}^{\infty} (u_i - \bar{u}_i(t,x))(u_j - \bar{u}_j(t,x)) f^e_\alpha(t,x,\mathbf{u}) \, d\mathbf{u} \int_0^{\infty} e^{-s} \, ds = -\bar{p} \delta_{ij} \tag{4.21}
\]

where \( \delta_{ij} \) is the Kronecker delta, and \( \bar{p} = \rho U^2 \) is the ensemble-averaged static pressure. This result is identical to that obtained at the leading order of the Chapman-Enskog expansion, with the value \( \rho U^2 \) derived herein by carrying out the integrations in (4.21).‡

To focus on the frictional shear stress, we decompose the total stress as the sum of the **pressure** and **deviatoric stress** in the usual way (Kundu et al. 2012):

\[
\sigma_{ij} = -\bar{p} \delta_{ij} + \tau_{ij}. \tag{4.22}
\]

Since \( f^e_\alpha(t-s\tau,x-s\tau \mathbf{u}, \mathbf{u}) = f^e_\alpha(t,x,\mathbf{u}) + \{ f^e_\alpha(t-s\tau,x-s\tau \mathbf{u}, u) - f^e_\alpha(t,x,\mathbf{u}) \} \), the deviatoric stress is

\[
\tau_{ij}(t,x) = -\iiint (u_i - \bar{u}_i(t,x))(u_j - \bar{u}_j(t,x)) \left\{ f^e_\alpha(t-s\tau,x-s\tau \mathbf{u}, \mathbf{u}) - f^e_\alpha(t,x,\mathbf{u}) \right\} \, e^{-s} \, ds \, d\mathbf{u} \tag{4.23}
\]

or, using the notation from (3.5) for \( f^e_\alpha \):

\[
\tau_{ij}(t,x) = -\rho \frac{1}{U^3} \iiint (u_i - \bar{u}_i(t,x))(u_j - \bar{u}_j(t,x)) \left\{ F(\Delta(t-s\tau,x-s\tau \mathbf{u}, \mathbf{u})) - F(\Delta(t,x,\mathbf{u})) \right\} \, e^{-s} \, ds \, d\mathbf{u}. \tag{4.24}
\]

The challenge in evaluating the stress (4.24) is in untangling \( F(\Delta(t-s\tau,x-s\tau \mathbf{u}, \mathbf{u})) \) from appearing in both the \( ds \) and \( d\mathbf{u} \) integrals. Two key insights are needed to untangle \( F(\Delta(t-s\tau,x-s\tau \mathbf{u}, \mathbf{u})) \): (1) since collisions occur on a much faster timescale than changes in the macroscopic flow, the time shift in \( F(\Delta(t-s\tau, \ldots)) \) may be ignored; and (2) the \( d\mathbf{u} \) integral in (4.24) can be made tractable by breaking it into two pieces, corresponding to “small” and “large” displacements.

† Note that \( f^e_\alpha \) is an even function of each \((u_i - \bar{u}_i)\), so integrals over odd powers of \((u_i - \bar{u}_i)\) integrate to zero.

‡ While not framed as a Chapman-Enskog expansion, our method is rooted in the same assumption of a small \( \tau \) and generates the same levels of description: At leading order, Eulerian hydrodynamics are recovered, while at the next order (first non-equilibrium correction) we recover viscous dynamics if the equilibrium distribution is Maxwellian or arrive at the fractional Laplacian if the equilibrium distribution is stable with heavy tails. In that sense, our representation of turbulence is a non-equilibrium contribution.
4.5.1. Removing the temporal shift

First, we demonstrate that the time-and-space-shifted \( F(\Delta(t-s\tau,x-s\tau u,u)) \) in (4.24) can be replaced by the space-only-shifted \( F(\Delta(t,x-s\tau u,u)) \). From the assumptions (4.14) and (4.19), which prescribe \( s \sim O(1) \) and small \( \tau \), we are authorized to perform a Taylor expansion in time:

\[
F(\Delta(t-s\tau,x-s\tau u,u)) = F(\Delta(t,x-s\tau u,u)) + \frac{dF}{d\Delta} \frac{\partial \Delta(t,x-s\tau u,u)}{\partial t} (-s\tau) + O(\tau^2)
\]

where from (3.5)

\[
\frac{\partial \Delta(t,x-s\tau u,u)}{\partial t} = -2 \frac{U^2}{3} \sum_{k=1}^{3} (u_k - \bar{u}_k(t,x-s\tau u)) \frac{\partial \bar{u}_k(t,x-s\tau u)}{\partial t}.
\]

Upon inserting (4.25) and (4.26) into (4.24), it is evident that \( dF/d\Delta \) always multiplies odd powers of velocity differences like \((u_i - \bar{u}_i)\); since \( dF/d\Delta \) itself is even in such velocity differences, the \( dF/d\Delta \) terms always integrate to zero, and (4.24) simplifies to

\[
\tau_{ij}(t,x) = -\frac{\rho}{U^3} \iiint \int (u_i - \bar{u}_i(t,x))(u_j - \bar{u}_j(t,x)) \left\{ F(\Delta(t,x-s\tau u,u)) - F(\Delta(t,x,u)) \right\} e^{-s} \, ds \, du
\]

with no longer a time shift in \( F \), although the spatial shift remains.

Since all terms are now evaluated at the present time \( t \), we choose to drop from here onward the mention of \( t \), with no implication of a steady state.

4.5.2. Splitting of the spatial shift

One key step in order to untangle the \( \int ds \) and \( \iiint du \) integrals in (4.27) is to split the \( \iiint du \) integral into two pieces, corresponding to "small" and "large" displacements \((x - x' = s\tau u)\). Formally, we write

\[
\iiint_{-\infty}^{\infty} \ldots du = \iiint_{s\tau |u| \leq \ell} \ldots du + \iiint_{s\tau |u| \geq \ell} \ldots du
\]

Since the demarcation length \( \ell \) is very short compared to the length scale \( L \) of the flow by virtue of assumption (4.16), the first integral captures a minor redistribution of particles and may be identified as a viscous shear stress, whereas the second integral (corresponding to large displacements) may be identified as a turbulent shear stress:

\[
\tau_{ij}^{\text{visc}} \equiv -\frac{\rho}{U^3} \iiint_{s\tau |u| \leq \ell} \int (u_i - \bar{u}_i(x))(u_j - \bar{u}_j(x)) \left\{ F(\Delta(x-s\tau u,u)) - F(\Delta(x,u)) \right\} e^{-s} \, ds \, du
\]

(4.29)

\[
\tau_{ij}^{\text{turb}} \equiv -\frac{\rho}{U^3} \iiint_{s\tau |u| \geq \ell} \int (u_i - \bar{u}_i(x))(u_j - \bar{u}_j(x)) \left\{ F(\Delta(x-s\tau u,u)) - F(\Delta(x,u)) \right\} e^{-s} \, ds \, du.
\]

(4.30)

For the viscous stress, small displacements will permit Taylor expansions of \( F \). For the turbulent stress, large displacements will permit consideration of only the tails of \( F \). In both cases, the problem simplifies considerably.

4.6. Small Displacements: \( s\tau |u| \leq \ell \)

Consider first the case of small displacements \(|x - x'| = s\tau |u| \leq \ell \), wherein we seek to evaluate the viscous stress (4.29). Since \( \ell \ll L \) by virtue of (4.16), the displacements are
much smaller than the length scale \( L \) over which the flow varies, and a Taylor expansion is permissible:

\[
F(\Delta(x-s\tau u)) = F(\Delta(x)) + \frac{dF(\Delta(x))}{d\Delta} \frac{\partial\Delta(x)}{\partial x_i} (-s\tau u_i) + O((s\tau |u|^2))
\]

(4.31)

with implied sum over \( l = 1, 2, 3 \). Using (3.5), the spatial derivative of \( \Delta \) is:

\[
\frac{\partial \Delta}{\partial x_i} = -\frac{2}{U^2} (u_k - \bar{u}_k(x)) \frac{\partial \bar{u}_k}{\partial x_i}
\]

(4.32)

with implied sum over \( k = 1, 2, 3 \). Now inserting (4.31) and (4.32) into (4.29), we obtain

\[
\tau_{ij}^{\text{visc}} = -\frac{\rho}{U^3} \int \int \int \int \frac{(u_i - \bar{u}_i)(u_j - \bar{u}_j)}{s\tau |u| \leq \ell} \left\{ \frac{dF}{d\Delta} \left[ -\frac{2}{U^2} (u_k - \bar{u}_k) \frac{\partial \bar{u}_k}{\partial x_i} \right] (-s\tau u_i) \right\} e^{-s} \; ds \; du
\]

(4.33)

in which \( i, j \) are fixed, summation is implied over \( k, l = 1, 2, 3 \), and all \( \bar{u} \) and \( \Delta \) are evaluated at \( x \). Note that since \( dF/d\Delta \) is even with respect to any \( (u_i - \bar{u}_i) \), only even terms survive the integral over \( du \) (and only when \( u_l \) is shifted to \( (u_l - \bar{u}_l) \)). Thus, for \( i \neq j \), the only terms that survive are \( (k = i, l = j) \) and \( (k = j, l = i) \). For \( i = j \), all three \( k = l \) terms survive:

\[
\tau_{ij}^{\text{visc}} = -\frac{\rho}{U^5} \int \int \int \int \frac{(u_i - \bar{u}_i)^2 (u_j - \bar{u}_j)^2}{s\tau |u| \leq \ell} \frac{dF}{d\Delta} \; du \\
+ \frac{3}{2} \frac{\partial u_k}{\partial x_i} \int \int \int \int \frac{(u_i - \bar{u}_i)^2 (u_k - \bar{u}_k)^2}{s\tau |u| \leq \ell} \frac{dF}{d\Delta} \; du
\]

(4.34)

with no summation implied over \( i, j \).

We can make some additional headway in simplifying (4.34) by realizing that \( dF/d\Delta \) is only a function of the velocity magnitude, so the integrals in (4.34) can be converted into spherical coordinates, and the angle integrals can be evaluated without specification of \( dF/d\Delta \). Using spherical coordinates \((r, \theta, \phi)\) such that \((u - \bar{u}, v - \bar{v}, w - \bar{w}) = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi, d(u - \bar{u}) (d(v - \bar{v}) d(w - \bar{w}) = r^2 \sin \phi d\phi d\theta dr, and \Delta = r^2/U^2, consider the following two prototypical integrals:

\[
I_{22} \equiv \int \int \int \frac{(u - \bar{u})^2 (v - \bar{v})^2}{s\tau |u| \leq \ell} \frac{dF}{d\Delta} \; d(u - \bar{u}) = \frac{4\pi}{15} \int_0^{\ell/s\tau} r^6 \frac{dF}{d\Delta} \; dr,
\]

(4.35)

\[
I_4 \equiv \int \int \int \frac{(u - \bar{u})^4}{s\tau |u| \leq \ell} \frac{dF}{d\Delta} \; d(u - \bar{u}) = \frac{4\pi}{5} \int_0^{\ell/s\tau} r^6 \frac{dF}{d\Delta} \; dr = 3I_{22}.
\]

Since \( I_4 = 3I_{22} \), assuming incompressibility \((\frac{\partial u_k}{\partial x_k} = 0)\), the \( i = j \) case in (4.34) simplifies to be identical in form to the \( i \neq j \) case. Therefore, (4.34) can be written as follows for any \( i, j \)

\[
\tau_{ij}^{\text{visc}}(x) = \mu \left[ \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right], \quad \mu = -\frac{\rho}{U^5} \int_0^{\infty} s e^{-s} \; ds, \quad I_{22} = \frac{4\pi}{15} \int_0^{\ell/s\tau} r^6 \frac{dF}{d\Delta} \; dr
\]

(4.36)

which is the stress for a Newtonian fluid. The viscosity expression in (4.36) provides
a relation between the viscosity \( \mu \), agitation speed \( U \), and mixing length \( \ell \) that can be written explicitly upon evaluation of integral \( I_{22} \). Evaluation of \( I_{22} \) depends on the choice of the equilibrium distribution \( F(\Delta) \).

### 4.6.1. Maxwell-Boltzmann distribution (\( \alpha = 2 \))

With the Maxwell-Boltzmann distribution (3.6), \( F(\Delta) = (2\pi)^{-3/2}e^{-\Delta/2} \), \( dF/d\Delta = -\frac{1}{2}(2\pi)^{-3/2}e^{-\Delta/2} \), with \( \Delta = r^2/U^2 \). Since this integrand of \( I_{22} \) decays exponentially and \( \ell/s\tau \) is very large (since \( s \sim O(1) \) by virtue of (4.14)), it is reasonable to approximate \( I_{22} \) by extending the integral to infinity. Then, \( I_{22} \approx -\frac{1}{2}U^7 \), \( \int_0^{\infty} se^{-s} \, ds = 1 \), and the viscosity from (4.36) becomes \( \mu = \rho U^2 \tau \).

\[ \text{(4.37)} \]

This end result is identical to that obtained via the Chapman-Enskog expansion, which also assumes small \( \tau \) and a Maxwell-Boltzmann distribution (Chapman & Cowling 1991). Equation (4.37) also verifies the scale parameter is \( U^2 = \sqrt{\nu/\tau} \), as listed in Table 2.

### 4.6.2. Cauchy distribution (\( \alpha = 1 \))

If instead we select the Cauchy distribution (3.7), \( F(\Delta) = \pi^{-2}(\Delta+1)^{-2} \), then \( dF/d\Delta = -\frac{2}{\pi^2}(\Delta+1)^{-3} \), and

\[ I_{22} = \frac{4\pi}{15} \int_0^{\ell/s\tau} \frac{-2}{\pi^2(\Delta+1)^3} \, dr = -\frac{8U^6}{15\pi} \int_0^{\ell/s\tau} \frac{r^6 \, dr}{(r^2+U^2)^3} \]

\[ = -\frac{8U^6}{15\pi} \left[ \frac{8r^5 + 25r^3U^2 + 15rU^4}{8(r^2+U^2)^2} - \frac{15U}{8} \tan^{-1} \frac{r}{U} \right]_{r=0}^{\ell/s\tau} \approx -\frac{8U^6}{15\pi} \frac{\ell}{s\tau} \]

where the last simplification was made possible because \( \frac{\ell}{s\tau} \gg U \). The \( s \) integral in (4.36) becomes \( \int_0^{\infty} e^{-s} \, ds = 1 \), and the viscosity relation from (4.36) now yields \( \mu = 16 \frac{\rho U}{15\pi} \ell_1 \).

\[ \text{(4.39)} \]

Since the viscosity \( \mu = \rho \nu \) is a property of the fluid (which doesn’t change with the turbulence level for a Newtonian fluid) and Table 2 already prescribes \( U_1 = \gamma_1 u_\tau \), then the only free parameter in (4.39) is \( \ell_1 \). Thus, equation (4.39) prescribes

\[ \ell_1 = \frac{15\pi}{16} \frac{\nu}{\gamma_1 u_\tau} \approx \frac{\nu}{u_\tau}, \]

which is consistent with the assertion (4.17).

### 4.6.3. General Lévy \( \alpha \)-stable distribution

To evaluate the general case, we first note that the result in (4.38) could have been equivalently obtained by only considering the tails of the \( F(\Delta) \) distribution. In the general case, the tails of the 3D multivariate Lévy \( \alpha \)-stable distributions behave asymptotically as (Nolan 2006)

\[ F(\Delta) \approx \frac{C_\alpha}{\Delta^{(\alpha+3)/2}} \quad \text{for} \quad \Delta \gg 1, \quad \text{where} \quad C_\alpha = \frac{2^\alpha \Gamma(\frac{\alpha+3}{2})}{\pi^{\frac{3}{2}} \Gamma(-\frac{\alpha}{2})}. \]

\[ \text{(4.41)} \]

† The subscript 2 reminds the reader that this \( U \) corresponds to \( \alpha = 2 \).

‡ The subscript 1 reminds the reader that these \( U \) and \( \ell \) correspond to \( \alpha = 1 \).
Thus, the derivative of $F$ is asymptotically
\[
\frac{dF}{d\Delta} \approx -\frac{\alpha + 3}{2} \frac{\bar C_\alpha}{\Delta^{(\alpha+5)/2}} \quad \text{for} \quad \Delta \gg 1.
\]
Inserting (4.42) and $\Delta = r^2/U^2$ into (4.36), we have
\[
I_{22} \approx \frac{4\pi}{15} \int_0^{\ell/s\tau} r^6 \left( -\frac{\alpha + 3}{2} \frac{\bar C_\alpha U^{\alpha+5}}{r^{\alpha+5}} \right) dr = -\frac{4\pi}{15} \frac{\alpha + 3}{2} \frac{\bar C_\alpha U^{\alpha+5}}{2 - \alpha} \left( \frac{\ell}{s\tau} \right)^{2-\alpha}.
\]
With the result $\int_0^\infty s^{\alpha-1} e^{-s} ds = \Gamma(\alpha)$, the viscosity relation (4.36) now yields
\[\mu = \frac{4\pi}{15} \frac{\alpha + 3}{2} \frac{\bar C_\alpha (U\tau)^\alpha}{\tau} \ell^{2-\alpha} \Gamma(\alpha),\]
and with $(U\tau)^\alpha/\tau = \gamma^\alpha q$ from (3.2), equation (4.44) yields the final result:
\[
\ell = \left( \frac{15}{4\pi} \frac{2 - \alpha}{\alpha + 3} \frac{\nu}{\Gamma(\alpha)\bar C_\alpha \gamma^\alpha q} \right)^{1/(2-\alpha)}.
\]
Evaluation of (4.45) for the $\alpha = 1$ and $\alpha = \frac{2}{3}$ cases yields $\ell_1 \sim \nu/u_\tau$ and $\ell_2 \sim \nu^2 \epsilon^{-\frac{1}{2}}$, respectively, consistent with the values given in (4.17) and (4.18).

4.6.4. Friction force (small displacements)

The friction force appearing in the momentum equation is then obtained by taking the divergence of the stress (4.36) in the usual way: $(\nabla \cdot \bar \tau^\text{visc})_j(x) = \frac{\partial \bar \tau^\text{visc}_{ij}(x)}{\partial x_i} = \mu \left[ \frac{\partial^2 u_i}{\partial x_i \partial x_j} + \frac{\partial^2 u_j}{\partial x_i x_i} \right]$. The first term is zero by incompressibility $\frac{\partial u_i}{\partial x_j} = 0$, which leaves the usual result for a Newtonian fluid:
\[
(\nabla \cdot \bar \tau^\text{visc})_j(x) = \mu \nabla^2 \bar u_j.
\]

4.7. Large Displacements: $s\tau|\mathbf{u}| \gg \ell$

We now turn our attention to the case of large displacements, $s\tau|\mathbf{u}| \gg \ell$, wherein we seek to evaluate the turbulent stress (4.30), which is repeated here for convenience:
\[
\tau^\text{turb}_{ij}(x) \equiv -\frac{\rho}{U^2} \iiint_{s\tau|\mathbf{u}| \gg \ell} \int \{ u_i - \bar u_i(x) \} \{ u_j - \bar u_j(x) \} \{ F(\Delta(x-s\tau\mathbf{u})) - F(\Delta(x)) \} e^{-s} ds \, d\mathbf{u}.
\]
For large displacements $(s\tau|\mathbf{u}| > \ell)$, the assumption $\lambda \ll \ell$ from (4.16) has two important implications that will simplify the analysis of (4.30):
\[
\text{large speeds: } |\mathbf{u}| \gg |\bar \mathbf{u}|,
\]
\[
\text{large velocity deviations: } \Delta \equiv |\mathbf{u} - \bar \mathbf{u}|^2/U^2 \gg 1.
\]

Proof: Since $s \sim \mathcal{O}(1)$ by virtue of (4.14), a large displacement $s\tau|\mathbf{u}| > \ell$ implies $|\mathbf{u}| > \mathcal{O}(\ell/\tau) = \mathcal{O}(\ell/\lambda)U_2 \gg U_2 \gg U$. Since $|\bar \mathbf{u}| \leq \mathcal{O}(U_2)$ by virtue of (4.15), $|\mathbf{u}| \gg U_2$ then implies $|\mathbf{u}| \gg |\bar \mathbf{u}|$. Then, $\Delta \equiv |\mathbf{u} - \bar \mathbf{u}|^2/U^2 \approx |\mathbf{u}|^2/U^2 \gg 1$.

† The last inequality holds, because $U$ decreases as $\alpha$ decreases: $U_2 > U_1 > U_\frac{2}{3}$. Given that $w_\tau/V \sim 1/\ln(Re)$ and $\epsilon \sim V^3/L$, we have the estimates $U_2/V = 1/(Re Kn)$, $U_1/V \sim 1/\ln(Re)$, and $U_\frac{2}{3}/V \sim Re Kn/\sqrt{Re}$. 

To begin the analysis of (4.30), note that large speeds \(|u| \gg |\bar{u}|\) permits the neglect of the \(\bar{u}_i\) and \(\bar{u}_j\) terms:

\[
\tau_{ij}^{\text{turb}}(x) = -\frac{\rho}{U^3} \iint \int u_i u_j \left\{ F(\Delta(x - s\tau u)) - F(\Delta(x)) \right\} e^{-s} \, ds \, du.
\]  

(4.48)

Our strategy now is to decouple the time and velocity integrals with the following substitution, \(u_i = (x_i - x'_i)/(s\tau)\) and \(du = dx'/(s\tau)^3\), which replaces velocity with its corresponding displacement over flight time:

\[
\tau_{ij}^{\text{turb}}(x) = -\frac{\rho}{U^3} \iint \int \frac{x_i - x'_i}{s\tau} \frac{x_j - x'_j}{s\tau} \left\{ F(\Delta(x')) - F(\Delta(x)) \right\} e^{-s} \, ds \, \frac{dx'}{(s\tau)^3}.
\]  

(4.49)

The negative signs in the \(dx'/(s\tau)^3\) terms have been used to flip the integration limits in (4.49).

Note that time still appears implicitly in the \(F(\Delta)\) terms (by virtue of the velocities therein), so the time and space integrals are not yet decoupled. To proceed, recall that a large displacement \(s\tau|u| \gg \ell\) implies \(\Delta \gg 1\), so we need only consider the tails of \(F(\Delta)\) when evaluating (4.49). Moreover, since \(|u| \gg |\bar{u}|\), we can further simplify the form of \(\Delta\) and \(F(\Delta)\) as follows: Rewriting definition (3.5), we have

\[
\Delta(x) = \sum_{k=1}^{3} \left\{ \frac{u_k^2}{U^2} - 2 \frac{\bar{u}_k \bar{u}(x)}{U^2} + \bar{u}_k(x)^2 \right\} = A - 2B + C
\]  

(4.50)

\[
\Delta(x') = \sum_{k=1}^{3} \left\{ \frac{u_k^2}{U^2} - 2 \frac{\bar{u}_k \bar{u}(x')}{U^2} + \bar{u}_k(x')^2 \right\} = A - 2B' + C'.
\]  

(4.51)

Since \(|u| \gg |\bar{u}|\), we note the ordering \(A \gg B \gg C\) and \(A \gg B' \gg C'\). Further, since the difference \(\Delta(x') - \Delta(x) \approx 2(B - B')\) is much smaller than \(\Delta\) itself (since \(\Delta \approx A\)), the difference \(F(\Delta(x')) - F(\Delta(x))\) may be simplified via Taylor series expansion:

\[
F(\Delta(x')) - F(\Delta(x)) \approx \frac{dF(\Delta(x))}{d\Delta} (\Delta(x') - \Delta(x)) \approx \frac{dF(\Delta(x))}{d\Delta} 2(B - B').
\]  

(4.52)

Upon inserting (4.52) into (4.49) and making the substitution \(u = (x - x')/(s\tau)\), we have

\[
\tau_{ij}^{\text{turb}}(x) = -\frac{\rho}{U^3 s^6} \iint \int (x'_i - x_i)(x'_j - x_j) \left\{ 2 \frac{dF}{d\Delta}(x - x') \cdot (\bar{u}(x) - \bar{u}(x')) \right\} e^{-s} \, ds \, \frac{dx'}{s^6}.
\]  

(4.53)

Note how at this stage the stress expression (4.53), originally quadratic in velocity differences, has morphed into an expression that is linear in velocity differences. The quadratic factors have now been expressed in terms of displacements, while a velocity difference re-emerged from the Taylor expansion of the distribution function.

Further progress can only be made by specifying the equilibrium distribution \(F(\Delta)\).

4.7.1. Maxwell-Boltzmann distribution \((\alpha = 2)\)

The tails of the Maxwell-Boltzmann distribution fall of exponentially, so the turbulent stress \(\tau_{ij}^{\text{turb}}\) is effectively zero. To show this formally, recall that for the Maxwell-
Boltzmann distribution, \( F(\Delta) = \frac{1}{(2\pi)^3/2} e^{-\Delta/2} \), so
\[
d\overline{F} = -\frac{1}{2} \frac{1}{(2\pi)^{3/2}} e^{-\Delta/2} \approx -\frac{1}{2} \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2} \frac{|\mathbf{x} - \mathbf{x}'|^2}{\overline{u}^2}} = -\frac{1}{2} \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2} \left(\frac{|\mathbf{x} - \mathbf{x}'|^2}{\overline{u}^2}\right)}.
\] (4.54)

Thus, the time integral in (4.53) is
\[
\int_0^\infty e^{-\frac{1}{2} \left(\frac{|\mathbf{x} - \mathbf{x}'|^2}{\overline{u}^2}\right)} s^{-2-s} ds = \frac{\sqrt{\frac{2\pi}{3}}}{\overline{u}^2} \left(\frac{|\mathbf{x} - \mathbf{x}'|}{\overline{u}^2}\right)^{-11/3} e^{-\frac{3}{2}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{\overline{u}^2}\right)^{2/3}}.
\] (4.55)

For proof of (4.55), see Appendix A.3. The stress (4.53) then becomes
\[
\tau_{ij}^{\text{turb}}(\mathbf{x}) \leq \frac{\rho}{\overline{u}^5 \tau^6} \int \int \int (x'_i - x_i)(x'_j - x_j) \left\{ \frac{1}{(2\pi)^{3/2}} (\mathbf{x} - \mathbf{x}') \cdot (\overline{\mathbf{u}}(\mathbf{x}) - \overline{\mathbf{u}}(\mathbf{x}')) \right\}
\cdot \left\{ \sqrt{\frac{2\pi}{3}} \left(\frac{|\mathbf{x} - \mathbf{x}'|}{\overline{u}^2}\right)^{-11/3} e^{-\frac{3}{2}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{\overline{u}^2}\right)^{2/3}} \right\} d\mathbf{x}'.
\] (4.56)

Since the exponent \(-\frac{3}{2}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{\overline{u}^2}\right)^{2/3}\) is very large negative, the exponential is vanishingly small. The conclusion is that the Maxwell-Boltzmann distribution, with its steep exponential tail, does not contribute in any significant way to the stress tensor in the range of large displacements. In other words, the Maxwell-Boltzmann distribution is associated with laminar viscous stress only and produces no turbulent stress.

One point must be clear. The absence of large-displacement, turbulent-like frictional stress with the use of the Maxwell-Boltzmann distribution is not equivalent to stating that the Maxwell-Boltzmann distribution precludes turbulence. It only says that all turbulent fluctuations then arise from other terms in the momentum equations (i.e. the advective terms). Turbulence is kept explicit and unresolvable (by direct numerical simulation for high-Reynolds number flows) even with today’s computer power. What we will show next is that the consideration of large displacements with a heavy-tail distribution allows for the capture of (at least some) turbulent fluctuations in some statistical way without need to resolve them in all their details.

4.7.2. General Lévy \(\alpha\)-stable distribution (\(\alpha < 2\))

We now proceed by deriving the turbulent stress for the general Lévy \(\alpha\)-stable distribution with \(0 < \alpha < 2\). The Cauchy distribution follows suit with \(\alpha = 1\). Our starting point is equation (4.53), which is repeated here for convenience
\[
\tau_{ij}^{\text{turb}}(\mathbf{x}) = -\frac{\rho}{\overline{u}^5 \tau^6} \int \int \int (x'_i - x_i)(x'_j - x_j) \left\{ \frac{1}{(2\pi)^{3/2}} (\mathbf{x} - \mathbf{x}') \cdot (\overline{\mathbf{u}}(\mathbf{x}) - \overline{\mathbf{u}}(\mathbf{x}')) \right\} e^{-s} \frac{ds}{s^6} d\mathbf{x}'.
\] (4.53)

Inserting the asymptotic expression for \(d\overline{F}/d\Delta\) from (4.42) into (4.53), and with the asymptotic expression \(\Delta \approx A = |\mathbf{u}|^2/\overline{u}^2 = |\mathbf{x} - \mathbf{x}'|^2/(\overline{u}^2)\), we have
\[
\tau_{ij}^{\text{turb}} = \frac{\rho}{\overline{u}^5 \tau^6} \int \int \int (x'_i - x_i)(x'_j - x_j) \left\{ (\alpha + 3)\overline{C}_\alpha \frac{\overline{u}^{\alpha+5}}{|\mathbf{x} - \mathbf{x}'|^{\alpha+5}} (\overline{\mathbf{u}}(\mathbf{x}) - \overline{\mathbf{u}}(\mathbf{x}')) \right\} e^{-s} \frac{ds}{s^6} d\mathbf{x}'.
\] (4.57)
or upon simplification

\[
\tau_{ij}^{turb} = \rho \frac{(U\tau)^{\alpha}}{\tau} (\alpha + 3) \bar{C}_\alpha \int \int \int \frac{(x'_i - x_i)(x'_j - x_j)}{|x'| - |x|^{\alpha+5}} \frac{(x'_i - x_i) \cdot (\bar{u}(x') - \bar{u}(x)) e^{-s} ds}{s^{1-\alpha}} dx'.
\]

\[(4.58)\]

The time integral now decouples, and its value is \(\int_0^\infty s^{\alpha-1} e^{-s} ds = \Gamma(\alpha).\) From (3.2), we have \((U\tau)^{\alpha}/\tau = \gamma^{\alpha} q.\) Thus, for \(\text{“large”}\) displacements we have finally

\[
\tau_{ij}^{turb}(x) = \rho \gamma^{\alpha}(\alpha + 3) \Gamma(\alpha) \bar{C}_\alpha \int \int \int \frac{(x'_i - x_i)(x'_j - x_j)}{|x'| - |x|^{\alpha+5}} \frac{(x'_i - x_i) \cdot (\bar{u}(x') - \bar{u}(x))}{x' - x} \, dx'.
\]

\[(4.59)\]

Appendix D provides a useful alternate form of the turbulent stress in (4.59).

4.7.3. Friction force (large displacements)

The friction force appearing in the momentum equation is the divergence of this stress tensor, \((\nabla \cdot \tau^{turb})_j(x) = \frac{\partial \tau_{ij}^{turb}(x)}{\partial x_i}.\) Taking derivatives of (4.59) term by term, and summing over \(i = 1, 2, 3,\) we have:

\[
(\nabla \cdot \tau^{turb})_j(x) = \rho \gamma^{\alpha}(\alpha + 3) \Gamma(\alpha) \bar{C}_\alpha \int \int \int \sum_{i=1}^3 \frac{(x'_i - x_i)(x'_j - x_j)}{|x'| - |x|^{\alpha+5}} \frac{(x'_i - x_i) \cdot (\bar{u}(x') - \bar{u}(x))}{|x' - x|^{\alpha+5}} dx'.
\]

\[(4.60)\]

\[
(\nabla \cdot \tau^{turb})_j(x) = \rho \gamma^{\alpha}(\alpha + 3) \Gamma(\alpha) \bar{C}_\alpha \int \int \int \frac{(x'_i - x_i)(x'_j - x_j)}{|x'| - |x|^{\alpha+5}} dx'.
\]

\[(4.61)\]

Equation (4.61) can be written as a fractional Laplacian upon integrating by parts: \(\int a \, db = ab - \int b \, da.\) First, write the dot product out into three terms \((x'_k - x_k)(\bar{u}_k(x') - \bar{u}_k(x)),\) with implied summation over \(k = 1, 2, 3.\) Now, for each of the three terms, perform partial integration in \(dx'_k\)

\[
a = (x'_j - x_j)(\bar{u}_k(x') - \bar{u}_k(x)) , \quad db = \frac{x'_k - x_k}{|x' - x|^{\alpha+5}} \, dx'_k ,
\]

\[
da = \{ \delta_{jk}(\bar{u}_k(x') - \bar{u}_k(x)) + (x'_j - x_j) \frac{\partial \bar{u}_k(x')}{\partial x_k} \} dx'_k , \quad b = \frac{-1}{(\alpha + 3)|x' - x|^{\alpha+3}},
\]

with no summation implied in each partial integration (fixed \(k).\) Since \(ab|_{-\infty}^\infty = 0\) for each of these partial integrations (so long as \(|\bar{u}(x')/|x'|^{\alpha+2} \to 0\) as \(|x'| \to \infty\)), the other
terms can be recombined (by now implying summation over $k$):

$$\left( \nabla \cdot \tau^{\text{turb}} \right)_j(x) = \rho q \gamma^\alpha \Gamma(\alpha+1) \bar{C}_\alpha \iiint_{|x' - x| \geq \ell} \delta_{jk} ( \bar{u}_k(x') - \bar{u}_k(x)) + (x'_j - x_j) \frac{\partial \bar{u}_k(x')}{\partial x'_k} \frac{1}{|x' - x|^{\alpha+3}} \, dx' ,$$  

(4.62)

where we used the identity $\alpha \Gamma(\alpha) = \Gamma(\alpha+1)$. For incompressible flow, $\frac{\partial \bar{u}_k}{\partial x'_k} = 0$, so this term is eliminated, and we have finally:

$$\left( \nabla \cdot \tau^{\text{turb}} \right)_j(x) = \rho q \gamma^\alpha \Gamma(\alpha+1) \bar{C}_\alpha \iiint_{|x' - x| \geq \ell} \frac{\bar{u}_j(x') - \bar{u}_j(x)}{|x' - x|^{\alpha+3}} \, dx' .$$  

(4.63)

Note that $\ell/L \to 0$ as $Re \to \infty$, so (4.63) is asymptotically equivalent to a Cauchy principal value integral:

$$\left( \nabla \cdot \tau^{\text{turb}} \right)_j(x) = \rho q \gamma^\alpha \Gamma(\alpha+1) \bar{C}_\alpha \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{|x' - x| \geq \ell} \frac{\bar{u}_j(x') - \bar{u}_j(x)}{|x' - x|^{\alpha+3}} \, dx' .$$  

(4.64)

Within a coefficient of proportionality, equation (4.64) prescribes the turbulent force as the “singular integral” form of the fractional Laplacian (Kwa´snicki 2017).

Note that the derivation herein was presented for 3D space, and the derivation for 1D or 2D space follows similarly, with the constant $\bar{C}_\alpha$ and exponents adjusted accordingly. Appendix E shows the reduction in order of the fractional Laplacian if the flow is two dimensional.

4.8. Momentum Equation

The momentum equation can now be formulated by inserting the sum of the small-displacement forcing (4.46) and large-displacement forcing (4.64) into momentum equation (4.8)

$$\frac{D \bar{u}}{Dt} = -\nabla \bar{p} + \mu \nabla^2 \bar{u} + \rho C_\alpha \iiint_{|x' - x| \geq \ell} \frac{\bar{u}(x') - \bar{u}(x)}{|x' - x|^{\alpha+3}} \, dx' ,$$  

(4.65)

where $C_\alpha \equiv \Gamma(\alpha+1) \bar{C}_\alpha \gamma^\alpha q$ is a turbulent mixing coefficient, with units (length)$^\alpha$/time). This is the result reported in equation (0.1).

Note that the Navier-Stokes equations are recovered from (4.65) if the equilibrium distribution is assumed to be Maxwell-Boltzmann, since $\bar{C}_\alpha = 0$ for $\alpha = 2$.

The relative magnitude of the advective inertial force to the turbulent friction force can be assessed as follows: Given characteristic velocity and length scales, $V$ and $L$, the scales of the inertial force and turbulent friction force in (4.65) are

advective inertial force: $\rho \bar{u} \cdot \nabla \bar{u} \sim \rho V^2/L$ ,

turbulent friction force: $\rho C_\alpha \iiint_{|x' - x| \geq \ell} \frac{\bar{u}(t,x') - \bar{u}(t,x)}{|x' - x|^{\alpha+3}} \, dx' \sim \rho q V/L^\alpha .$  

(4.66)

Therefore, the nondimensional number is

$$\frac{\text{advection inertial force}}{\text{turbulent friction force}} = \frac{VL^{\alpha-1}}{q} .$$  

(4.67)
which is identical to the ratio (2.43) considering turbulent transport. For $\alpha = 2$, this ratio is the Reynolds number $Re = VL/\nu$, and for $\alpha = 1$, this ratio is $V/u_\tau$.

4.9. Properties of the Fractional Laplacian

In this section, we consider some properties of the friction force (4.64):

$$F(x) = \rho C_\alpha \int_{-\infty}^{\infty} \frac{\bar{u}(x') - \bar{u}(x)}{|x' - x|^{\alpha+3}} \, dx'. \quad (4.68)$$

4.9.1. Conservation of momentum

The fractional Laplacian respects conservation of momentum, since its spatial integral across the entire domain vanishes

$$\int_{\mathbb{R}^3} F(x) \, dx = \int_{\mathbb{R}^3} \rho C_\alpha \int_{-\infty}^{\infty} \frac{\bar{u}(x') - \bar{u}(x)}{|x' - x|^{\alpha+3}} \, dx' \, dx = 0. \quad (4.69)$$

The proof is obvious from the symmetry of the integrand; substitute $x'$ for $x$ and vice versa, and find that the integral is equal to minus itself. Thus, a net momentum gain or loss occurs only at the boundaries.

4.9.2. Energy dissipation

The fractional Laplacian dissipates kinetic energy. The rate of work done on the fluid (per unit volume) is $F(x) \cdot \bar{u}(x)$, so the total work rate for the whole fluid domain is

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F(x) \cdot \bar{u}(x) \, dx = \int_{\mathbb{R}^3} \rho C_\alpha \int_{-\infty}^{\infty} \frac{\bar{u}(x') - \bar{u}(x)}{|x' - x|^{\alpha+3}} \cdot \bar{u}(x) \, dx' \, dx$$

$$= -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho C_\alpha \int_{-\infty}^{\infty} \frac{\bar{u}(x') - \bar{u}(x)}{|x' - x|^{\alpha+3}} \cdot \bar{u}(x') \, dx' \, dx$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho C_\alpha \int_{-\infty}^{\infty} \frac{|\bar{u}(x') - \bar{u}(x)|^2}{|x' - x|^{\alpha+3}} \, dx' \, dx \leq 0,$$

where the first equality comes from the anti-symmetry of the integrand, and the second equality comes from averaging the two previous expressions. Since the work rate is negative (energy is being dissipated), the total kinetic energy can only decrease over time (unless energy is provided at the boundaries).

5. Boundary Conditions and Solutions for Unidirectional Shear Flow

We preface this section by recalling that the above derivation of (4.65) was restricted to unbounded flows. The first assumption of an unbounded domain was made in equation (4.10), which only included the particular solution to the Boltzmann equation (i.e. it excluded the homogeneous solution that would exist to enforce boundary conditions). Treatment of boundary conditions for the general 3D case is beyond the scope of this article. In this section, we consider unidirectional shear flow with one and two boundaries. We then use these results to model Couette flow (two boundaries) and a boundary layer flow (one boundary, but with inertial terms).

5.1. Problem Setup

In this section, we consider steady, incompressible, unidirectional shear flow. Aligning the $x$-axis with the flow and the $z$-axis with its shear, we make the following starting
assumptions:
\[ \bar{u} = \bar{u}(z), \quad \bar{v} = \bar{w} = 0, \quad \frac{\partial}{\partial t} = 0, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0. \] (5.1)

With these assumptions, both the mass distribution function \( f \) and the equilibrium distribution \( f_{eq}^\alpha \) are reduced to functions of four variables, \( f = f(z, u, v, w) \) and \( f_{eq}^\alpha = f_{eq}^\alpha(z, u, v, w) \). The Boltzmann equation (4.4) reduces to:
\[ w\tau \frac{\partial f}{\partial z} + f = f_{eq}^\alpha. \] (5.2)

Assume the spatial domain is \( 0 \leq z \leq \H \) (where \( \H \) may be finite as in Couette flow or infinite as in a boundary layer flow). The velocity domain is triply infinite, with each of \( u, v, \) and \( w \) varying from \(-\infty\) to \(\infty\). Solutions of (5.2) are given in the following subsections.

### 5.2. Solution in a Semi-infinite Domain (\( 0 \leq z < \infty \))

Consider first the simple case of a semi-infinite domain \( (0 \leq z < \infty) \). Since equation (5.2) is linear in its unknown, \( f \), its solution can be written as the sum of the particular and homogeneous solutions:
\[ f(z, u, v, w) = \begin{cases} 
- \frac{1}{w\tau} \int_z^\infty f_{eq}^\alpha(z', u, v, w) e^{\frac{z' - z}{w\tau}} dz' & \text{for } w < 0 \\
\frac{1}{w\tau} \int_0^z f_{eq}^\alpha(z', u, v, w) e^{\frac{z' - z}{w\tau}} dz' + A(u, v, w) e^{-\frac{z}{w\tau}} & \text{for } w \geq 0
\end{cases}. \] (5.3)

Note that the homogeneous solution is zero for the \( w < 0 \) case in (5.3), because otherwise the associated exponential \( e^{\frac{-z}{w\tau}} \) would blow up as \( w \to 0^- \) or \( z \to \infty \).

Clearly, there is a physical process at play: For \( w > 0 \), the fluid comes upward from below, and \( 0 \leq z' \leq z \). For \( w < 0 \), the fluid comes downward from above, and \( z \leq z' \leq \infty \). In both cases, the starting point of the integral term is the boundary (either \( z = 0 \) or \( \infty \)). Note that all exponentials remain well behaved as \( w \to \pm 0 \).

The homogeneous solution \( A(u, v, w) e^{-\frac{z}{w\tau}} \) exists so as to enforce the following boundary condition: At a solid boundary, the particles must ‘bounce back’ (Maxwell 1879), and this implies symmetry of \( f \) with respect to the normal velocity, \( w \), at the boundary:
\[ f(z = 0, u, v, w) = f(z = 0, u, v, -w). \] (5.4)

In order to enforce this symmetry, set \( w \to -w \) in the \( w < 0 \) expression in (5.3), and set that expression equal to the \( w \geq 0 \) expression. The equilibrium function \( f_{eq}^\alpha \) is symmetric in \( w \), so it remains unchanged. Thus, equation (5.4) yields
\[ A(u, v, w) = \frac{1}{w\tau} \int_0^\infty f_{eq}^\alpha e^{-\frac{z'}{w\tau}} dz', \] (5.5)
and the solution (5.3) for \( f \) that meets the boundary condition (5.4) is:
\[ f(z, u, v, w) = \begin{cases} 
- \frac{1}{w\tau} \int_z^\infty f_{eq}^\alpha e^{\frac{z' - z}{w\tau}} dz' & \text{for } w < 0 \\
\frac{1}{w\tau} \int_0^z f_{eq}^\alpha e^{\frac{z' - z}{w\tau}} dz' + \frac{1}{w\tau} \int_0^\infty f_{eq}^\alpha e^{-\frac{z'}{w\tau}} dz' & \text{for } w \geq 0
\end{cases}. \] (5.6)

Equation (5.6) is the solution of the Boltzmann equation for the unidirectional shear flow problem with a single boundary (at \( z = 0 \)).
Two relevant questions that are important to ask at this point are, does the solution from (5.6) respect the density constraint (4.1)? Also, what are the ensemble-averaged velocity field $\bar{u}(z)$ (4.2) and stress field $\tau_{zz}(z)$ (4.9) corresponding to this $f(z,u,v,w)$?

With the $f$ given by (5.6), it is easy to verify that the density constraint (4.1) is indeed satisfied. Straightforward algebra also shows that the resulting velocity (4.2) and stress (4.9) are given by:

$$0 = \int_{0}^{\infty} [\bar{u}(z') - \bar{u}(z)] \, dz' \int_{0}^{\infty} f_1^{eq}(w;\alpha) \left[ e^{-\frac{|z'-z|}{w}} + e^{-\frac{z'+z}{w}} \right] \, \frac{dw}{w} \quad , \quad (5.7)$$

$$\tau_{zz}(z) = \frac{1}{\tau} \int_{0}^{\infty} [\bar{u}(z') - \bar{u}(z)] \, dz' \int_{0}^{\infty} f_1^{eq}(w;\alpha) \left[ \text{sign}(z'-z)e^{-\frac{|z'-z|}{w}} - e^{-\frac{z'+z}{w}} \right] \, dw \quad , \quad (5.8)$$

where we have defined the 1D distribution $f_1^{eq}(w;\alpha) \equiv \int_{-\infty}^{\infty} f^{eq}_\alpha(z',u,v,w) \, du \, dv$ and used the relation $\int_{-\infty}^{\infty} u f^{eq}_\alpha(z',u,v,w) \, du \, dv = \bar{u}(z') f_1^{eq}(w;\alpha)$ (See Appendix F). Equations (5.7) and (5.8) are the governing equations for the ensemble-averaged velocity field and its resulting stress field for the case of a single boundary. They are exact and can be evaluated upon specifying the equilibrium distribution $f_1^{eq}(w;\alpha)$.

Note that the integrals over $w$ now appear in one of two nondimensional forms:

$$g_0(\chi;\alpha) \equiv \int_{0}^{\infty} \left[ \frac{U}{\tau} f_1^{eq}(U\beta;\alpha) \right] e^{-\chi/\beta} \, d\beta \quad ,$$

$$g_1(\chi;\alpha) = \int_{0}^{\infty} \left[ \frac{U}{\tau} f_1^{eq}(U\beta;\alpha) \right] e^{-\chi/\beta} \, d\beta \quad . \quad (5.9)$$

where we have defined $\beta \equiv w/U$. Moreover, note that because $U\tau$ is very small, the argument $\chi = \frac{|z'-z|}{U\tau}$ or $\frac{z'+z}{U\tau}$ is very large, except for a small range near $z' = z$. Therefore, one can easily evaluate $g_0$ and $g_1$ using the asymptotic formulae derived in Appendix G.

Note that the stress $\tau_{zz}$ (5.8) is independent of $z$. For proof, take the $z$-derivative of (5.8), use (5.7) to eliminate one set of terms, and use the asymptotic expressions for $g_0$ in Appendix G to verify that the other set of terms cancel. Thus, the stress is constant and equal to its wall value, $\tau_{zz} = \rho \bar{u}_z^2$.

5.2.1. Example: Velocity and Stress for the Cauchy $f_\alpha^{eq}$ distribution

For the Cauchy distribution $\frac{U}{\rho} f_1^{eq}(U\beta;\alpha=1) = \frac{1}{\pi} \frac{1}{\beta^2+1}$, Appendix G gives $g_0(\chi) \approx 1/(\pi \chi)$ and $g_1(\chi) \approx 1/(\pi \chi^2)$. Thus, equations (5.7) and (5.8) evaluate to

$$0 = \frac{\rho U_1}{\pi} \int_{0}^{\infty} [\bar{u}(z') - \bar{u}(z)] \left[ \frac{1}{(z'-z)^2} + \frac{1}{(z'+z)^2} \right] \, dz' \quad , \quad (5.10)$$

$$\tau_{zz}(z) = \frac{\rho U_1}{\pi} \int_{0}^{\infty} [\bar{u}(z') - \bar{u}(z)] \left[ \frac{1}{z'-z} - \frac{1}{z'+z} \right] \, dz' \quad . \quad (5.11)$$

Equation (5.10) is the governing equation for the ensemble-averaged flowfield, and equation (5.11) provides the associated shear stress. Note that equation (5.10) allows for two degrees of freedom: an additive constant and a multiplicative constant. Thus, two boundary conditions must be supplied to yield a unique solution.

Allowing for the fluid to slip along the boundary (i.e. not prescribing $\bar{u}(0)$), the solution
to (5.10) is the logarithmic profile

\[ \tilde{u}(z) = \frac{u_\tau}{\kappa} \ln \left( \frac{z}{z_0} \right). \]  \hspace{1cm} (5.12)

For proof, see Appendix A.4. Here \( u_\tau \) is the friction velocity, \( \kappa \) is the von Kármán constant, and \( z_0 \) is the roughness height. The stress associated with this log profile is

\[ \tau_{zz}(z) = \frac{\rho U_1 u_\tau}{\pi \kappa} \int_0^\infty \ln(z'/z) \left[ \frac{1}{z' - z} - \frac{1}{z' + z} \right] dz' = \frac{\rho U_1 u_\tau}{\pi \kappa} \int_0^\infty \ln(z'/z) \frac{2z}{z'^2 - z^2} dz'. \]  \hspace{1cm} (5.13)

The last integral was evaluated via change of variable \( z' = z\zeta \), which eliminates the \( z \) variable and verifies that the stress is independent of \( z \), as expected. Thus, \( \tau_{zz} = \frac{\pi}{2} \rho U_1 \frac{u_\tau}{\kappa} \).

Now since the friction velocity is defined from the wall stress as \( \tau_{zz} = \frac{\pi}{2} \rho U_1 \frac{u_\tau}{\kappa} \) and the agitation velocity is given by \( U_1 = \gamma_1 u_\tau \), we have \( \rho u_\tau^2 = \frac{\pi}{2} \rho (\gamma_1 u_\tau) \frac{u_\tau}{\kappa} \), which can be rearranged as:

\[ \gamma_1 = \frac{U_1}{u_\tau} = \frac{2\kappa}{\pi}. \]  \hspace{1cm} (5.14)

With \( \kappa = 0.41 \) this ratio is \( \gamma_1 = 0.261 \), and using this \( \gamma_1 \) in (4.40) yields \( \ell_1 = 11.3\nu/u_\tau \). Alternatively, if we had a theoretical justification to \textit{a priori} prescribe \( \ell_1 = 11.3\nu/u_\tau \), then equations (4.40) and (5.14) would predict the von Kármán constant to be \( \kappa = 0.41 \).

Note that the intercept between the laminar sublayer \( \tilde{u}(z)/u_\tau = z^+ \) and the log law \( \tilde{u}(z)/u_\tau = \ln(z^+)/\kappa + C^+ \), with \( \kappa = 0.41 \) and \( C^+ = 5.0 \), occurs at \( z^+ \equiv zu_\tau/\nu = 10.8 \), which is very close to our theoretical prediction of 11.3.

### 5.3. Solution in a Finite Domain \( 0 \leq z \leq H \)

Turning our attention to a finite domain \( 0 \leq z \leq H \), the solution of the Boltzmann equation (5.2) is

\[ f(z,u,v,w) = \begin{cases} 
\frac{1}{w\tau} \int_H^z f_{\alpha}^{eq}(z',u,v,w) e^{z'/w\tau} dz' + A_H(u,v,w) e^{|w-z|/w\tau} & \text{for } w < 0 \\
\frac{1}{w\tau} \int_0^z f_{\alpha}^{eq}(z',u,v,w) e^{z'/w\tau} dz' + A_0(u,v,w) e^{-|z|/w\tau} & \text{for } w \geq 0
\end{cases}. \]  \hspace{1cm} (5.15)

Similar to (5.3), the solution (5.15) is composed of particular and homogeneous parts, and again the particular solution involves integration in \( z' \) from a boundary (at 0 or \( H \)) to the point of interest \( z \). Because there are two boundaries, the homogenous solution has contributions for both \( w \) positive and negative; the functions \( A_0 \) and \( A_H \) will now be determined so as to enforce the bounce-back boundary condition at both walls.

The ‘bounce-back’ boundary condition implies symmetry of \( f \) with respect to the normal velocity, \( w \), now at both boundaries:

\[ f(z=0,u,v,w) = f(z=0,u,v,-w), \]  \hspace{1cm} (5.16)

\[ f(z=H,u,v,w) = f(z=H,u,v,-w). \]

These two constraints in (5.16) determine the two unknown functions, \( A_H \) and \( A_0 \), in
(5.15). Evaluating \( f \) at \( z = 0 \) and \( H \), equation (5.15) yields

\[
\begin{align*}
f(z=0,u,v,w) &= \begin{cases} \frac{-1}{w^T} \int_0^H f_{\alpha} e^{\frac{z'}{w^T}} dz' + A_H(u,v,w) e^{\frac{H}{w^T}} & \text{for } w < 0 \\ A_0(u,v,w) & \text{for } w \geq 0 \end{cases} \\
f(z=H,u,v,w) &= \begin{cases} A_H(u,v,w) & \text{for } w < 0 \\ \frac{1}{w^T} \int_0^H f_{\alpha} e^{\frac{z'-H}{w^T}} dz' + A_0(u,v,w) e^{\frac{-H}{w^T}} & \text{for } w \geq 0 \end{cases} 
\end{align*}
\]  

(5.17)

In order to enforce (5.16), set \( w \rightarrow -w \) in each of the \( w < 0 \) expressions in (5.17), and set them equal to the corresponding \( w \geq 0 \) expressions. Then for \( w \geq 0 \), we have

\[
\begin{align*}
\frac{1}{w^T} \int_0^H f_{\alpha} e^{\frac{z'}{w^T}} dz' + A_H(u,v,-w) e^{\frac{-H}{w^T}} &= A_0(u,v,w) , \\
\frac{1}{w^T} \int_0^H f_{\alpha} e^{\frac{z'-H}{w^T}} dz' + A_0(u,v,w) e^{\frac{-H}{w^T}} &= A_H(u,v,-w) .
\end{align*}
\]  

(5.18)

Equations (5.18) form a 2 by 2 system of equations for the unknown functions \( A_0(u,v,w) \) and \( A_H(u,v,-w) \). The solution to this system of equations is

\[
\begin{align*}
A_H(u,v,-w) &= \frac{1}{w^T \sinh \left( \frac{H}{w^T} \right)} \int_0^H f_{\alpha} e^{\frac{z'}{w^T}} dz' , \\
A_0(u,v,w) &= \frac{1}{w^T \sinh \left( \frac{H}{w^T} \right)} \int_0^H f_{\alpha} e^{\frac{z'-H}{w^T}} dz' .
\end{align*}
\]  

(5.19)

(5.20)

In order to form \( A_H(u,v,w) \), we substitute \( w \rightarrow -w \) into equation (5.19), and we find that the result is unchanged, \( A_H(u,v,w) = A_H(u,v,-w) \). With equations (5.19) and (5.20) into (5.15), the final expression for \( f \) is:

\[
\begin{align*}
f(z,u,v,w) &= \begin{cases} \frac{-1}{w^T} \int_z^H f_{\alpha} e^{\frac{z'-z}{w^T}} dz' + \frac{e^{\frac{H-z}{w^T}}}{w^T \sinh \left( \frac{H}{w^T} \right)} \int_0^H f_{\alpha} e^{\frac{z'}{w^T}} dz' & \text{for } w < 0 \\ \frac{1}{w^T} \int_0^z f_{\alpha} e^{\frac{z'-z}{w^T}} dz' + \frac{e^{-\frac{z}{w^T}}}{w^T \sinh \left( \frac{H}{w^T} \right)} \int_0^H f_{\alpha} e^{\frac{z'-H}{w^T}} dz' & \text{for } w \geq 0 \end{cases} 
\end{align*}
\]  

(5.21)

where \( f_{\alpha} = f_{\alpha}(z',u,v,w) \). Equation (5.21) is the solution of the Boltzmann equation (5.2) for the case of two boundaries (at \( z = 0 \) and \( H \)).

With the \( f \) given by (5.21), it is straightforward to verify that the density constraint (4.1) is again satisfied. With some effort, the velocity \( \bar{u}(z) \) (4.2) can be shown to obey

\[
0 = \int_0^H [\bar{u}(z') - \bar{u}(z)] dz' \int_0^\infty f_{1;\alpha}(w;\alpha) e^{-\frac{|z-z'|}{w^T}} dw \bigg/ w \\
+ \int_0^H [\bar{u}(z') - \bar{u}(z)] dz' \int_0^\infty f_{1;\alpha}(w;\alpha) \left\{ \frac{e^{\frac{z-H}{w^T}}}{w^T \sinh \left( \frac{H}{w^T} \right)} \cosh \left( \frac{z'}{w^T} \right) + \frac{e^{-\frac{z}{w^T}}}{w^T} \cosh \left( \frac{z'-H}{w^T} \right) \right\} \bigg/ w .
\]  

(5.22)

We can simplify further by writing the \( \cosh(\ldots) \) terms as exponentials and expanding the \( \sinh(\ldots) \) term in a Taylor series

\[
\frac{1}{\sinh \left( \frac{H}{w^T} \right)} = \frac{2}{e^{\frac{H}{w^T}} - e^{-\frac{H}{w^T}}} = \frac{2}{1 - e^{-2\frac{H}{w^T}}} = \frac{2}{1 - e^{-2\frac{H}{w^T}}} \sum_{n=0}^\infty e^{-2n\frac{H}{w^T}} ,
\]  

(5.23)
which allows us to write equation (5.22) as

$$
0 = \int_0^H \left[ \bar{u}(z') - \bar{u}(z) \right] dz' \int_0^\infty f_1^{eq}(w;\alpha) e^{-|z'-z| \frac{w}{\pi \chi}} \frac{dw}{w}
+ \int_0^H \left[ \bar{u}(z') - \bar{u}(z) \right] dz' \int_0^\infty f_1^{eq}(w;\alpha) \sum_{n=0}^{\infty} e^{-\frac{2nH}{w}} \left\{ e^{-\frac{z'+z}{w}} + e^{-\frac{2H-z'-z}{w}} - e^{-\frac{2nH-z'}{w}} + e^{-\frac{2nH+z'}{w}} \right\} \frac{dw}{w}
$$

(5.24)

The stress (4.9) is evaluated in much the same way: Upon inserting (5.21) into (4.9), manipulating the \( \cosh(\cdots) \) terms, and making use of (5.23), the stress evaluates to

$$
\tau_{zz}(z) = \frac{1}{\tau} \int_0^H \left[ \bar{u}(z') - \bar{u}(z) \right] \sigma(z'-z) dz' \int_0^\infty f_1^{eq}(w;\alpha) e^{-|z'-z| \frac{w}{\pi \chi}} \frac{dw}{w}
+ \frac{1}{\tau} \int_0^H \left[ \bar{u}(z') - \bar{u}(z) \right] dz' \int_0^\infty f_1^{eq}(w;\alpha) \sum_{n=0}^{\infty} e^{-\frac{2nH}{w}} \left\{ - e^{-\frac{z'+z}{w}} + e^{-\frac{2H-z'-z}{w}} - e^{-\frac{2nH-z'}{w}} + e^{-\frac{2nH+z'}{w}} \right\} \frac{dw}{w}
$$

(5.25)

The solution of equation (5.24) yields the ensemble-averaged velocity field, and equation (5.25) provides the associated stress field. Note that in the limit \( H \rightarrow \infty \), equations (5.24) and (5.25) reduce to (5.7) and (5.8), as expected. Note also that the integrals in (5.24) and (5.25) are all of the form of \( g_0(\chi;\alpha) \) and \( g_1(\chi;\alpha) \) from (5.9), so these can be evaluated easily using the results from Appendix G.

Note that similar to the single-boundary case, the stress \( \tau_{zz} \) (5.25) can be shown to be independent of \( z \) and thus equal to its wall value, \( \tau_{zz} = \rho \bar{u}_z^2 \). The proof follows as in the single-boundary case.

5.3.1. Example: Velocity and Stress for the Cauchy \( f_0^{eq} \) distribution

For the Cauchy distribution \( f_1^{eq}(w;\alpha=1) \), Appendix G gives \( g_0(\chi) \approx 1/(\pi \chi) \) and \( g_1(\chi) \approx 1/(\pi \chi^2) \). Thus, equations (5.24) and (5.25) evaluate to

$$
0 = \frac{\rho \bar{U}_1}{\pi} \int_0^H \left[ \bar{u}(z') - \bar{u}(z) \right] \left[ \frac{1}{(z'-z)^2} + \sum_{n=0}^{\infty} \left\{ \frac{1}{(2nH+z'+z)^2} + \frac{1}{((2n+2)H-z'-z)^2} \right\} \right] dz'
$$

(5.26)

$$
\tau_{zz}(z) = \frac{\rho \bar{U}_1}{\pi} \int_0^H \left[ \bar{u}(z') - \bar{u}(z) \right] \left[ \frac{1}{z'-z} + \sum_{n=0}^{\infty} \left\{ - \frac{1}{2nH+z'+z} + \frac{1}{(2n+2)H-z'-z} - \frac{1}{(2n+2)H-z'+z} + \frac{1}{(2n+2)H+z'-z} \right\} \right] dz'
$$

(5.27)
Figure 3: Illustration of the denominators appearing in (5.26) and (5.27) as the lengths of possible paths from point \( z \) to point \( z' \). The path lengths repeat to infinity.

Equation (5.26) is the governing equation for the ensemble-averaged flowfield, and equation (5.27) provides the associated shear stress. Note that (5.26) and (5.27) are mirror symmetric about the channel midplane; that is, the substitution \( z \rightarrow H - z \) and \( z' \rightarrow H - z' \) (i.e. a change of coordinate system from being attached to the bottom wall pointing up to being attached to the top wall pointing down) leaves the velocity equation unchanged and results in \( \tau_{xx} \rightarrow -\tau_{xx} \) (with the negative sign accounting for the change in direction of the coordinate system), as expected.

The denominators in (5.26) and (5.27) represent the lengths of all possible paths from point \( z \) to point \( z' \), as illustrated in Figure 3. The parameter \( n \) can be interpreted as the number of additional bounces off both walls (before starting one of the \( n = 0 \) paths), thus adding \( 2Hn \) to the path lengths.

As with (5.10), equation (5.26) allows for two degrees of freedom: an additive constant and a multiplicative constant. Thus, two boundary conditions must be supplied to yield a unique solution.

Although an analytic solution of (5.26) is not presently known to the authors, an analytic solution can be found for the leading-order operator:

\[
\int_0^H \frac{\bar{u}(z') - \bar{u}(z)}{(z' - z)^2} \, dz' = 0 . \tag{5.28}
\]

Allowing the fluid to slip along the walls, the exact solution of (5.28) is the double-logarithmic profile:

\[
\bar{u}(z) = \frac{V}{2} + \frac{u_\tau}{\kappa} \ln \left( \frac{z}{H - z} \right) . \tag{5.29}
\]

For proof, see see §A.5. In (5.29), the quantities \( V/2 \) (velocity at mid-height) and \( u_\tau/\kappa \) are the allowed degrees of freedom. Appendix H verifies that this double-log profile (5.29) is nearly an exact solution of the full momentum equation (5.26), in agreement with experimental observations for turbulent Couette flow.

5.4. Solution in a Finite Domain with Laminar Sublayers \((b \leq z \leq H - b)\)

The reality of turbulent flows is that regardless of the agitation within the bulk of the fluid, there exists a laminar sublayer next to each wall. Modeling these laminar sublayers requires a non-uniform \( \alpha(z) \) distribution, with \( \alpha = 2 \) in the laminar sublayers and \( \alpha < 2 \) within the bulk of the fluid (Epps 2017). However, modeling a non-uniform \( \alpha(z) \) distribution poses a significant mathematical challenge. In order to break ground herein, we consider piecewise constant \( \alpha(z) \), which is the simplest non-uniform \( \alpha(z) \) that
models the essential physics:

\[
\alpha(z) = \begin{cases} 
\alpha_2 & \text{for } 0 \leq z \leq b \text{ or } H - b \leq z \leq H \\
\alpha_1 & \text{for } b \leq z \leq H - b
\end{cases} \quad \text{ laminar sublayers}
\]

\[
\text{turbulent core flow} \ .
\]

(5.30)

where \(\alpha_2 \equiv 2\) and say \(\alpha_1 = 1\).

In order to conserve mass and momentum between the three zones, we apply the constraint on \(f\) that the probability of vertical velocities bringing particles into each laminar sublayer equals that bringing them out. This anstaz implies symmetry of \(f\) with respect to \(w\) at the edge of each laminar sublayer:

\[
f(z = b, u, v, w) = f(z = b, u, v, -w)
\]

\[
f(z = H - b, u, v, w) = f(z = H - b, u, v, -w) \ .
\]

(5.31)

Note that these constraints imply that \(\bar{w}(z = b) = \bar{w}(z = H - b) = 0\), consistent with our assumptions in (5.1) above.

In Appendix I, we show that the starting equation (5.15) with boundary conditions from (5.31) results in a mass distribution \(f_{\text{eq}}\) within the core zone \(b \leq z \leq H - b\). In fact, we can define a shifted coordinate system \(\bar{z} \equiv z - b\) and write the solution with laminar sublayers (110) in the same form as our prior solution without laminar sublayers (5.21). In other words, modeling laminar sublayers via piecewise constant \(\alpha(z)\) is equivalent to modeling a flow with uniform \(\alpha_1\) in a channel of reduced height \(\bar{H} \equiv H - 2b\). Figure 4 provides an illustration.

Thus, equations (5.26) and (5.27) (which were derived from (5.21) for flows without laminar sublayers) can be adapted to the case of laminar sublayers by making the following substitutions:

\[
z \rightarrow z - b
\]

\[
z' \rightarrow z' - b
\]

\[
H \rightarrow H - 2b \ .
\]

(5.32)

These substitutions are carried out in equations (6.8) and (6.9) below.

The boundary conditions to (5.26) are also modified to account for the position and velocity at the edge of the laminar sublayer, which are \(\bar{u}(z = b) = \bar{u}_b\) and \(\bar{u}(z = H - b) = V - \bar{u}_b\) (rather than the actual surface speeds \(\bar{u}(z = 0) = 0\) and \(\bar{u}(z = H) = V\)). The wall shear stress, which is due only to viscous stress, is \(\tau_w = \mu \bar{u}_b/b\).

5.5. Incorrect Boundary Conditions

The development in §5.4 is complicated and begs comparison to a simple alternative. One common way to apply boundary conditions is to evaluate the fractional Laplacian in
the infinite domain, but to assume that the velocity outside of the physical domain equals that at the boundary (Lischke et al. 2018). In this case, we would have \( \bar{u}(z>H) = \bar{u}(z=H) \) and \( \bar{u}(z<0) = \bar{u}(z=0) \). In the Cauchy case, the momentum equation would be

\[
0 = \int_{-\infty}^{\infty} \frac{u(z') - \bar{u}(z)}{(z' - z)^2} dz' \quad \text{(5.33)}
\]

\[
0 = \int_{0}^{H} \frac{u(z') - \bar{u}(z)}{(z' - z)^2} dz' + \int_{-\infty}^{0} \frac{u(0) - \bar{u}(z)}{(z' - z)^2} dz' + \int_{H}^{\infty} \frac{\bar{u}(H) - \bar{u}(z)}{(z' - z)^2} dz' \quad \text{(5.34)}
\]

It is worth noting that one can also obtain (5.34) by assuming homogeneous solutions \( A_{H}(u,v,w) = f_{\alpha}^{q}(z=H,u,v,w) \) and \( A_{0}(u,v,w) = f_{\alpha}^{q}(z=0,u,v,w) \), such that (5.15) becomes

\[
f(z,u,v,w) = \begin{cases} 
- \frac{1}{w_T} \int_{z}^{H} f_{\alpha}^{eq} e^{\frac{z' - z}{\mu}} dz' + f_{\alpha}^{eq}(z=H,u,v,w) e^{\frac{H-z}{\mu}} & \text{for } w < 0 \\
\frac{1}{w_T} \int_{0}^{z} f_{\alpha}^{eq} e^{\frac{z' - z}{\mu}} dz' + f_{\alpha}^{eq}(z=0,u,v,w) e^{\frac{z}{\mu}} & \text{for } w \geq 0
\end{cases}
\]

(5.35)

With \( f \) given by (5.35) the density constraint (4.1) is satisfied, and the velocity constraint (4.2) becomes the governing equation:

\[
0 = \int_{0}^{H} \left[ \bar{u}(z') - \bar{u}(z) \right] g_{1} \left( \frac{|z' - z|}{U_T:\alpha} \right) \frac{dz'}{U_T} + \left[ \bar{u}(0) - \bar{u}(z) \right] g_{0} \left( \frac{z}{U_T:\alpha} \right) + \left[ \bar{u}(H) - \bar{u}(z) \right] g_{0} \left( \frac{H-z}{U_T:\alpha} \right)
\]

(5.36)

For the Cauchy case, \( g_{0}(\chi;\alpha=1) \approx 1/(\pi \chi) \) and \( g_{1}(\chi;\alpha=1) \approx 1/(\pi \chi^2) \), so (5.36) yields (5.34).

We will show in §6.2 that equation (5.34) yields a velocity profile that does not match experimental observations.

6. Numerical Implementation and Examples

6.1. Numerical Viscosity

Note that viscosity does not appear explicitly in the the stress expressions (5.11) and (5.27). In these equations, we expect to see terms of the form \( \tau_{zz}^{visc} = \mu \frac{\partial u(z)}{\partial z} \). Recall, the viscous stress \( \tau_{zz}^{visc} \) (4.29) arises due to agitation at small distances \(|z' - z| \leq \ell\). As \(|z' - z|\) tends to zero, the leading terms in (5.11) and (5.27) become indeterminate (zero divided by zero), so let us consider these terms more closely:

\[
\tau_{zz}(z) = \frac{\rho U_1}{\pi} \int_{0}^{H} \frac{\bar{u}(z') - \bar{u}(z)}{z' - z} dz' + \text{the other terms in (5.27)}
\]

(6.1)

Recalling from (4.28) the partitioning of the stress \( \tau_{zz} = \tau_{zz}^{visc} + \tau_{zz}^{turb} \) into that due to small displacements \(|z' - z| \leq \ell_1\) and large displacements \(|z' - z| \geq \ell_1\), we can write

\[
\tau_{zz}^{turb}(z) = \frac{\rho U_1}{\pi} \left\{ \int_{0}^{z-\ell_1} + \int_{z+\ell_1}^{H} \right\} \frac{\bar{u}(z') - \bar{u}(z)}{z' - z} dz' + \text{the other terms in (5.27)}
\]

(6.2)

\[
\tau_{zz}^{visc}(z) = \frac{\rho U_1}{\pi} \int_{z-\ell_1}^{z+\ell_1} \frac{\bar{u}(z') - \bar{u}(z)}{z' - z} dz'.
\]
This small-displacement integral leads to the desired viscous term. Upon Taylor series expansion $\bar{u}(z') - \bar{u}(z) = \frac{d\bar{u}(z)}{dz}(z' - z) + \frac{1}{2} \frac{d^2\bar{u}(z)}{dz^2}(z' - z)^2 + \ldots$, we have

$$\tau^{\text{visc}}_{xz}(z) = \frac{\rho U_1}{\pi} \int_{z-\ell_1}^{z+\ell_1} \frac{d\bar{u}(z)}{dz}(z' - z) + \frac{1}{2} \frac{d^2\bar{u}(z)}{dz^2}(z' - z)^2 + \ldots \, dz' = 2\ell_1 \frac{\rho U_1}{\pi} \frac{d\bar{u}(z)}{dz} + O(\ell_1^3),$$

(6.3)

Thus, similar to (4.39) (but here with a different front factor, due to assuming unidirectional shear flow), we find the viscosity relation

$$\mu = \frac{2}{\pi} \rho U_1 \ell_1.$$  

(6.4)

In a discretized numerical solution, one would a priori choose the fluid viscosity $\mu$, the density $\rho$, and the grid spacing $\delta z$. This leaves the length $\ell_1$ and agitation velocity $U_1$ as free parameters to be chosen to (i) facilitate convenient numerical integration, and (ii) satisfy (6.4). The most straightforward approach to facilitate numerical integration is to set $\ell_1 = \delta z/2$; then evaluation of the large displacement integrals $\left\{\int_0^{\delta z/2} + \int_{\delta z/2}^H\right\}$ via trapezoidal rule is equivalent to a single trapezoidal integration over the entire domain except omitting the $z' = z$ term, which we hereafter denote $\int_0^H$. The agitation velocity is then set to

$$U_1 = \mu \pi/(\rho \delta z),$$

(6.5)

so as to enforce (6.4). Care must be taken a priori to choose a grid spacing to yield an appropriate $\ell_1$ consistent with (4.17).

### 6.2. Example: Couette Flow

Couette flow is shear-driven flow between parallel plates, consistent with the starting assumptions of (5.1). Thus, the ensemble-averaged flowfield is $\bar{u} = (\bar{u}(z),0,0)$, with boundary conditions $\bar{u}(z=0) = 0$ and $\bar{u}(z=H) = V$, where $H$ is the gap height, and $V$ is the speed of the top plate. Here, we consider flow at Reynolds number $Re = \rho V H/\mu = 66,000$, which sets the desired viscosity $\mu$ (with $\rho$, $V$, and $H$ set to unity). In this section, we compare results using two methods: (1) the “laminar sublayer” method (§5.4), and (2) the “incorrect boundary conditions” method (§5.5).

Consider first the “incorrect boundary conditions” method (§5.5). After desingularizing the integral, equation (5.34) becomes:

$$\mu \frac{d^2\bar{u}(z)}{dz^2} + \frac{\rho U_1}{\pi} \left\{ \int_0^H \frac{\bar{u}(z') - \bar{u}(z)}{(z' - z)^2} \, dz' + \frac{\bar{u}(0) - \bar{u}(z)}{z} + \frac{\bar{u}(H) - \bar{u}(z)}{H - z} \right\} = 0, $$

(6.6)

Equation (6.6) can be discretized and solved numerically. The dashed integral $\int$ in (6.6) indicates omitting the $z' = z$ term when discretizing the integral (via trapezoidal rule). Define $\delta z = H/(M - 1)$, $z_i = (i - 1/2)\delta z$ for $i = 1, \ldots, M$, and weights (consistent with trapezoidal rule) $W_{ij} = (1 - \delta_{ij})(1 - \frac{1}{2}\delta_{j1})(1 - \frac{1}{2}\delta_{jM})\delta z/(z_j - z_i)^2$. In order to match the numerical viscosity with the actual viscosity $\mu$, we set $U_1 = \mu \pi/(\rho \delta z)$. Then, we can write (5.34) as

$$\frac{1}{\delta z} (\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}) + \sum_{j=1}^M W_{ij} (\bar{u}_j - \bar{u}_i) + \frac{\bar{u}_1 - \bar{u}_i}{z_i} + \frac{\bar{u}_M - \bar{u}_i}{H - z_i} = 0,$$

(6.7)

with boundary conditions $\bar{u}_1 = 0$ and $\bar{u}_M = 1$.

In the laminar sublayer method (§5.4), the free parameters include the laminar sublayer thickness $b$ and edge velocity $\bar{u}_b$, in addition to $\delta z$, $U_1$. We found best agreement with
the data by making the following choices: (1) set $U_1 = \mu \pi/(\rho \delta z)$ per (6.5); (2) choose the ratio \( \bar{u}_b/b \) in order to achieve the desired wall shear stress (to match the data), since the wall shear stress is $\tau_w = \mu \bar{u}_b/b$; (3) set $\delta z = b$ and choose $b$ such that the stress is constant across the channel (as it should be), which results in $b$ being approximately the thickness of the laminar sublayer. In this method, we seek the solution of (5.26) with the substitution from (5.32), which yields:

$$
0 = \mu \frac{d^2 \bar{u}(z)}{dz'^2} + \frac{\rho U_1}{\pi} \int_b^{H-b} \frac{\bar{u}(z') - \bar{u}(z)}{(z' - z)^2} dz' \\
+ \frac{\rho U_1}{\pi} \int_b^{H-b} [\bar{u}(z') - \bar{u}(z)] \sum_{n=0}^N \frac{1}{(2n H + z' + z - (4n + 2)b)^2} \\
+ \frac{1}{((2n + 2)H - z' - z - (4n + 2)b)^2} + \frac{1}{((2n + 2)H - z' + z - (4n + 4)b)^2} \\
+ \frac{1}{((2n + 2)H + z' - z - (4n + 4)b)^2} dz',
$$

(6.8)

with “boundary” conditions $\bar{u}(z=b) = \bar{u}_b$ and $\bar{u}(z=H-b) = V - \bar{u}_b$. Equation (6.8) can be discretized via trapezoidal rule, analogous to (6.7). The stress (5.27) with (5.32) becomes

$$
\tau_{xx}(z) = \mu \frac{d\bar{u}(z)}{dz} + \frac{\rho U_1}{\pi} \int_b^{H-b} \frac{\bar{u}(z') - \bar{u}(z)}{(z' - z)^2} dz' \\
+ \frac{\rho U_1}{\pi} \int_b^{H-b} [\bar{u}(z') - \bar{u}(z)] \sum_{n=0}^N \left[ \frac{1}{(2n H + z' + z - (4n + 2)b)} - \frac{1}{(2n + 2)H - z' - z - (4n + 2)b)} \right] dz',
$$

(6.9)

which can be discretized in a similar fashion (via centered differences and trapezoidal rule). The singularities $z' = z = b$ and $z' = z = H$ cancel and can be omitted.

Since $\mu/(\rho U_1/\pi) = \delta z \ll 1$, the viscous friction force is of lower order than the turbulent friction force, except perhaps in a transition layer of height proportional to $\delta z$. Thus, our expectation is that solution of (6.8) will yield a log profile that is desingularized at the wall. One such desingularized double-log profile is

$$
\bar{u}(z) = \frac{V}{2} + \frac{V}{2} \ln[(d + z)/(d + H - z)]
$$

(6.10)

where $d$ is a small number ($d \ll 1$) that represents a roughness height. Equation (6.10) simultaneously meets the boundary conditions ($\bar{u}(z=0) = 0$ and $\bar{u}(z=H) = V$) and has the proper asymptotic behavior away from the walls $d \ll z \ll 1 - d$.

Figure 5 shows the velocity profile predicted by numerical solutions of (6.7) and (6.8), as well as a best fit log profile (6.10) and experimental data from (Robertson & Johnson 1970). The log profile fits the numerical solution of (6.8) very well, and the agreement with the experiment is excellent. Here, $M = 154$ grid nodes yields $\delta z = b = 0.0065H = 9.85\nu/u_{\tau}$, which is roughly the thickness of the laminar sublayer. In this example, the wall stress is $\tau_w = 1.04 \times 10^{-3}(\frac{1}{2}\rho V^2)$, yielding a friction velocity of $u_{\tau} \equiv \sqrt{\tau_w/\rho} = \sqrt{\frac{1.04 \times 10^{-3}(\frac{1}{2}\rho V^2)}{\rho}}$. 


Figure 5: (a) Velocity profile for Couette flow at Re = 66,000: ‘—’ “laminar sublayer” method equation (6.8); ‘—·—’ best fit log profile (6.10) with \( d = 1.907 \times 10^{-6}; \) ‘—’ “incorrect boundary condition” equation (6.7); and ‘□’ experimental data (Robertson & Johnson 1970). The vertical and horizontal lines at half-value are included to illustrate mirror-symmetry about the midplane. The laminar flow solution (diagonal dashed line) is also included for reference. Horizontal dashed lines at \( z = b \) and \( H - b \) indicate the laminar sublayers. (b) Velocity profiles plotted on log scale, with horizontal lines to indicate \( z = b, 0.3H, 0.5H, \) and \( 0.7H. \) (c)/(d) Stress (6.9) plotted as the sum of viscous \( \mu d\bar{u}/dz \) and turbulent contributions.

0.0228\( V \). The agitation speed is \( U_1 = \mu \pi/(\rho \delta z) = 0.00728V \), which interestingly yields \( \gamma_1 = U_1/u_\tau = 0.3187 \approx 1/\pi \). Here, \( N = 100 \) terms were used in the summations in (6.8) and (6.9), which was sufficient to yield converged results.

It is important to note that \( M = 154 \) grid nodes and \( \delta z = 9.85\nu/u_\tau \) is the proper resolution for this calculation; i.e. increasing the grid resolution will not improve the solution, contrary to conventional wisdom. The reason that a finite grid resolution is necessary is that the fractional Laplacian integral \( \int_b^H \ldots )dz' \) in (6.8) is supposed to exclude the zone \( |z' - z| < \ell_1 \), where \( \ell_1 \) is a finite quantity. Thus, if \( \delta z \) were made too small, then more than just the \( i = j \) grid point would need to be excluded in the discretized (summation) form of the fractional Laplacian integral.
Figure 6: (a) Couette flow velocity profile: ‘—’ solution of the momentum equation (6.8); “- -” solution of constant stress equation \( \tau_{zx}(z) = \tau_w \) (6.11); (b) Stress (6.9) plotted as the sum of viscous \( \mu \ddot{u}/dz \) and turbulent contributions.

The dash-dot curve on Figure 5a is the velocity profile from the “incorrect boundary conditions” equation (6.7). The incorrect boundary conditions give too much weight to the end point values, \( \bar{u}(z=0) \) and \( \bar{u}(z=H) \), thus drawing the solution more towards the laminar solution (straight diagonal line) than it should be.

Figure 5c shows that the stress distribution matches the experimental data at the wall and is nearly constant across the channel. Here, \( \bar{u}_b/b \) was fixed such that the wall stress \( \tau_w = \mu \bar{u}_b/b \) matched the experimental data, and \( b \) was chosen such that the stress turned out to be uniform across (most of) the channel. Future efforts aim to create a numerical algorithm capable of predicting these parameters without aid of experimental data.

The slight discrepancy in \( \tau_{zx}(z) \) not being constant near the walls is due to the inconsistencies that arise in the numerical discretization of the momentum and stress equations (i.e. the discretized form of (6.8) is not exactly a centered difference approximation to (6.9)). As a check, we solved a system of equations

\[
\tau_{zx}(z_i) = \tau_w \tag{6.11}
\]

for \( i = 2, \ldots, N-1 \) (with \( \tau_{zx}(z_i) \) from (6.9)) augmented with boundary conditions \( \bar{u}_1 = \bar{u}_b \) and \( \bar{u}_N = V - \bar{u}_b \). As expected, the solution of equation (6.11) yields a velocity profile that nearly overlays on the solution of (6.8) (see Figure 6). The turbulent stress field from these two velocity profiles differs slightly, with the solution of (6.11) producing constant stress, by construction.

6.3. 2D Boundary Layer Flow

Now consider a 2D turbulent boundary layer flow over a flat plate, with plate length \( L \) and freestream speed \( V \). Assume \( \alpha = 1 \) consistent with a Cauchy equilibrium distribution. Assume the boundary layer thickness is small \( \delta_{(x)} \ll L \) and consequently \( \frac{\partial}{\partial x} \ll \frac{\partial}{\partial z} \). With this assumption, the results from unidirectional shear flow apply, and the governing
equations become
\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0
\]
\[
\rho \left( \frac{\partial \bar{u}}{\partial x} + \bar{w} \frac{\partial \bar{u}}{\partial z} \right) = \frac{\rho U_1 \delta z}{\pi} \frac{\partial^2 \bar{u}}{\partial z^2} + \frac{\rho U_1}{\pi} \int_b^\infty \frac{\bar{u}(x,z') - \bar{u}(x,z)}{(z' - z)^2} dz' + \frac{\rho U_1}{\pi} \int_b^\infty \frac{\bar{u}(x,z') - \bar{u}(x,z)}{(z' + z - 2b)^2} dz'.
\]
with boundary conditions \(\bar{u}(x,z=b) = \bar{u}_b\), \(\bar{u}(x=0,z) = V\), and \(\bar{u}(x,z=\infty) = V\). We also make the approximation \(\bar{w}(x,z=b) = 0\), consistent with the unidirectional shear flow assumptions. Here, we have written the viscosity as \(\rho U_1 \delta z/\pi\) to indicate its relation to the finite length \(\delta z\). The form of (6.12) suggests a similarity solution
\[
\eta = (z - b)/x, \quad \bar{u}/V = f'(\eta), \quad \bar{w}/V = \eta f'(\eta) - f(\eta)
\]
with boundary conditions \(f(0) = 0\), \(f'(0) = \bar{u}_b/V\), and \(f'(\infty) = 1\). In (6.13), the form of \(\bar{w}\) was selected so as to satisfy the continuity equation and boundary condition \(\bar{w}(\eta=0) = 0\).

Note that the similarity parameter \(\eta\) is formed by stretching \(z\) linearly with \(x\), which is consistent with the \(\alpha = 1\) scaling for turbulent transport. If \(t\) is the time of flight of a particle in the freestream, then \(x \sim Vt\) and \(\eta \sim z/Vt \sim t^{-1}\). Thus, \(\alpha = 1\) describes turbulent transport with a boundary layer thickness that grows linearly with distance downstream.

Inserting (6.13) into (6.12) and dividing by \(\rho V^2/x\), one obtains the momentum equation in similarity form
\[
-f(\eta) f''(\eta) = \frac{U_1}{\pi V} f''(\eta) + \frac{U_1}{\pi V} \int_0^\infty \frac{f'(\eta') - f'(\eta)}{(\eta' - \eta)^2} d\eta' + \frac{U_1}{\pi V} \int_0^\infty \frac{f'(\eta') - f'(\eta)}{(\eta' + \eta)^2} d\eta'.
\]
Since the solution domain is finite \(0 \leq \eta \leq H\) but we know that \(f'(\eta>H) \approx 1\), we can split \(f_0^\infty = f_H^0 + f_H^\infty\) and evaluate the infinite integrals as follows
\[
-f(\eta) f''(\eta) = \frac{U_1}{\pi V} f''(\eta) + \frac{U_1}{\pi V} \left\{ \int_0^H \frac{f'(\eta') - f'(\eta)}{(\eta' - \eta)^2} d\eta' + \int_0^H \frac{f'(\eta') - f'(\eta)}{(\eta' + \eta)^2} d\eta' \right\} + \frac{1 - f'(\eta)}{(H - \eta)} + \frac{1 - f'(\eta)}{(H + \eta)}
\]
Taking the \(f_i' = f'(\eta_i)\) as the unknowns of interest, equation (6.15) can be discretized as follows:
\[
\frac{f_i'_{i+1} - f_i'_{i-1}}{2 \delta \eta} + \frac{U_1}{\pi V} \frac{f_i'_{i+1} - 2f_i'_{i} + f_i'_{i-1}}{\delta \eta^2} + \sum_{j=1}^M W_{ij} (f_j' - f_i')
\]
\[
- \frac{U_1}{\pi V} \left\{ \frac{1}{(H - \eta_i)} + \frac{1}{(H + \eta_i)} \right\} f_i' = - \frac{U_1}{\pi V} \left\{ \frac{1}{(H - \eta_i)} + \frac{1}{(H + \eta_i)} \right\}
\]
where
\[
W_{ij} = \frac{U_1}{\pi V} \left( 1 - \frac{1}{2} \delta_{ij} \right) \left( 1 - \frac{1}{2} \delta_{ij} \right) \left( \frac{1 - \delta_{ij}}{(\eta_j - \eta_i)^2} + \frac{1}{(\eta_j + \eta_i)^2} \right)
\]
and the boundary conditions become \(f_i' = \bar{u}_b/V\) and \(f_M' = 1\). The first term in (6.16) is nonlinear, so the equation must be solved iteratively, with \(f(\eta) = \int_0^\eta f'(\eta') d\eta'\) updated between iterations.
Here, we consider a boundary layer flow with $Re = \rho V L/\mu = 10^6$, $\rho = 1$ kg/m$^3$, $V = 1$ m/s, and $L = 1$ m. For comparison, we use as reference the Law of the Wall:

$$\bar{u}(x,z) = \frac{1}{\kappa} \ln \left( \frac{z u_\tau(x)}{\nu} \right) + C^+,$$

where $u_\tau \equiv \sqrt{\tau_w/\rho}$ is the friction velocity, $\tau_w(x)$ is the wall shear stress, $\kappa = 0.41$ is the von Kármán constant, and $C^+ = 5.0$ is a constant (Schlichting & Gersten 2000). The friction velocity $u_\tau(x)$ was computed herein from (Schlichting & Gersten 2000) equations [2.13], [17.60], and [18.99]. By definition, $Re_x = Re \cdot x/L$, $\Lambda(x) \equiv \ln(Re_x)$, and $D \equiv 2 \ln \kappa + \kappa(C^+ - 3.0)$. Then, function $G(x)$ is determined from solution of implicit equation $\frac{A}{\Delta} + 2 \ln \frac{A}{\Delta} - D = A$. These data yield the friction velocity $u_\tau(x) = V \kappa G(x)/\Lambda(x)$ and velocity profile $\bar{u}(x,z)$ (6.18). For the laminar sublayer $z^+ \equiv zu_\tau/\nu < 8$, the velocity profile (6.18) is replaced with $\bar{u}/u_\tau = z^+$, and for the overlap layer $8 < z^+ < 20$, the velocity profile is blended with piecewise cubic Hermite interpolation. The 99% boundary layer thickness $\delta(x)$ (of the velocity profiles from (6.18)) was computed numerically by interpolation of $\bar{u}(x,z)$ and is nearly equal to $\delta(x) \approx 0.11 x G(x)/\Lambda(x) = 0.11 x u_\tau(x)/(V \kappa)$. At the plate end, $\delta(x=L) = 0.0129$ m, $\tau_w = 0.0023$ Pa, and $u_\tau(x=L) = 0.0479$ m/s.

In the numerical solution, the simulation domain height was set to $H = 0.11$ (which is about $9\delta(x=L)/L$). As with the Couette flow problem, we found good agreement with the experimental data by setting the grid resolution equal to the laminar sublayer thickness, $\delta\eta = b/L = (8.00u_\tau/u_\tau)/L = 1.67 \times 10^{-4}$ ($M = 659$ grid nodes), and by setting the agitation velocity $U_1$ so as to match the numerical viscosity with the actual viscosity $U_1 = 2\pi \mu/\rho \delta\eta = 0.0188$ m/s. This yields $\gamma_1 = U_1/u_\tau = 0.3928$, just slightly larger than that from Couette flow. The velocity at the edge of the laminar sublayer is taken to match that of the laminar sublayer, $\bar{u}_b = 8.00u_\tau = 0.3831$ m/s, which then correctly prescribes the wall shear stress.

Figure 7: 2D boundary layer velocity profile: ‘–’ numerical solution of (6.16); ‘-.-’ Law of the Wall (6.18) with $\kappa = 0.41$ and $C^+ = 5.0$ (Schlichting & Gersten 2000); ‘-’ velocity profiles predicted by XFOIL subroutine UWALL, which blends the Law of the Wall with the Spalding formula for the outer layer (Drela 1989). (a) Boundary layer profiles are plotted in outer units $z/L$, in order to illustrate the linear growth of the 99% boundary layer thickness (dashed line), $\delta \sim x$; (b) The $x/L = 1$ profile is plotted in inner units $\bar{z}/\nu$ versus $u_\tau = \bar{u}/u_\tau$ with log-linear scale in order to illustrate the logarithmic nature of the velocity profiles.
Figure 7 shows excellent agreement between our numerical solution of equation (6.16) and the logarithmic velocity profile predicted by the Law of the Wall (6.18). For reference, Figure 7 also shows the velocity profile predicted by using the Spalding formula for the outer layer, which is done in the widely-used airfoil analysis program XFOIL (Drela 1989). Clearly, the velocity profiles predicted by the present theory are within the level of uncertainty between the Law of the Wall and XFOIL predictions.

We justify the use of the Cauchy $f_\alpha^q$ distribution in this problem by the fact that the boundary layer thickness grows nearly linearly with respect to distance along the plate $\delta \sim x$. This scaling can be seen in Figure 7a, where the boundary layer thickness is shown as a dashed line. Since $G(x)/\Lambda(x)$ asymptotes to nearly constant value, $u_\tau(x)$ is nearly constant, and $\delta(x) \sim x u_\tau/V = u_\tau t$, where $t = x/V$ is the freestream flight time. Note that $\delta$ represents a distance of momentum diffusion, so an appropriate similarity variable describing turbulent transport is $\eta = \delta/(u_\tau t)$, which has linear scaling with time, consistent with $\alpha = 1$. So, once again, we conclude that the choice $\alpha = 1$ is particularly well suited for the modeling of wall turbulence.

Analytically predicting the von Kármán constant $\kappa$ is a continued subject of inquiry on our part. While Figures 5 and 7b show excellent agreement in both the velocity profile and the wall stress (the two ingredients needed to predict $\kappa$), this agreement was brought about by choosing the laminar sublayer velocity gradient $\bar{u}_b/b$ to match the wall stress predicted by the data. In order to theoretically predict the von Kármán constant, we will need to impose another relation to predict the wall stress. One possibility for this closure equation could be matching the slope of the velocity profile at the junction between the laminar sublayer and the outer layer, as in imposing the requirement $\lim_{z \to b^+} \frac{d\bar{u}(z)}{dz} = \frac{\bar{u}_b}{b}$ in the numerical solution of the momentum (or stress) equation.

7. Conclusions

This paper has provided a mathematical formalism (fractional Calculus) that we propose an alternative to RANS/LES closure schemes. Our theory is shown herein to be successful in application to shear turbulence, and we hope that this paper will spur additional investigations to turbulence modeling using fractional Calculus.

Starting from the framework of Boltzmann kinetic theory and calculating its associated stress field, we showed that the friction force in the momentum equations may be conveniently split into a first contribution from small displacements, yielding a viscous component, and a second contribution from large displacements, yielding a fractional Laplacian component. The demarcation length that separates small from large displacements is flow dependent and can be chosen as the laminar sublayer thickness or the Kolmogorov microscale. Thus, the fractional Laplacian can be viewed as the part of the friction force that is generated by turbulent motions.

Boltzmann kinetic theory presumes the existence of an equilibrium distribution for the velocity fluctuations, which is traditionally taken as the Maxwell-Boltzmann (normal) distribution. Through consideration of turbulent transport, we showed here that this traditional choice is not necessary and that any Lévy $\alpha$-stable distribution is an acceptable alternative. Among the possible values of dimensionless parameter $0 < \alpha \leq 2$, three stand out. For $\alpha = 2$, the equilibrium distribution is the traditional Maxwell-Boltzmann, and friction is accomplished solely by molecular viscosity. For $\alpha = 1$, the equilibrium distribution is the Cauchy distribution, and the fractional Laplacian reproduces the logarithmic velocity profile of a turbulent flow along a wall. The formalism is also shown to properly represent the growth of a turbulent boundary layer along a plate
and its accompanying logarithmic velocity profile (Law of the Wall). In the case $\alpha = \frac{2}{3}$, the formalism should have the potential to represent inertial turbulence (Kolmogorov cascade), but this has not yet been verified.

Thus, the selection of the $\alpha$ value must reflect the nature of the flowfield under consideration: $\alpha = 2$ for laminar flows controlled by molecular viscosity $\nu$, $\alpha = 1$ for shear turbulence characterized by a friction velocity $u_\tau$, and $\alpha = 2/3$ for inertial turbulence characterized by an energy dissipation $\epsilon$. A formalism suitable for the calculation of a flowfield with all three processes acting simultaneously remains to be developed.

The fractional Laplacian formalism raises a few ancillary questions, the answer to which requires additional work. First, a physical connection should exist between the agitation energy $\frac{3}{2} \rho U^2$ of Boltzmann kinetics and the turbulent kinetic energy. A further line of inquiry is whether Boltzmann kinetic energy could also provide an equation governing the evolution of the turbulent kinetic energy. Treatment of the boundary conditions for a general 3D flow also requires additional work, as does treatment of a continuously-varying distribution of $\alpha(t, x)$. In particular, it is not clear what physics should be invoked to obtain an equation governing the variation of $\alpha$. Finally, while the results herein suggest that a theoretical prediction of the von Kármán constant will be possible, additional work remains to be done to make this prediction.

The results presented herein for wall turbulence are most encouraging.

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**Appendix A. Proofs**

A.1. Proof that (2.20) Satisfies Stability with Unequal Timesteps

In §2, we derived

$$p_{\tilde{\eta}}(\tilde{\eta}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-|\gamma k|^\alpha} e^{i k \cdot \tilde{\eta}} dk$$

as the probability distribution that satisfies the half-timestep stability constraint (2.9).

Here, we show that probability distributions (2.20) also satisfy the following stability constraint with unequal timesteps:

$$p_{\tilde{\xi}}(\tilde{\xi}; \delta t) = \iiint_{-\infty}^{\infty} p_{\tilde{\xi}}(\tilde{\xi}; \delta t') p_{\tilde{\xi}}(\tilde{\xi} - \tilde{\xi}'; \delta t - \delta t') d\tilde{\xi}' .$$

(A 1)

Equation (A 1) prescribes that the probability of a jump of displacement $\tilde{\xi}$ in time $\delta t$ is equal to the integral sum of the probabilities of sequential jumps of initial displacement $\tilde{\xi}'$ in time $\delta t'$ and then complementary $\tilde{\xi} - \tilde{\xi}'$ in time $\delta t - \delta t'$.

As in §2, we first put (A 1) in similarity form by recalling from (2.10) that the similarity parameter for a jump $\tilde{\xi}$ in time $\delta t$ is $\tilde{\eta} \equiv \tilde{\xi}/(q \delta t)^{1/\alpha}$. With $\delta t' = r \delta t$, we have

$$\tilde{\xi}'/(q r\delta t)^{1/\alpha} = r^{-1/\alpha} \tilde{\eta}' ,$$

$$\tilde{\xi} - \tilde{\xi}'/(q (1 - r)\delta t)^{1/\alpha} = (1 - r)^{-1/\alpha}(\tilde{\eta} - \tilde{\eta}') .$$

(A 2)

We recall from (2.11), the probability conversion is: $p_{\xi}(\xi; \delta t) = p_{\eta}(\eta = \xi/(q \delta t)^{1/\alpha})/(q \delta t)^{3/\alpha}$
and \( d\xi = (q \delta t)^{3/\alpha} d\eta \). Then, using the conversions from (A 2), equation (A 1) becomes:

\[
\frac{p\tilde{\eta}(\tilde{\eta})}{(q \delta t)^{3/\alpha}} = \iiint_{-\infty}^{\infty} \frac{p\tilde{\eta}(r^{-1/\alpha} \tilde{\eta})}{(q r \delta t)^{3/\alpha}} \frac{p\tilde{\eta}((1-r)^{-1/\alpha}(\tilde{\eta} - \tilde{\eta}'))}{(q (1-r) \delta t)^{3/\alpha}} \, d\tilde{\eta}' (q \delta t)^{3/\alpha},
\]

which simplifies to

\[
p\tilde{\eta}(\tilde{\eta}) = \frac{1}{r^{3/2} (1-r)^{3/2}} \iiint_{-\infty}^{\infty} p\tilde{\eta}(r^{-1/\alpha} \tilde{\eta}) \, p\tilde{\eta}((1-r)^{-1/\alpha}(\tilde{\eta} - \tilde{\eta}')) \, d\tilde{\eta}'.
\]

The question of interest is: Does the solution (2.20) satisfy the general divisibility constraint (A 4)? We now insert (2.20) into (A 4) and show that the right hand side reduces back to \( p\tilde{\eta}(\tilde{\eta}) \):

\[
p\tilde{\eta}(\tilde{\eta}) \overset{?}{=} \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \left[ \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-|\gamma k_1| \alpha} e^{i k_1 \cdot \left(r^{-\frac{1}{\alpha}} \tilde{\eta}\right)} \, dk_1 \right] \cdot \left[ \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-|\gamma k_2| \alpha} e^{i k_2 \cdot \left((1-r)^{-\frac{1}{\alpha}}(\tilde{\eta} - \tilde{\eta}')\right)} \, dk_2 \right] \, d\tilde{\eta}',
\]

in which \( k_1 \) and \( k_2 \) are dummy variables of integration. With \( k = k_1 r^{-1/\alpha} \) and \( k' = k_2 (1-r)^{-1/\alpha} \) such that \( dk = dk_1 r^{-3/\alpha} \) and \( dk' = dk_2 (1-r)^{-3/\alpha} \), then after rearranging

\[
p\tilde{\eta}(\tilde{\eta}) \overset{?}{=} \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \iiint_{-\infty}^{\infty} e^{-r |\gamma k| \alpha - (1-r) |\gamma k'| \alpha} e^{i k' \cdot \tilde{\eta}} \left[ \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{i (k-k') \cdot \tilde{\eta}' \, dk'} \right] \, dk \, dk'
\]

\[
\overset{\delta(k-k')}{=} \delta(k-k')
\]

The \( d\tilde{\eta}' \) integral results in \( \delta(k-k') \), and then in the \( dk' \) integral, the \( \delta(k-k') \) transforms all \( k' \) into \( k \). The exponent then simplifies to the desired result:

\[
p\tilde{\eta}(\tilde{\eta}) \overset{?}{=} \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-|\gamma k| \alpha} e^{-i k \cdot \tilde{\eta}} \, dk = p\tilde{\eta}(\tilde{\eta}) \quad \checkmark
\]

Thus, we have proven that the solution of the half-timestep divisibility problem (2.20) satisfies the constraint for the general timestep divisibility problem (A 1).

A.2. Proof that (2.22) follows from (2.20) with \( \alpha = 2 \)

Equation (2.20) was

\[
p\tilde{\eta}(\tilde{\eta}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-|\gamma k| \alpha} e^{-i k \cdot \tilde{\eta}} \, dk.
\]

Upon making the substitutions \( k' = \gamma k \) and \( dk' = \gamma^3 \, dk \) and then immediately replacing \( k' \) with \( k \), we can write

\[
p\tilde{\eta}(\tilde{\eta}) = \frac{1}{\gamma^3} \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-|k| \alpha - i k \cdot \tilde{\eta}/\gamma} \, dk.
\]

With \( \alpha = 2 \), \( |k|^2 = k_1^2 + k_2^2 + k_3^2 \), so the integrals decouple:

\[
p\tilde{\eta}(\tilde{\eta}) = \frac{1}{(2\pi \gamma)^3} \int_{-\infty}^{\infty} e^{-k_1^2 - i k_1 \tilde{\eta}_1/\gamma} \, dk_1 \int_{-\infty}^{\infty} e^{-k_2^2 - i k_2 \tilde{\eta}_2/\gamma} \, dk_2 \int_{-\infty}^{\infty} e^{-k_3^2 - i k_3 \tilde{\eta}_3/\gamma} \, dk_3.
\]
Consider the first integral. Note $-k_1^2 - ik_1\tilde{n}_1/\gamma = -(k_1 + i\tilde{n}_1/2\gamma)^2 - \tilde{n}_1^2/4\gamma^2$, so
\[ I_1 = \frac{1}{2\pi\gamma} e^{-\tilde{n}_1^2/4\gamma^2} \int_{-\infty}^{\infty} e^{-(k_1 + i\tilde{n}_1/2\gamma)^2} dk_1. \] (A 10)

Use the substitution $u = k_1 + i\tilde{n}_1/2\gamma$, $du = dk_1$, and evaluate the resulting integral $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$. Then, $I_1 = \frac{1}{\sqrt{2\pi}(\sqrt{2}\gamma)} e^{-\tilde{n}_1^2/2(\sqrt{2}\gamma)^2}$. Following suit for the other two integrals and reforming $\tilde{n}_1^2 + \tilde{n}_2^2 + \tilde{n}_3^2 = |\tilde{n}|^2$, we have the final result:
\[ p\tilde{n}(\tilde{n}) = \frac{1}{(2\pi)^3/2(\sqrt{2}\gamma)^3} \exp\left(-\frac{|\tilde{n}|^2}{2(\sqrt{2}\gamma)^2}\right) \]  
(2.22)

A.3. Evaluation of the Integral Appearing in (4.55)

Here, we evaluate the integral defined in equation (4.55):
\[ I = \int_0^{\infty} e^{-\frac{1}{2} D s^{-2} - s} ds \]  
(4.55)

where we have defined $D \equiv |x - x'|^2/(\tau U^2)$ to simplify the algebra. A change of variable $\beta = 1/s^5$ transforms (4.55) into
\[ I = \frac{1}{5} \int_0^{\infty} e^{-\frac{1}{2} D \beta^{2/5} - \beta^{-1/5}} d\beta. \]  
(A 11)

This integrand decays to zero rapidly for $\beta$ away from both sides of the value $\beta_0$ that minimizes the negative of the exponent,
\[ \frac{1}{2} D \beta^{2/5} + \beta^{-1/5}. \]  
(A 12)

This (negative) exponent reaches a minimum (of $\frac{3}{2} D^{1/3}$) at $\beta_0 = D^{-5/3}$. An estimate of the integral can be made by expanding the exponent (A 12) in a Taylor series (with $\beta = \beta_0 + \bar{\beta}$ for small $\bar{\beta}$):
\[ \frac{1}{2} D \beta_0^{2/5} + \beta_0^{-1/5} = \frac{1}{2} \beta_0^{-3/5}(\beta_0 + \bar{\beta})^{2/5} + (\beta_0 + \bar{\beta})^{-1/5} = \frac{3}{2} \beta_0^{-1/5} + \frac{3}{50} \beta_0^{-11/5} \bar{\beta}^2 + O(\bar{\beta}^3). \]  
(A 13)

Retaining only the leading terms and inserting (A 13) into (A 11), we obtain
\[ I = \frac{1}{5} e^{-\frac{3}{2} \beta_0^{1/5}} \int_{-\beta_0}^{\infty} e^{-\frac{3}{50} \beta_0^{-11/5} \beta^2} d\beta \leq \sqrt{\frac{2\pi}{3}} \beta_0^{11/10} e^{-\frac{3}{2} \beta_0^{-1/5}}, \]  
(A 14)

with the upper bound obtained by extending the lower limit of integration all the way to $-\infty$. Upon replacing $\beta_0^{11/10} = (|x - x'|/\tau U)^{-11/3}$ and $\beta_0^{-1/5} = (|x - x'|/\tau U)^{2/3}$, one verifies the final result in (4.55).

A.4. Proof that (5.12) is an exact solution of (5.10)

We need to show:
\[ 0 = \frac{1}{2} \int_0^5 [\ln(z' - z) - \ln(z)] \left[ \frac{1}{(z' - z)^2} + \frac{1}{(z' + z)^2} \right] dz' = \int_0^\infty \left[ \frac{\ln(z'/z)}{(z' - z)^2} + \frac{\ln(z'/z)}{(z' + z)^2} \right] dz'. \]

We break the integral into two parts $\int_0^\infty = \int_0^\infty + \int_\infty^\infty$, set $\zeta = z'/z$ in the first integral, and set $\zeta = z'/z$ in the second. Then,
\[ 0 = \frac{1}{2} \int_0^1 \left[ \frac{\ln(z)}{(z - 1)^2} - \frac{\ln(z)}{(z + 1)^2} \right] d\zeta - \frac{1}{2} \int_0^1 \left[ \frac{\ln(1/z)}{(1 - z)^2} + \frac{\ln(1/z)}{1 + z^2} \right] d\zeta = 0. \]  
(A 15)
A.5. Proof that (5.29) is an exact solution of (5.28).

Without loss of generality, assume $H = 1$. We need to show:

\[ 0 \equiv \int_0^1 \left[ \ln \left( \frac{z'}{1 - z'} \right) - \ln \left( \frac{z}{1 - z} \right) \right] \frac{dz'}{(z' - z)^2} = \int_0^1 \ln \left( \frac{z'}{1 - z'} \frac{1 - z}{z'} \right) \frac{dz'}{(z' - z)^2} . \]  
(A 16)

We break the integral into two parts $\int_0^1 = \int_0^z + \int_z^1$, and in the second integral, we make the substitution

\[ \frac{z' - z}{1 - z'} = \frac{z}{1 - z} \frac{1 - \zeta}{\zeta} . \]  
(A 17)

Then define $D = \zeta(1 - \zeta) + (\zeta - z)^2$ such that $z' = z^2(1 - \zeta)/D$. Then $1 - z' = \zeta(1 - z^2)/D$, and

\[ dz' = -d\zeta \frac{z^2(1 - z)^2}{\zeta^2(1 - z^2)} = -d\zeta \frac{z^2(1 - z^2)}{D^2} . \]  
(A 18)

Inserting these results in (A 16), we have

\[ 0 \equiv \int_0^z \ln \left( \frac{z'}{1 - z'} \frac{1 - z}{z} \right) \frac{dz'}{(z' - z)^2} - \int_z^1 \ln \left( \frac{z'}{1 - z'} \frac{1 - \zeta}{\zeta} \right) \frac{1}{(z' - z)^2} \frac{z^2(1 - z^2)}{D^2} d\zeta . \]  
(A 19)

The integration limits and logarithm fractions both flip, with no net change in sign. Inserting the above expression for $z'$, the fractions simplify as follows:

\[ \frac{1}{(z^2(1 - \zeta)/D - z)^2} \frac{z^2(1 - z^2)}{D^2} = \frac{(1 - z)^2}{(z(1 - \zeta) - D)^2} = \frac{1}{(\zeta - z)^2} . \]  
(A 20)

Finally, we have

\[ 0 \equiv \int_0^z \ln \left( \frac{z'}{1 - z'} \frac{1 - z}{z} \right) \frac{dz'}{(z' - z)^2} - \int_0^z \ln \left( \frac{1 - z}{z} \frac{1 - \zeta}{\zeta} \right) \frac{d\zeta}{(\zeta - z)^2} = 0 \]  
(A 21)

Appendix B. Lévy $\alpha$-Stable Distributions

The following is a primer on (isotropic) Lévy $\alpha$-stable distributions (Lévy 1925; Shintani & Umeno 2017; Nolan 2006). In the univariate case, the probability density function $f_X(x;\alpha,\beta,\gamma,\mu)$ of random variable $X$ is said to follow a Lévy $\alpha$-stable distribution if it is given by

\[ f_X(x;\alpha,\beta=0,\gamma,\delta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\gamma k|\alpha} e^{-ik(x-\delta)} \, dk . \]  
(B 1)

The four parameters $\{\alpha, \beta, \gamma, \delta\}$ are the power-law index $0 < \alpha \leq 2$ that characterizes the tail of the distribution, the skewness $-1 \leq \beta \leq 1$ that characterizes asymmetry ($\beta = 0$ for isotropic distributions), scale parameter $0 < \gamma < \infty$ (generalization of the standard deviation), and location $-\infty < \delta < \infty$ (generalization of the mean).

Upon replacing $k' = \gamma k$ (and then immediately dropping the primes), one finds

\[ f_X(x;\alpha,0,\gamma,\delta) = \frac{1}{\gamma} f_X\left(\frac{x}{\gamma};\alpha,0,1,0\right) = \frac{1}{\gamma} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k|\alpha} e^{-ik\frac{x}{\gamma}} \, dk . \]  
(B 2)

The integral in (B 2) can be written in terms of the standard variable $z \equiv (x - \delta)/\gamma$ and the standard distribution $f_Z(z;\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k|\alpha} e^{-ikz} \, dk$. The only two cases where (B 2) can be evaluated analytically are the normal distribution $f_Z(z;\alpha=2) = \frac{1}{2\sqrt{\pi}} e^{-z^2/4}$ (with standard deviation $\sqrt{2}$) and the Cauchy distribution $f_Z(z;\alpha=1) = \frac{1}{\pi (z^2+1)}$. 
The multivariate extension of (B 1) is

\[ f_\mathbf{x}(\mathbf{x};\alpha,\beta=0,\gamma,\delta) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-|\gamma \mathbf{k}|^\alpha} e^{-i\mathbf{k} \cdot (\mathbf{x} - \delta)} d\mathbf{k} \]  

with random vector \( \mathbf{x} \in \mathbb{R}^n \), vectors \( \mathbf{k}, \delta \in \mathbb{R}^n \), and \( d\mathbf{k} \) shorthand for \( dk_1 dk_2 \ldots dk_n \). Again, upon setting \( \mathbf{k}' = \gamma \mathbf{k} \), then \( d\mathbf{k}' = \gamma^n d\mathbf{k} \), and it is easy to see that

\[ f_\mathbf{Z}(\mathbf{z};\alpha,0,\gamma,\delta) = \frac{1}{\gamma^n} f_\mathbf{Z}(\mathbf{z} = \frac{\mathbf{z}}{\gamma};\alpha) \]  

with the standard distribution

\[ f_\mathbf{Z}(\mathbf{z};\alpha) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-|\mathbf{k}|^\alpha} e^{-i\mathbf{k} \cdot \mathbf{z}} d\mathbf{k} \]  

as \( |\mathbf{z}| \to \infty \). A plot of \( \bar{C}_{\alpha,n} \) is given in Figure 8.

Herein, we let \( f_\mathbf{Z}^q = \frac{1}{|\mathbf{z}|^{\alpha+n}} F(\Delta) \), with \( F(\Delta = |\mathbf{u} - \bar{\mathbf{u}}|^2) = f_\mathbf{Z}(\mathbf{z} = \frac{\mathbf{u} - \bar{\mathbf{u}}}{\Delta};\alpha) \). Thus, \( \Delta = |\mathbf{z}|^2 \), and the tails of \( F(\Delta) \) are

\[ F(\Delta) \to \frac{\bar{C}_{\alpha,n}}{\Delta^{(\alpha+n)/2}} \text{ as } \Delta \to \infty \]  

as given in (4.41).

**Appendix C. Demarcation Scale \( \ell \) as the Kolmogorov Microscale**

Kolmogorov’s theory of the turbulence cascade (“interial turbulence”) is centered on the existence of a scale-independent transfer of energy, from the largest scale where turbulence is stirred by one or several instability mechanisms down to the shortest scale where viscous dissipation takes hold. The *energy dissipation rate* (rate of kinetic energy dissipation per mass) is denoted \( \epsilon \) and has the dimensions of energy per mass per time.

From elementary dimensional analysis, it follows that given any intermediate length scale (“eddy diameter”) \( d \), the associated turbulent velocity fluctuations \( u \) are given by

\[ u = A (\epsilon d)^{1/3} \approx (\epsilon d)^{1/3} \]  

in which the dimensionless coefficient \( A \) is a universal constant close to unity (Kundu *et al.* 2012) and ignored hereafter. The value of \( d \) ranges from the largest scale of the
flow, which we denote \( L \), down to the shortest eddy scale, which we denote \( \ell \) and which is called the Kolmogorov microscale. Applying (C 1) at both extreme scales, we obtain:

- At the largest scale \( L \) of the flow, the corresponding velocity scale is \( V \). Thus, by virtue of (C 1), we have \( V \sim (\epsilon L)^{1/3} \), from which we can extract the energy dissipation:

\[
\epsilon \simeq V^3 / L .
\]  

(C 2)

- At the shortest scale \( \ell \), with associated minimum turbulent velocity \( u_{min} \simeq (\epsilon \ell)^{1/3} \), molecular viscosity takes over because the Reynolds number at that scale becomes on the order of unity: \( Re = \frac{\rho u_{min} \ell}{\mu} = \frac{\rho(\epsilon \ell)^{1/3} \ell}{\mu} \simeq 1 \). This implies

\[
\ell / L \simeq Re^{-3/4} .
\]  

(C 3)

Combining (C 2) and (C 3), we have \( \ell = \nu^{3/4} \epsilon^{-1/4} \).

### Appendix D. Alternative Form of the Turbulent Stress Tensor

The stress (4.59)

\[
\tau_{ij}^{turb}(\mathbf{x}) = \rho q \gamma^\alpha (\alpha+3) \Gamma(\alpha) \bar{C}_\alpha \int \int \int_{|\mathbf{x}' - \mathbf{x}| \geq \ell} (x'_i - x_i)(x'_j - x_j) \frac{(\mathbf{x}' - \mathbf{x}) \cdot (\bar{u}(\mathbf{x}') - \bar{u}(\mathbf{x}))}{|\mathbf{x}' - \mathbf{x}|^{\alpha+5}} d\mathbf{x}'
\]  

(4.59)

can be simplified by integrating by parts: \( \int a \, db = ab - \int b \, da \). First, write the dot product out into three terms \((x'_i - x_i)(\bar{u}_k(\mathbf{x}') - \bar{u}_k(\mathbf{x}))\), with implied summation over \( k = 1, 2, 3 \). Now, for each of the three terms, perform partial integration in \( dx'_k \)

\[
a = (x'_i - x_i)(x'_j - x_j)(\bar{u}_k(\mathbf{x}') - \bar{u}_k(\mathbf{x}))
\]

\[
da = \left\{ \delta_{ik}(x'_j - x_j)(\bar{u}_k(\mathbf{x}') - \bar{u}_k(\mathbf{x})) + \delta_{jk}(x'_i - x_i)(\bar{u}_k(\mathbf{x}') - \bar{u}_k(\mathbf{x})) + (x'_i - x_i)(x'_j - x_j) \frac{\partial \bar{u}_k(\mathbf{x}')}{\partial x'_k} \right\} dx'_k
\]

\[
 db = \frac{(x'_k - x_k)}{|\mathbf{x}' - \mathbf{x}|^{\alpha+3}} dx'_k
\]

\[
b = \frac{-1}{(\alpha+3)|\mathbf{x}' - \mathbf{x}|^{\alpha+3}}
\]

with no summation implied in each partial integration (fixed \( k \)). Since \( ab \big|_{-\infty}^{\infty} = 0 \) for each of these partial integrations (so long as \(|\bar{u}(\mathbf{x}')|/|\mathbf{x}'|^{\alpha+1} \to 0 \) as \(|\mathbf{x}'| \to \infty \)), the other terms can be recombined (by now implying summation over \( k \)):

\[
\tau_{ij}^{turb}(\mathbf{x}) = \rho q \gamma^\alpha \Gamma(\alpha) \bar{C}_\alpha \int \int \int_{|\mathbf{x}' - \mathbf{x}| \geq \ell} \left\{ \frac{1}{|\mathbf{x}' - \mathbf{x}|^{\alpha+3}} \right\} \cdot \left\{ \delta_{ik}(x'_j - x_j)(\bar{u}_k(\mathbf{x}') - \bar{u}_k(\mathbf{x})) + \delta_{jk}(x'_i - x_i)(\bar{u}_k(\mathbf{x}') - \bar{u}_k(\mathbf{x})) + (x'_i - x_i)(x'_j - x_j) \frac{\partial \bar{u}_k(\mathbf{x}')}{\partial x'_k} \right\} d\mathbf{x}' .
\]  

(D 1)

Assuming incompressible flow, \( \frac{\partial \bar{u}_k}{\partial x'_k} = 0 \), we are left with

\[
\tau_{ij}^{turb}(\mathbf{x}) = \rho q \gamma^\alpha \Gamma(\alpha) \bar{C}_\alpha \int \int \int_{|\mathbf{x}' - \mathbf{x}| \geq \ell} \frac{(x'_j - x_j)(\bar{u}_i(\mathbf{x}') - \bar{u}_i(\mathbf{x})) + (x'_i - x_i)(\bar{u}_j(\mathbf{x}') - \bar{u}_j(\mathbf{x}))}{|\mathbf{x}' - \mathbf{x}|^{\alpha+3}} d\mathbf{x}' .
\]  

(D 2)
In the special case of the Cauchy distribution \((\alpha = 1, q = u_\tau)\), the stress \((D 2)\) becomes
\[
\tau_{ij}^{\text{turb}}(x) = \frac{\rho u_{\gamma}}{\pi^2} \iint \frac{(x_j' - x_j)(\tilde{u}_i(x') - \tilde{u}_i(x)) + (x_i' - x_i)(\tilde{u}_j(x') - \tilde{u}_j(x))}{|x' - x|^4} \, dx'.
\]

\[\text{(D 3)}\]

**Appendix E. Reduction of Order of the Fractional Laplacian**

The fractional Laplacian of scalar function \(\tilde{u}(x)\) of vector \(x\) in \(\mathbb{R}^n\) space is defined as (Kwaśnicki 2017):
\[
\mathcal{L}_n \tilde{u}(x) \equiv \bar{C}_{\alpha,n} \iint_{-\infty}^{\infty} \frac{\tilde{u}(x') - \tilde{u}(x)}{|x' - x|^\alpha + n} \, dx', \quad \bar{C}_{\alpha,n} \equiv \frac{2\alpha \Gamma \left( \frac{\alpha + n}{2} \right)}{\pi \frac{\alpha}{2} \Gamma \left( \frac{\alpha + n}{2} - \frac{n}{2} \right)}.
\]

\[\text{(E 1)}\]

Note that the coefficient \(\bar{C}_{\alpha,n}\) in the fractional Laplacian \((E 1)\) is identical to the coefficient in the tail of the Lévy \(\alpha\)-stable distribution \((B 5)\).

Consider the case of \(x \in \mathbb{R}^n\) where scalar function \(\tilde{u}(x)\) is not a function of \(x_n\). In this case, we expect that we can simply apply the definition of the fractional Laplacian \((E 1)\) on the \(\mathbb{R}^{n-1}\) subspace spanned by the components of \(x\) that \(\tilde{u}(x)\) actually depends on. Equivalently, we expect that integration over \(dx_n\) in \(\mathcal{L}_n \tilde{u}(x)\) should leave remaining integrals identical to the formula for \(\mathcal{L}_{n-1} \tilde{u}(x_1, \ldots, x_n)\). Using this thought experiment, we can derive the relation between constants \(\bar{C}_{\alpha,n}\) and \(\bar{C}_{\alpha,n-1}\). Whence, we expect
\[
\int_{-\infty}^{\infty} \frac{\bar{C}_{\alpha,n} \, dx_n}{[(x_1' - x_1)^2 + \cdots + (x_n' - x_n)^2]^{\frac{\alpha + n}{2}}} = \frac{\bar{C}_{\alpha,n-1}}{[(x_1' - x_1)^2 + \cdots + (x_{n-1}' - x_{n-1})^2]^{\frac{\alpha + n - 1}{2}}}.
\]

\[\text{(E 2)}\]

Upon setting \(a^2 = (x_1' - x_1)^2 + \cdots + (x_{n-1}' - x_{n-1})^2, b^2 = (x_n' - x_n)^2\), so \(\alpha b = dx_n', \) and \(b = a \tan \theta\), we have \(a^2 + b^2 = a^2(1 + \tan^2 \theta) = a^2 / \cos^2 \theta\) and \(\alpha b = a \theta / \cos^2 \theta\), such that the left hand side of \((E 2)\) evaluates to
\[
\int_{-\infty}^{\infty} \frac{\bar{C}_{\alpha,n} \, db}{[a^2 + b^2]^{\frac{\alpha + n}{2}}} = \int_{-\pi/2}^{\pi/2} \frac{\bar{C}_{\alpha,n} \, a \, d\theta / \cos^2 \theta}{[a^2 / \cos^2 \theta]^{\frac{\alpha + n}{2}}} = \frac{\bar{C}_{\alpha,n}}{[a^2]^{\frac{\alpha + n - 1}{2}}} \sqrt{\pi} \frac{\Gamma \left( \frac{\alpha + n - 1}{2} \right)}{\Gamma \left( \frac{\alpha + n}{2} \right)}.
\]

\[\text{(E 3)}\]

The right hand side of \((E 2)\) is \(\frac{\bar{C}_{\alpha,n-1}}{[a^2]^{\frac{\alpha + n - 1}{2}}}\). For this expression to be equal to the right hand side of \((E 3)\), we require
\[
\bar{C}_{\alpha,n-1} = \bar{C}_{\alpha,n} \sqrt{\pi} \frac{\Gamma \left( \frac{\alpha + n - 1}{2} \right)}{\Gamma \left( \frac{\alpha + n}{2} \right)}.
\]

\[\text{(E 4)}\]

This requirement is consistent with the definition of \(\bar{C}_{\alpha,n}\) in \((E 1)\). So to recapitulate, if scalar function \(\tilde{u}(x)\) is not a function of \(x_n\), that variable can be integrated out of the fractional Laplacian by simply applying the definition \((E 1)\) in the \(\mathbb{R}^{n-1}\) subspace spanned by the remaining \(n - 1\) variables. Moreover, if \(\tilde{u}(x)\) only depends on \(k\) elements of \(x\), then the fractional Laplacian can be applied only on that \(\mathbb{R}^{n-k}\) subspace. This is useful when we consider shear flows such as the Couette problem, since the velocity field has only \(z\) dependency, \(\tilde{u} = \tilde{u}(z)\).
Appendix F. Useful Notation

In the unidirectional shear flow problem, define for convenience the following reduced equilibrium functions,

\[ f_{eq}^3(u - \bar{u}(z), v, w; \alpha) = f_{eq}^\alpha(z, u, v, w) \]  
\[ f_{eq}^2(u - \bar{u}(z), w; \alpha) = \int_{-\infty}^{\infty} f_{eq}^3(u - \bar{u}(z), v, w; \alpha) \, dv \]  
\[ f_{eq}^1(w; \alpha) = \int_{-\infty}^{\infty} f_{eq}^2(u - \bar{u}(z), v, w; \alpha) \, du \]

For example, Maxwell-Boltzmann has \( f_{eq}^1(w; \alpha=2) = \frac{\rho}{U} \frac{1}{\sqrt{2\pi}} e^{-w^2/2U^2} \) and Cauchy has \( f_{eq}^1(w; \alpha=1) = \frac{\rho}{U} \frac{1}{\pi} \frac{1}{w^2 + 1} \).

The following results are also useful,

\[ \int_{-\infty}^{\infty} f_{eq}^1(w; \alpha) \, dw = \rho \]  
\[ \int_{-\infty}^{\infty} u f_{eq}^2(u - \bar{u}(z), v, w; \alpha) \, du = \bar{u}(z) f_{eq}^1(w; \alpha) \]

Appendix G. Evaluation of Integrals \( g_0 \) and \( g_1 \)

The following definitions appeared in the Couette flow velocity and stress calculations:

\[ g_0(\chi; \alpha) \equiv \int_{0}^{\infty} \left( \frac{U}{\rho} f_{eq}^1(U\beta; \alpha) \right) \exp \left( -\frac{\chi}{\beta} \right) \, d\beta \]  
\[ g_1(\chi; \alpha) \equiv \int_{0}^{\infty} \left( \frac{U}{\rho} f_{eq}^1(U\beta; \alpha) \right) \exp \left( -\frac{\chi}{\beta} \right) \, d\beta \]

where \( \beta = w/U \). Here, we provide approximations for integrals \( g_0 \) and \( g_1 \) under the assumption: \( \chi \gg 1 \). The two equilibrium distributions of interest are the Cauchy \( \frac{U}{\rho} f_{eq}^1(U\beta; \alpha=1) = \frac{1}{\pi} \frac{1}{\beta^2 + 1} \) and Maxwell-Boltzmann \( \frac{U}{\rho} f_{eq}^1(U\beta; \alpha=2) = \frac{1}{\sqrt{2\pi}} e^{-\beta^2/2} \). Because \( \chi \gg 1 \), only large values of \( \beta \) contribute to these integrals, so we need only consider the asymptotic behavior of these \( f_{eq}^\alpha \) distributions for \( \beta \gg 1 \).

G.0.1. Cauchy \( f_{eq}^\alpha \)

For the Cauchy case, \( \frac{U}{\rho} f_{eq}^1(U\beta; \alpha=1) = \frac{1}{\pi} \frac{1}{\beta^2 + 1} \), equation (G 1) evaluates to the following analytic formula:

\[ g_0(\chi; \alpha=1) = \frac{1}{\pi} \text{imag} \left\{ e^{i\chi} \text{Ei}(-i\chi) \right\} + \cos \chi , \]  
\[ g_1(\chi; \alpha=1) = -\frac{1}{\pi} \text{real} \left\{ e^{i\chi} \text{Ei}(-i\chi) \right\} + \sin \chi , \]

where \( \text{Ei}(\bullet) \) is the exponential integral (Matlab \( \text{ei} \) function).

For \( \chi \gg 1 \), the Cauchy distribution behaves asymptotically as \( \frac{U}{\rho} f_{eq}^1(U\beta; \alpha=1) \rightarrow \frac{1}{\pi} \frac{1}{\beta^2} \), and the expressions in (G 2) simplify considerably. Inserting \( \frac{1}{\pi} \frac{1}{\beta^2} \) into (G 1), one finds:

\[ g_0(\chi; \alpha=1) \approx \frac{1}{\pi} \frac{1}{\chi} , \]  
\[ g_1(\chi; \alpha=1) \approx \frac{1}{\pi} \frac{1}{\chi^2} . \]
Figure 9: The functions \( g_0(\chi) \) and \( g_1(\chi) \) for the Maxwell-Boltzmann and Cauchy distributions. Note that both \( g_0(\chi) \) and both \( g_1(\chi) \) functions are similar near the origin but differ in their tails. Also shown are the analytic formulae (G 2) and asymptotic approximations (G 3) and (G 10).

As shown in Figure 9, these formulae are accurate for \( \chi > 10 \).

G.0.2. 1D Lévy \( \alpha \)-stable distribution \( 0 < \alpha < 2 \)

The 1D Lévy \( \alpha \)-stable distribution (with \( |z| = \beta \) and \( n = 1 \) in (B 5)) has tails that decay as

\[
f(\beta) \approx \frac{\bar{C}_{\alpha,1}}{\beta^{\frac{\alpha}{\alpha+1}}} , \quad \bar{C}_{\alpha,1} = \frac{2^\alpha}{\pi^{\frac{1}{2}}} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{|\Gamma\left(-\frac{\alpha}{2}\right)|} = \frac{1}{\pi} \Gamma(\alpha + 1) \sin\left(\frac{\pi}{2} \alpha\right)
\]  

for \( \beta \to \infty \). Inserting (G 4) into (G 1), one finds:

\[
\begin{align*}
g_0(\chi; \alpha) & \approx \bar{C}_{\alpha,1} \int_0^\infty \frac{e^{-\frac{\beta}{\beta^{\frac{\alpha}{\alpha+1}}}}}{\beta^{\frac{\alpha}{\alpha+1}}} d\beta = \bar{C}_{\alpha,1} \frac{\Gamma(\alpha)}{\chi^\alpha} , \\
g_1(\chi; \alpha) & \approx \bar{C}_{\alpha,1} \int_0^\infty \frac{e^{-\frac{\chi}{\beta^{\frac{\alpha}{\alpha+1}}}}}{\beta^{\frac{\alpha}{\alpha+1}}} d\beta = \bar{C}_{\alpha,1} \frac{\Gamma(\alpha + 1)}{\chi^{\alpha+1}} .
\end{align*}
\]  

For \( \alpha = 1 \), we have \( \Gamma(1) = \Gamma(2) = 1 \) and \( \bar{C}_{\alpha,1} = 1/\pi \), so (G 5) predicts \( g_0(\chi;\alpha=1) = 1/(\pi \chi) \) and \( g_1(\chi;\alpha=1) = 1/(\pi \chi^2) \), in agreement with (G 3).

G.0.3. Maxwell-Boltzmann \( f_\rho^{eq} \)

In general, integrals \( g_0(\chi;\alpha) \) and \( g_1(\chi;\alpha) \) can not be evaluated analytically. For the Maxwell-Boltzmann case, \( \frac{U}{\rho} f_\rho^{eq}(U;\beta;\alpha=2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2}} \), we have

\[
\begin{align*}
g_0(\chi;\alpha=2) & = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{\beta^2}{2} - \frac{\chi}{\beta}\right) d\beta , \\
g_1(\chi;\alpha=2) & = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{\beta^2}{2} - \frac{\chi}{\beta}\right) \frac{d\beta}{\beta} .
\end{align*}
\]  

Note that, for \( \chi \gg 1 \), the exponent term in (G 6) is nearly zero for all but a narrow range of \( \beta \) about the maximum (minimum negative value) of the argument of the exponent. The (negative of the) argument is,

\[
\frac{\beta^2}{2} + \frac{\chi}{\beta} = \frac{\beta^3 + 2\chi}{2\beta} .
\]
The minimum of \((G7)\) occurs at \(\beta_0 = \chi^{\frac{3}{2}}\) and is \(\frac{3}{2} \chi^{\frac{3}{2}}\). An approximation of \((G6)\) can be made by Taylor expanding \((G7)\) about \(\beta_0\), which yields:

\[
\frac{\beta^2 + 2 \chi}{2 \beta} \approx \frac{\beta^0 + 2 \chi}{2 \beta_0} + \left[ \frac{3 \beta^0}{2 \beta_0} - \frac{3 \beta^0 + 2 \chi}{2 \beta_0} \right] (\beta - \beta_0) + \frac{1}{2} \left[ \frac{6\beta^0}{2 \beta_0} - \frac{3 \beta^0}{2 \beta_0} - \frac{3 \beta^0 + 2 \chi}{2 \beta_0} \right] (\beta - \beta_0)^2
\]

\[
= \frac{3 \chi^{\frac{3}{2}}}{2} + \frac{3}{2} (\beta - \beta_0)^2.
\]

Inserting \((G8)\) into \((G6)\), we have

\[
g_1(\chi; \alpha = 2) \approx \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp \left( -\frac{3}{2} \chi^{\frac{3}{2}} - \frac{3}{2} (\beta - \beta_0)^2 \right) \frac{d\beta}{\beta}
\]

\[
\approx \exp \left( -\frac{3}{2} \chi^{\frac{3}{2}} \right) \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{3}} \int_0^\infty \exp \left( -\frac{3}{2} \beta - \frac{3}{2} (\beta - \beta_0)^2 \right) d\beta.
\]

The evaluation of \(g_0\) proceeds similarly and results is \(\beta_0\) times this expression for \(g_1\). To summarize, for \(\chi \gg 1\), we find

\[
g_0(\chi; \alpha = 2) \approx \frac{1}{\sqrt{3}} \exp \left( -\frac{3}{2} \chi^{\frac{3}{2}} \right),
\]

\[
g_1(\chi; \alpha = 2) \approx \frac{1}{\sqrt{3}} \exp \left( -\frac{3}{2} \chi^{\frac{3}{2}} \right) \chi^{-\frac{1}{3}}.
\]

Note that because \(\chi \gg 1\), these integrals are always exponentially small. Thus, they produce a negligible contribution to the velocity and stress calculations.

Appendix H. Solution of Equation (5.26) versus Double-Log Profile

Figure 10 provides verification that the double-log profile \(\bar{u}(z) = \ln (z/(1 - z))\) from (5.29) is a solution to momentum equation (5.26). In this example, the “boundary conditions” applied were an additive constant set such that \(\bar{u}(z=1/2) = 0\) and a multiplicative constant set such that slope \(d\bar{u}(z=1/2)/dz = 0.3\) (value immaterial). Here, we set \(H = 1\), \(N = 50\) terms retained in each of the 4 summations in (5.26), \(M = 501\) grid points, and retained the viscous-like term in the vicinity of \(z' = z\). Solution of (5.26) was obtained iteratively, starting from the initial profile \(\bar{u}(z) = 0.3 (z - \frac{1}{2})\) and iterating \(\bar{u}(z) = \bar{u}(z) + \text{RHS of (5.26)}, \text{until convergence. The boundary values } \bar{u}(z=0) \text{ and } \bar{u}(z=H) \text{ were free to vary naturally until the solution converged.}

Appendix I. Solution in a Finite Domain with Piecewise \(\alpha(z)\)

In §5.3, we presented the solution of the Boltzmann equation (5.2) in a finite domain \((0 \leq z \leq H)\), which was

\[
f(z,u,v,w) = \begin{cases} 
\frac{1}{w^r} \int_{H}^{z} f^{eq}_{\alpha}(z',u,v,w) e^\frac{z'-z}{w^r} dz' + A_H(u,v,w) e^\frac{H-z}{w^r} & \text{for } w < 0 \\
\frac{1}{w^r} \int_{0}^{z} f^{eq}_{\alpha}(z',u,v,w) e^\frac{z'-z}{w^r} dz' + A_0(u,v,w) e^\frac{z}{w^r} & \text{for } w > 0
\end{cases}.
\]

Equation (5.15) is composed of particular and homogeneous parts, \(f = f_p + f_h\), where the particular solution involves integration in \(z'\) from a boundary (at 0 or \(H\)) to the point of interest \(z\). Because there are two boundaries, the homogenous solution has contributions for both \(w\) positive and negative; the functions \(A_0\) and \(A_H\) are determined so as to
enforce the desired boundary conditions. In §5.3, we derived the appropriate $A_0$ and $A_H$ so as to enforce “bounce back” symmetry at the walls, also with the assumption that the $\alpha$ parameter was constant throughout the flow domain.

In this Appendix, we consider the case of piecewise-constant $\alpha(z)$, so as to model the laminar sublayer very near each wall:

$$\alpha(z) = \begin{cases} \alpha_2 & \text{for } 0 \leq z \leq b \text{ or } H - b \leq z \leq H \text{ laminar sublayers} \\ \alpha_1 & \text{for } b \leq z \leq H - b \text{ turbulent core flow} \end{cases} \quad (5.30)$$

where $\alpha_2 = 2$ and say $\alpha_1 = 1$. In order to conserve mass and momentum between the three zones, we apply the constraint on $f$ that the probability of vertical velocities bringing particles into the laminar sublayer equals that bringing them out:

$$f(z = b, u, v, w) = f(z = b, u, v, -w)$$
$$f(z = H - b, u, v, w) = f(z = H - b, u, v, -w) \quad (5.31)$$

The constraints (5.31) will now be used to determine the two unknown functions in (5.15), $A_H$ and $A_0$. Evaluating $f$ at $z = b$ and $H - b$, equation (5.15) yields

$$f(z = b, u, v, w) = \begin{cases} \frac{1}{w^{\alpha}} \int_{H}^{H-b} f_{eq} e^{z'/w} \, dz' + A_H(u,v,w) e^{H-b/w} & \text{for } w < 0 \\
\frac{1}{w^{\alpha}} \int_{0}^{b} f_{eq} e^{z'/w} \, dz' + A_0(u,v,w) e^{-b/w} & \text{for } w \geq 0 \end{cases}$$

$$f(z = H - b, u, v, w) = \begin{cases} \frac{1}{w^{\alpha}} \int_{H}^{H-b} f_{eq} e^{z'-(H-b)/w} \, dz' + A_H(u,v,w) e^{b/w} & \text{for } w < 0 \\
\frac{1}{w^{\alpha}} \int_{0}^{b} f_{eq} e^{z'-(H-b)/w} \, dz' + A_0(u,v,w) e^{-(H-b)/w} & \text{for } w \geq 0 \end{cases} \quad (I1)$$

where $f_{eq} = f_{eq}^{\alpha,q}(z',u,v;\alpha(z'))$. In order to enforce symmetry, set $w \rightarrow -w$ in each of the $w < 0$ expressions in (I1), and set that expression equal to the corresponding $w \geq 0$ expression. The equilibrium function $f_{eq}^{\alpha}$ is symmetric in $w$ so it remains unchanged.
Now for \( w \geq 0 \), equations (I 11) become

\[
-\frac{1}{w\tau} \int_{H}^{b} f_{\alpha} e^{-\frac{z'-b}{w\tau}} \, dz' + A_{H} e^{-\frac{H-b}{w\tau}} = \frac{1}{w\tau} \int_{0}^{b} f_{\alpha} e^{-\frac{z'-b}{w\tau}} \, dz' + A_{0} e^{-\frac{b}{w\tau}}
\]

\[
-\frac{1}{w\tau} \int_{H}^{H-b} f_{\alpha} e^{-\frac{z'-(H-b)}{w\tau}} \, dz' + A_{H} e^{-\frac{b}{w\tau}} = \frac{1}{w\tau} \int_{0}^{H-b} f_{\alpha} e^{-\frac{z'-(H-b)}{w\tau}} \, dz' + A_{0} e^{-\frac{(H-b)}{w\tau}}
\]

Equations (I 12) form a 2 by 2 system of equations for the unknown functions \( A_{0}(u,v,w) \) and \( A_{H}(u,v,-w) \). With careful attention to the fact that \( f_{\alpha}^{eq}(z',u,v,w;\alpha(z')) \) changes form with \( \alpha(z') \), but that \( \alpha(z') \) is assumed piecewise constant, the solution to this system of equations can be found to be

\[
A_{H}(u,v,-w) = \frac{e^{\frac{b}{w\tau}}}{w\tau \sinh\left(\frac{H-2b}{w\tau}\right)} \int_{b}^{H-b} f_{\alpha_{1}}^{eq} \cosh\left(\frac{z'-b}{w\tau}\right) dz' - \frac{1}{w\tau} \int_{H-b}^{H} f_{\alpha_{2}}^{eq} e^{\frac{H-z'}{w\tau}} \, dz',
\]

(I 13)

\[
A_{0}(u,v,w) = \frac{e^{\frac{b}{w\tau}}}{w\tau \sinh\left(\frac{H-2b}{w\tau}\right)} \int_{b}^{H-b} f_{\alpha_{1}}^{eq} \cosh\left(\frac{z'-H+b}{w\tau}\right) dz' - \frac{1}{w\tau} \int_{0}^{b} f_{\alpha_{2}}^{eq} e^{\frac{z'}{w\tau}} \, dz'.
\]

(I 14)

where \( f_{\alpha_{1}}^{eq} \equiv f_{\alpha_{1}}^{eq}(z',u,v,w;\alpha_{1}) \) and \( f_{\alpha_{2}}^{eq} \equiv f_{\alpha_{2}}^{eq}(z',u,v,w;\alpha_{2}) \). In order to form \( A_{H}(u,v,w) \), substitute \( w \rightarrow -w \) into equation (I 13),

\[
A_{H}(u,v,w) = \frac{e^{-\frac{b}{w\tau}}}{w\tau \sinh\left(\frac{H-2b}{w\tau}\right)} \int_{b}^{H-b} f_{\alpha_{1}}^{eq} \cosh\left(\frac{z'-b}{w\tau}\right) dz' + \frac{1}{w\tau} \int_{H-b}^{H} f_{\alpha_{2}}^{eq} e^{\frac{-H-z'}{w\tau}} \, dz'.
\]

(I 15)

With equations (I 14) and (I 15) into (5.15), the final expression for the homogeneous solution \( f_{h} \) is:

\[
f_{h} = \begin{cases} 
A_{H}(u,v,w) e^{\frac{H-z}{w\tau}} & \text{for } w < 0 \\
A_{0}(u,v,w) e^{-\frac{z}{w\tau}} & \text{for } w \geq 0
\end{cases},
\]

\[
= \begin{cases} 
\frac{e^{\frac{H-z-b}{w\tau}}}{w\tau \sinh\left(\frac{H-2b}{w\tau}\right)} \int_{b}^{H-b} f_{\alpha_{1}}^{eq} \cosh\left(\frac{z'-b}{w\tau}\right) dz' + \frac{1}{w\tau} \int_{H-b}^{H} f_{\alpha_{2}}^{eq} e^{\frac{z'-z}{w\tau}} \, dz' & \text{for } w < 0 \\
\frac{e^{\frac{b-z}{w\tau}}}{w\tau \sinh\left(\frac{H-2b}{w\tau}\right)} \int_{b}^{H-b} f_{\alpha_{1}}^{eq} \cosh\left(\frac{z'-H+b}{w\tau}\right) dz' - \frac{1}{w\tau} \int_{0}^{b} f_{\alpha_{2}}^{eq} e^{\frac{z'-z}{w\tau}} \, dz' & \text{for } w \geq 0
\end{cases}.
\]

(I 16)

In the core region \( b \leq z \leq H - b \), the particular solution \( f_{p} \) from (5.15) is

\[
f_{p} = \begin{cases} 
\frac{1}{w\tau} \int_{H-b}^{z} f_{\alpha_{1}}^{eq} e^{\frac{z'-z}{w\tau}} \, dz' + \frac{1}{w\tau} \int_{H}^{H-b} f_{\alpha_{2}}^{eq} e^{\frac{z'-z}{w\tau}} \, dz' & \text{for } w < 0 \\
\frac{1}{w\tau} \int_{b}^{z} f_{\alpha_{1}}^{eq} e^{\frac{z'-z}{w\tau}} \, dz' + \frac{1}{w\tau} \int_{0}^{b} f_{\alpha_{2}}^{eq} e^{\frac{z'-z}{w\tau}} \, dz' & \text{for } w \geq 0
\end{cases}.
\]

(I 17)

Inspection of (I 16) and (I 17) reveals that all terms involving \( f_{\alpha_{2}}^{eq} \) exactly cancel upon
forming the complete solution \( f = f_h + f_p, \)

\[
\begin{align*}
\frac{1}{w_T} \int_{H-b}^{H} f_{\alpha_1}^{eq} e^{\frac{z'-z}{w_T}} dz' + \frac{e^{\frac{H-z-b}{w_T}}}{w_T \sinh\left(\frac{H-2b}{w_T}\right)} \int_0^{H-b} f_{\alpha_1}^{eq} \cosh\left(\frac{z'-b}{w_T}\right) dz' \quad (w < 0) \\
\frac{1}{w_T} \int_{z}^{H} f_{\alpha_1}^{eq} e^{\frac{z'-z}{w_T}} dz' + \frac{e^{\frac{b-z}{w_T}}}{w_T \sinh\left(\frac{H-2b}{w_T}\right)} \int_0^{H-b} f_{\alpha_1}^{eq} \cosh\left(\frac{z'-H+b}{w_T}\right) dz' \quad (w \geq 0)
\end{align*}
\]

Note that the only terms that remain in (I 8) involve \( f_{\alpha_1}^{eq} \) and integrals within the core region \( b \) to \( H-b \). Thus, the conclusion is that the laminar sublayer \( f_{\alpha_2}^{eq} \) terms do not affect the mass or momentum in the turbulent zone \( b \leq z \leq H-b \), as expected. The laminar sublayer merely serves to shift the effective boundary from the wall to the edge of the laminar sublayer.

Note that the solution (I 8) (which includes the effect of laminar sublayers) can be put in the same form as the solution (5.21) (which does not include laminar sublayers) by defining a shifted coordinate system as follows (see Figure 4),

\[
\begin{align*}
\bar{z} &\equiv z - b \\
\bar{z}' &\equiv z' - b \\
\bar{H} &\equiv H - 2b
\end{align*}
\]

Then, the solution (I 8) for the flow in the core region \( 0 \leq \bar{z} \leq \bar{H} \) is

\[
\begin{align*}
\frac{1}{w_T} \int_{\bar{z}}^{\bar{H}} f_{\alpha_1}^{eq} e^{\frac{\bar{z}'-\bar{z}}{w_T}} d\bar{z}' + \frac{e^{\frac{\bar{H}-\bar{z}}{w_T}}}{w_T \sinh\left(\frac{\bar{H}}{w_T}\right)} \int_0^{\bar{H}} f_{\alpha_1}^{eq} \cosh\left(\frac{\bar{z}'-\bar{H}}{w_T}\right) d\bar{z}' \quad (w < 0) \\
\frac{1}{w_T} \int_{0}^{\bar{z}} f_{\alpha_1}^{eq} e^{\frac{\bar{z}'-\bar{z}}{w_T}} d\bar{z}' + \frac{e^{\frac{-\bar{z}}{w_T}}}{w_T \sinh\left(\frac{\bar{H}}{w_T}\right)} \int_0^{\bar{H}} f_{\alpha_1}^{eq} \cosh\left(\frac{\bar{z}'-\bar{H}}{w_T}\right) d\bar{z}' \quad (w \geq 0)
\end{align*}
\]

which is identical in form to (5.21). Thus, modeling laminar sublayers via piecewise constant \( \alpha(z) \) is equivalent to modeling a flow with uniform \( \alpha_1 \) in a channel of reduced height \( \bar{H} \equiv H - 2b \), as illustrated in Figure 4.

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