Barotropic thin jets over arbitrary topography

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Received 20 June 1995; revised 29 March 1996; accepted 2 August 1996

Abstract

An existing theory for thin jets on the beta plane is modified for applications to arbitrary topography. While the resulting path equation bears close resemblance to that for the beta plane, the fact that bottom topography, unlike the Coriolis parameter, can vary non-monotonically and in both horizontal directions leads to interesting effects, such as retroflection and jet trapping.

Applications to a rectilinear ridge and a circular seamount lead to analytical criteria predicting under what conditions a jet, of known upstream profile, will be deflected, retroflected or trapped. Finally, an application to an elliptical seamount has possible implications for the ring current atop Georges Bank in the Gulf of Maine. © 1997 Elsevier Science B.V.

1. Introduction

In numerous atmospheric and oceanic situations, flow fields include recognizable jets that are substantially narrower than the domains which they traverse. These jets are so easily identified that, if they have any degree of permanency or recurrence, a specific name has been given to them, for example the ‘Jet Stream’ in the atmosphere, the ‘Gulf Stream’ in the North Atlantic Ocean (originating as the ‘Loop Current’ in the Gulf of Mexico), the ‘Kuroshio’ in the North Pacific Ocean and the ‘Undercurrent’ in the tropical ocean, and countless others. In the consideration of a particular jet, the first and fundamental question concerns the causes for its existence. Having answered that, one may wish to know why the jet flows in a particular direction, what characterizes its meandering pattern, and how it behaves upon encountering a changing environment (change of latitude or topography). In developing theories and models to answer such questions, it is tempting to consider the jet as a single entity endowed with certain attributes such as a known velocity profile (both vertically and across the stream), and

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then to reduce the problem from 3D to 1D, i.e. to track the jet’s main axis downstream
and, possibly, with time.

The idea of such reduction is classic and, in oceanography, dates back to the early
1960s when Warren (1963) first explored the influence of bottom topography on the
Gulf Stream. Generalizing that theory and correcting some of its inconsistencies,
Robinson and Niiler (1967) then formulated a path equation that relates the curvature of
the jet axis (however deformed) to $\beta$ (the meridional gradient of the Coriolis parameter)
and to the local bottom depth. The coefficients of their equation contained integration of
the velocity profile over the width and depth of the jet. Using the equation for a jet of
known velocity profile then permitted the determination of its curvature at any point
and, by downstream integration, the construction of its path. Applications were subse-
quently made to the Gulf Stream (Niiler and Robinson, 1967) and the Kuroshio
(Robinson and Taft, 1972).

This path equation was derived for a jet in steady state, and although both the beta
and topographic effects were simultaneously retained, the emphasis was placed on the
beta effect (because of the large-scale nature of the jets to which the model was applied).
The main weakness of the theory is its lumped approach: the path equation is obtained
by transverse integration of the vorticity equation. While this is a quick way to obtain an
equation that leaves only the downstream coordinate, the resulting equation includes,
beside the radius of curvature, some integral moments of the velocity profile within the
jet, such as the vertical and cross-stream integrals of the speed and its square. To solve
for the radius of curvature, it is then necessary to assume that these integrals are
constant, i.e. that they are not varying downstream. This is a crucial assumption, for it is
not at all evident a priori why a distortion induced by a change in curvature would not
also result in a change of the velocity profile and hence of its integral moments. For
example, curvature implies a certain degree of orbital vorticity (Cushman-Roisin, 1994,
pp. 243–244) which, by virtue of conservation of potential vorticity, must be at the
expense of shear vorticity, ambient vorticity and/or vertical stretching; if shear vorticity
is affected, the jet profile changes with path curvature.

Later, Robinson et al. (1975) generalized the theory to include time dependency. A
critique of this work was then published by Flierl and Robinson (1984). In that latter
work, topography was ignored and the emphasis was placed on the time-dependent
meandering and the wave field that it induces in surrounding waters. More recently, and
independently of this line of work, Pratt (1988) derived a path equation as a long-wave
asymptotic behavior of the contour-dynamics equation. This theory is, however, quite
restricted by the assumption of quasi-geostrophy (i.e. weak jets) and of piecewise
uniform potential-vorticity distributions (thus excluding the planetary beta effect).

Using a full curvilinear system of coordinates to avoid the possibility of multi-valued
functions, Cushman-Roisin et al. (1993) were able to derive a set of equations predicting
the temporal variations of the coordinates of points on the jet axis in a reduced-gravity
system. The first two equations are mere definitions of the stream coordinate (true
downstream length) and curvature, while the third equation is the path equation proper,
relating the rate of lateral displacement of the jet axis (i.e. sideways movement or
skidding) to vortex induction and the beta effect. Topographic effects were excluded by
virtue of the reduced-gravity nature of the dynamics. This path equation was obtained as
a compatibility condition on the dynamics of a higher-order perturbation and, viewed as such, is a cousin of the integral equation derived in the earliest theories. A major difference, however, is that the integral moments making up the coefficients of the equation relate to a reference jet profile and therefore do not depend on the local and instantaneous curvature. There is no more need for the additional assumption of constant velocity moments, which weakened the earlier theories. When expressed in the same coordinate system, the path equations are identical. In other words, the new theory does not correct but only validates the earlier theories.

A yet different tack was followed by Cushman-Roisin (1993) in the consideration of steady or steadily translating meanders on a thin reduced-gravity jet. The approach was to enforce conservation of potential vorticity on a particle-by-particle basis between a reference, uncurved state and any curved state. Again, the reduced-gravity model prevented the inclusion of bottom topography. The path equations of both Cushman-Roisin et al. (1993) and Cushman-Roisin (1993) were found to be identical when applied to those common situations for which they are both valid.

The approach followed by Cushman-Roisin (1993) is flexible, and topographic effects can easily be substituted for the beta effect. The purpose of the present paper is to perform such substitution and then to explore some of its consequences. Unlike the Coriolis parameter, topography is not restricted to vary monotonically and in a single direction, and we may therefore expect to encounter new types of solution, representing jet behaviors other than meandering, such as deflection, retroreflection and trapping.

Eq. (25) of Cushman-Roisin (1993) provides the speed of steadily translating meanders on an otherwise zonal jet on a beta plane

\[ c = -\beta R_d^2 + \frac{2 R_d^2 U}{3 R Y}, \]

where \( \beta \) is the gradient of the Coriolis parameter, \( R_d \) is a reference deformation radius (based on the reduced gravity and a layer thickness characterizing the stratification), \( U \) is the maximum speed at the jet center in the uncurved state, \( Y \) is the meridional excursion of the jet axis, and \( R \) is the local radius of curvature. (The factor \( 2/3 \) originates from the choice of a particular jet profile in the uncurved state.) In a steady state, the meander propagation speed \( c \) vanishes, and the local radius of curvature \( R \) (positive for clockwise curvature in the Northern Hemisphere) is a function of the jet excursion \( Y \) (positive northward): \( R = 2U/3\beta Y \). Noting \( \Delta f = \beta Y \), the change in Coriolis parameter experienced locally by the jet, and introducing a reference Coriolis parameter \( f_o \) for convenience, we can rewrite this last expression in a non-dimensional form:

\[ \frac{2}{3} \frac{U}{f_o R} = \frac{\Delta f}{f_o}, \]

which equates the jet Rossby number (with the curvature radius as the length scale) to the relative change in Coriolis parameter. Because topographic stretching plays a role analogous to the beta effect, we may anticipate that the preceding path equation can be
transposed for problems with topography by simply substituting a relative depth variation $\Delta h/h_o$ for $\Delta f/f_o$.

To validate somewhat this intuition, consider an isolated, steady, barotropic jet in an fluid of depth $h_o$ on a Northern Hemisphere $f$-plane impinging upon a step discontinuity of height $\Delta h$. The conservation of potential vorticity, $q = (f + \zeta)/h$, for the jet gives:

$$
\frac{f + \zeta_o}{h_o} = \frac{f + \zeta_o + \Delta\zeta}{h_o + \Delta h} \Rightarrow \Delta\zeta = \frac{\Delta h}{h_o}(f + \zeta_o),
$$

where $\zeta_o$ and $\zeta_o + \Delta\zeta$ are the relative vorticities in the upstream and downstream states, respectively. For a step down, $\Delta h > 0$, the jet will develop additional cyclonic vorticity, $\Delta\zeta > 0$, and tend to curve to the left. Assuming that the jet has no initial curvature and, for the moment, that the entire vorticity change, $\Delta\zeta$, is manifested as curvature of the jet, then for a jet with velocity scale $U$, we have:

$$
\Delta\zeta = \frac{U}{R},
$$

where $R$ is the radius of curvature (now defined in the accepted trigonometric sense as positive for counterclockwise curvature). Combining these two we find:

$$
\frac{U}{(f + \zeta_o)R} = \frac{\Delta h}{h_o},
$$

which relates the curvature of the jet, $1/R$, to the relative depth change and the jet's velocity profile. This path equation bears a strong resemblance to Eq. (1). A more rigorous derivation is, however, in order.

Similar arguments developed for the distinct case of a uniform flow encountering a topographic feature are classical (Holton, 1992, pp. 99–102).

2. Theory

In general, the structure of the jet itself may adjust to changes in fluid depth, so we need to proceed with caution as we relate two states of the jet. First, we define a depth, $h_o$, as the depth for which the jet has (or would have) no curvature. Over this depth the jet is straight and, therefore, can be simply represented in Cartesian coordinates (Fig. 1):

$$
v = v_o(n_o)
$$

$$
q = \frac{1}{h_o} \left( f + \frac{dv_o}{dn_o} \right).
$$

Here, $v_o$ is the unidirectional velocity profile and $n_o$ the cross-stream coordinate, increasing to the right of the jet facing downstream.

Then, as we consider the jet some distance away where the depth is different, $h \neq h_o$, we assume that it curves and/or deforms in a manner consistent with the conservation
Fig. 1. The geometry of the thin jet system.

of potential vorticity. The curved jet is more conveniently described in polar coordinates (Fig. 1) with \( r = R + n \):

\[
v = v(n)
\]

\[
q = \frac{1}{h} \left( f + \frac{dv}{dn} + \frac{v}{R+n} \right).
\]

Because paths of separate individual parcels do not cross, there exists a unique, continuous mapping function which relates a parcel’s initial position in the straight section of the jet, \( n_o \), to its subsequent position in the curved jet, \( n \). We can dynamically connect the jet, then, between the two locations via two conservation laws:

Vorticity:

\[
q(n) = q_o(n_o) \Rightarrow \frac{1}{h} \left( f + \frac{dv}{dn} + \frac{v}{R+n} \right) = \frac{1}{h_o} \left( f + \frac{dv_o}{dn_o} \right).
\]

Volume:

\[
hv(n) \, dn = h_o v_o(n_o) \, dn_o \Rightarrow hv(n) = h_o v_o(n_o) \frac{dn_o}{dn}.
\]

These equations form a \( 2 \times 2 \) system for \( v(n) \) and the mapping function, \( n_o(n) \), given an initial jet profile \( v_o(n_o) \) and bottom topography.

Conservation of potential vorticity relies on the absence of friction and external torques (such as a rotational wind stress), while conservation of volume indirectly assumes the absence of heating and cooling. Note that the conservation of the Bernoulli function, which could have been stated by virtue of the steadiness and frictionless nature of the system, is implied by the previous two statements. (This can be proven by recalling that the derivative of the Bernoulli function with respect to the volumetric-
transport streamfunction is the potential vorticity, and concluding that if two of these quantities are conserved, so is the third automatically.) Finally, note that the relative vorticity is not assumed small compared with the ambient vorticity; we therefore include in our study jets of finite-amplitude velocities (i.e. with Rossby numbers based on the jet width on the order of unity).

2.1. Thin-jet approximation

If we make an assumption that the jet width is small compared with the radius of curvature, $|n| \ll |R|$, then:

$$\frac{v(n)}{R + n} = \frac{v(n)}{R} = \frac{dv}{dn}. \quad (11)$$

If the curvature is to be small, it follows from Eq. (4) that there is a limit to the size of topographic variations, so that the local depth, $h = h_o + \Delta h$, does not differ much from the reference depth $h_o$, i.e. $|\Delta h| \ll h_o$. The immediate consequence of this is that $n_o$ is nearly $n$, and $\nu_o(n_o)$ is nearly $\nu(n)$. Therefore, we may seek a solution of the form:

$$n_o = n + n'(n) \text{ with } n' \ll n \quad (12)$$

$$\nu(n) = \nu_o(n) + \nu'(n) \text{ with } \nu' \ll \nu_o \quad (13)$$

and linearize the conservation laws, in the independent variable $n$:

$$\frac{dv'}{dn} + \frac{\nu_o}{R} = \frac{\Delta h}{h_o} \left( f + \frac{dv_o}{dn} \right) + \frac{d^2 v_o}{dn^2} n', \quad (14)$$

$$\frac{\Delta h}{h_o} \nu_o + \nu' = \frac{dv_o}{dn} n' + \nu_o \frac{dn'}{dn}. \quad (15)$$

Eliminating $\nu'$ in favor of $n'$ and integrating across the full width of the jet from $a$ (left) to $b$ (right), we obtain:

$$\left[ \frac{\nu_o^2}{dn} \right]_{a}^{b} = \frac{f \Delta h}{h_o} \int_{a}^{b} \nu_o(n) \, dn + \frac{\Delta h}{h_o} \left[ \nu_o^2 \right]_{a}^{b} - \frac{1}{R} \int_{a}^{b} \nu_o^2(n) \, dn. \quad (16)$$

Since $\nu_o^2 \to 0$ on both sides by definition of a jet, the boundary terms vanish and we have:

$$\frac{1}{R} \int_{a}^{b} \nu_o^2(n) \, dn = \frac{f \Delta h}{h_o} \int_{a}^{b} \nu_o(n) \, dn. \quad (17)$$

If we define the scale of the flow by:

$$U = \frac{\int_{a}^{b} \nu_o^2 \, dn}{\int_{a}^{b} \nu_o \, dn}, \quad (18)$$
then, the curvature of the jet is expressible as a function of the departure of the local depth, $h$, from the reference depth, $h_o$, and of an integral measure of the jet scale:

$$\frac{1}{R} = \frac{f\Delta h}{h_o U}. \quad (19)$$

This states that as a thin jet moves into shallower water, $\Delta h < 0$, it develops negative curvature ($1/R < 0$) and turns to the right (clockwise). In addition, for a fixed $\Delta h$, as the jet speed, $U$, increases, the curvature decreases. Therefore, $U$ can also be thought of as a measure of the stiffness of the jet. Note that Eq. (19) has the form of Eq. (1) and Eq. (4), as anticipated.

### 2.2. Path equations

The preceding relation can be readily utilized to construct the path of a thin jet over variable topography. To begin, define a local coordinate system $(s,n)$ such that $s$ is coaxial with the jet and $n$ is orthogonal to it (Fig. 2). If we define the angle $\theta$ between the $s$-axis of this coordinate system and the $x$-axis of a Cartesian reference framework, then the curvature of the jet path, $1/R$, is given simply by $d\theta/ds$. Elementary geometry then gives the equations for the path in the Cartesian frame:

$$\frac{dx}{ds} = \cos \theta. \quad (20a)$$

$$\frac{dy}{ds} = \sin \theta. \quad (20b)$$

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{f}{U} \left[ \frac{h(x,y)}{h_o} - 1 \right]. \quad (20c)$$

This $3 \times 3$ system can be integrated forward to determine the path of the jet from the initial position $(x_o,y_o)$ and direction $(\theta_o)$ with knowledge of the jet stiffness, $U$, reference depth, $h_o$, and the bottom topography, $h(x,y)$. Note that it is not necessary for
the jet to pass over a depth of \( h_o \) at any time. All we need is the knowledge of either \( h_o \), or, equivalently, the potential vorticity of a particle within the jet, to proceed.

3. Particular solutions

In this section, we investigate a series of solutions for the path of thin jets over various simple topographies. We restrict our attention to idealized situations that are representative of oceanic scale flows. In doing so, we obtain some general criteria for geophysically relevant solutions, which readily apply to atmospheric situations.

3.1. Rectilinear ridge: incident jet

The first case we consider is that of a jet approaching a rectilinear ridge. Due to the linear dependence of the path equations on topography, the results for a ridge apply in reverse to a canyon, which we will, therefore, not treat explicitly. Of primary importance, is the condition that determines whether a thin jet crosses the ridge or is retroflected back toward the basin of origin. From Eq. (19), we expect that if the jet is sufficiently stiff or the ridge sufficiently narrow and/or short, the curvature will be small, and the jet traverses the ridge; otherwise, it retroflects.

We assume that the ridge is aligned with the y-axis, for convenience, and rises above the reference depth, \( h_o \), with a height of \( \eta(x) \), so that the local depth at any point is \( h(x) = h_o - \eta(x) \) (see Fig. 3). The ridge is confined between \(-a < x < a\). For the rectilinear ridge we can derive the retroflection criterion from the first and third path equations without the use of the second. Here, these take the relatively simple form:

\[
\begin{align*}
\frac{dx}{ds} &= \cos \theta, \\
\frac{d\theta}{ds} &= -\frac{f}{h_o U} \eta(x).
\end{align*}
\]

Combining to eliminate the path variable, \( s \), gives:

\[
\frac{d\theta}{dx} = -\frac{f\eta(x)}{h_o U \cos \theta}.
\]

This can be separated and integrated along the path to give:

\[
\sin \theta(x) - \sin \theta_{-\infty} = -\frac{f}{h_o U} \int_{-a}^{x} \eta(\xi) d\xi,
\]

which relates the angle of the jet path, \( \theta(x) \), to the incidence angle, \( \theta_{-\infty} \) (\( |\theta_{-\infty}| < (\pi/2) \)), and the path turning as an integral over the topography. As long as \( \theta(x) \) does not reach \(-\pi/2\) over the topography, the jet traverses the ridge, departing with an exit angle, \( \theta \), found by evaluating Eq. (23) at \( x = a \). If, however, \( \theta(x) \) reaches \(-\pi/2\) within \( |x| < a \), then the jet retroflects (and the above expression is multivalued). The condition for retroflection from a ridge then is:

\[
\frac{f}{h_o U} \int_{-a}^{x} \eta(\xi) d\xi = 1 + \sin \theta_{-\infty},
\]
for a value of $x$ somewhere over the ridge ($-a < x < a$). Put another way, all impinging jets satisfying the condition:

$$\frac{f}{h_o U} \int_{-a}^{+a} \eta(x) \, dx > 1 + \sin \theta_{-\infty}$$

(25)

are retroflected by the ridge. For a trough, such as a trench or canyon, the angle of the retroflection is $+(\pi/2)$ and the analogous criterion is:

$$\frac{f}{h_o U} \int_{-a}^{+a} |\eta(x)| \, dx > 1 - \sin \theta_{-\infty}.$$  

(26)

For retroflecting jets, the effect of the topographic turning is 'symmetrical' (i.e. the path is symmetric about the $x$-axis), so that the exit angle, $\theta_e$, is the mirror of the entrance angle ($\theta_e = \pi - \theta_{-\infty}$). This implies, for an eastward jet impinging on a rectilinear ridge, that if $\theta_{-\infty} > 0$ and the jet retroflects, then the jet path will cross itself, which violates the present theory. This situation needs to be treated carefully and suggests the need for some higher-order dynamics, such as time dependence or the

![Fig. 3. Top. Path of various thin jets in the vicinity of a symmetric Gaussian rectilinear ridge topography of height 400 m and horizontal decay scale of 1000 m. The depth of no curvature is equal to the far field ocean depth of 1000 m. Upper three paths vary by their stiffness ($U = 0.08, 0.10, 0.12 \text{ m s}^{-1}$) but share the same normal incidence ($\theta_{-\infty} = 0^\circ$). Lower three have fixed stiffness ($U = 0.10 \text{ m s}^{-1}$) with variable incidence angles ($\theta_{-\infty} = -20^\circ, 0^\circ, 20^\circ$). Bottom. Depth profile across the ridge.](image-url)
possibility of spatial bifurcation of the jet, for resolution. In other words, some of the preceding assumptions (steadiness, single thin jet, quiescent surrounding fluid, and downstream length scale much longer than cross-stream length scale) may no longer be made. We have not resolved this problem, preferring to leave it for a separate investigation.

Fig. 3 shows some sample path integrations of the system Eq. (19) for thin jets approaching a ridge with a Gaussian profile. The upper trio of paths are for varying stiffness and orthogonal incidence upon the ridge. For this entrance angle \( \theta_{\infty} = 0 \), the critical stiffness for retroreflection by a Gaussian ridge of height \( \hat{h} \) is:

\[
U_{\text{crit}} = \frac{f \hat{h}}{h_o} \int_{-\infty}^{\infty} e^{-\xi^2/2} d\xi = \sqrt{2\pi} f x_0 \hat{h}/h_o.
\]  

In this figure, the ridge is of height \( \hat{h} = 400 \) m and has a width scale of \( x_0 = 1 \) km, so that the critical stiffness for retroreflection of orthogonally incident jets, \( U_{\text{crit}} = 0.10 \) m s\(^{-1}\), is very close to the intermediate value used. The stiffer jet crosses the ridge, the more pliable jet retroreflects, and the jet with critical stiffness just reaches the backside of the ridge, 'exiting' parallel to the ridge axis. The lower trio of paths are for jets with the identical stiffness (\( U = 0.10 \) m s\(^{-1}\)) but for varying entrance angles (\( -20^\circ, 0^\circ, 20^\circ \)).

3.2. Continental shelf: alongshore jet

Another application of the thin-jet theory is for flow along a continental shelf. For large scale, steady, geostrophic flow on an \( f \)-plane, where relative vorticity is unimportant, the flow is along the isobaths. Our interest here is in other possible steady paths for flow slightly away from the isobaths when the effects of relative vorticity are included.

Consider a thin jet flowing nearly parallel to a topographic slope over the isobath, \( h_o \), for which the jet has no curvature. Assume that the deviations from isobathic motion are small, \( \psi = \pm (\pi/2) \mp \alpha \), where \( \alpha \ll 1 \) and the upper (lower) signs are associated with a northward (southward) jet. The small angle, \( \alpha \), is defined such that positive \( \alpha \) is associated with an eastward deviation from north–south flow. Then the path equations, linearized about the north–south path, reduce to:

\[
\frac{dx}{ds} = \cos\left(\pm \frac{\pi}{2} \mp \alpha\right) = \sin \alpha = \alpha,
\]

\[
\frac{dy}{ds} = \sin\left(\pm \frac{\pi}{2} \mp \alpha\right) = \pm \cos \alpha = \pm 1,
\]

\[
\frac{d\theta}{ds} = \mp \frac{d\alpha}{ds} = \frac{f}{h_o U} \eta(x),
\]

Eq. (28b) states that, for small angles, the path coordinate, \( s \), is essentially equivalent to the downstream distance along the isobath, \( \pm y \). Differentiating Eq. (28c) with respect to \( s \) and then combining with Eq. (28a) gives one equation for the angle \( \alpha \):

\[
\frac{d^2\alpha}{ds^2} \mp \frac{f \eta'(x)}{h_o U} \alpha = 0,
\]
where $\eta'(x)$ is the zonal slope of the topography. The solutions to this equation have either oscillatory or exponential character depending on whether $\mp \eta'(x)$ is positive or negative, respectively. This implies that for an oscillatory (i.e. stable) solution, fluid must deepen towards the right of the jet. For constant $\eta'(x)$, the stable solutions are trigonometric with wavelength:

$$
\lambda = 2\pi \sqrt{\frac{h_o U}{f|\eta'|}} = 2\pi \sqrt{\frac{U}{\beta_c}},
$$

where $\beta_c (= f|\eta'|/h_o)$ is the effective topographic $\beta$. This wavelength is the topographic analogue of the planetary critical meander scale ($\sqrt{U/\beta}$), the length scale at which vorticity induction is arrested by Rossby wave propagation (Cushman-Roisin, 1994, p. 245).

In Fig. 4, we show the results for the path integration along a continental shelf of half-Gaussian profile. In this case, the southward jet (with deeper fluid on the left) is unstable and rapidly exits the shelf. Whereas, the northward jet (with deeper fluid on the right) is stable and possesses a meander wavelength in close agreement with Eq. (30) with $\eta'$ evaluated on the central isobath.
The instability of a thin jet with the deep fluid on its left (right) in the Northern (Southern) Hemisphere is believed to be a new result. The beta-plane analogous instability has, however, been known for some time (Philander, 1976).

3.3. Rectilinear ridge: closed paths

Another interesting limit occurs for jets initially parallel to the axis of a one-dimensional ridge (i.e. $\theta_o = \pm (\pi/2)$). If the integrated topographic turning over the ridge leads to retroflection, then there is a possibility that the path closes on itself to form a ring. For a symmetric ridge, we can derive an expression for the jet stiffness necessary for this ring current to occur. In this case, symmetry requires that the topographic turning lead to an angular change of $-(\pi/2)$ between the initial location $(x_o, y_o)$ and the crest of the ridge. So from Eq. (23), for a closed path to form over an axisymmetric ridge centered on $x = 0$ and lying between $-a \leq x \leq a$, the jet stiffness must satisfy:

$$\sin(0) - \sin\left(\frac{\pi}{2}\right) = -1 = \frac{-f}{h_o U} \int_{x_o}^{0} \eta(\xi) d\xi \text{ for } |x| \leq a.$$  \hspace{1cm} (31)

The ring current has a zonal extent of $2x_o$ where $x_o$ is directly related to the stiffness of the jet and the topography. For a particular ridge of width $2a$, the maximum stiffness which allows for a closed path is for a jet initially on the outer edge of the topography, $x_o \to a$, and is given by:

$$U_{\text{closed}} = \frac{f}{h_o} \int_{-a}^{0} \eta(x) dx = \frac{f}{h_o} \int_{0}^{a} \eta(x) dx.$$  \hspace{1cm} (32)

This case is shown for a Gaussian ridge in Fig. 5.

In the case of asymmetric ridges, there is no simple expression for the proper stiffness for closure. In order for the path to close, the net distance traveled by the jet along the ridge on one side of the crest must equal that traveled back on the other side. This means that the $y$-equation re-enters the problem and an entire class of solutions is possible. If the path does not close but remains trapped to the ridge, then the jet path will describe a series of loops with the jet crossing itself many times. If the meridional excursion of the jet path on the western slope exceeds (is less than) that on the eastern, these loops will translate northward up (southward down) the ridge.

3.4. Circular seamount: closed paths

For a circular seamount, $\eta = \mathcal{F}(r)$, the most obvious approach is to seek circular solutions for the jet path. For there to be a closed circular solution we require the curvature of the isobaths, $1/r$, to match that of the dynamic jet curvature due to the changing depth, $\eta = h_o - h$:

$$\frac{1}{r} = \frac{f\eta(r)}{h_o U}.$$  \hspace{1cm} (33)
Fig. 5. Top. Closed path solution on a symmetric Gaussian rectilinear ridge of height 400 m in an ocean of depth 1000 m. The jet stiffness \((U)\) is 0.05 m s\(^{-1}\) and the depth of no curvature \((h_0)\) is 1000 m. Bottom. Depth profile across the ridge.

By plotting the two curvatures, the possible solutions are easily seen graphically (Fig. 6). It is clear that for monotonic topography there can be 0, 1, 2 or 3 solutions for the radius of the closed jet path depending on the topographic profile of the seamount.

Fig. 6. Radial dependence of the dynamic curvature due to topography (solid line) and the curvature of the isobaths (dashed line) for a circular Gaussian seamount of height 600 m with radial decay scale of 1500 m. The jet stiffness \((U)\) is 0.025 m s\(^{-1}\) and the depth of no curvature \((h_0)\) is 1000 m.
With the possibility of multiple solutions, it is of interest to discern which ones are stable. A detailed linear ability analysis is presented in Appendix A, while only a physically motivated treatment is presented here for completeness. Except in the rare case that the topography, $\eta(r)$, falls off at exactly $h_0 U / fr$, Eq. (33) will not balance if the jet is displaced slightly from the solution radius, $r_s$, for which Eq. (33) is met. The linear stability question then becomes what is the tendency of each side of the equation for small displacements away from $r_s$. In general, for outward (inward) displacements of the jet axis, the jet will find itself in deeper (shallower) water and will curve less (more) than it did at $r = r_s$ whereas the radius of curvature of the isobaths will also be different. When the gradient of the curvature of the isobaths exceeds the gradient of the dynamic curvature:

$$\frac{1}{r_s^2} \left| \frac{f}{h_0 U} \frac{d\eta}{dr} \right| > \left| \frac{\left( \frac{x'}{c^2} + \frac{y'}{d^2} \right)}{\sqrt{\left( \frac{x'}{c^2} \right)^2 + \left( \frac{y'}{d^2} \right)^2}} \right|^{3/2} = \frac{cd}{\left( c^2 \sin^2 \phi + d^2 \cos^2 \phi \right)^{3/2}},$$ (34)

such as at $r = r_1$ in Fig. 6, an outward displacement moves the jet into a region where the dynamic curvature induced in the jet due to topography is greater than the local curvature of the isobaths, so the jet will spiral radially inward relative to the isobaths. If displaced inward, the dynamic curvature is less than that of the isobaths, so it spirals outward. In either case, it returns toward its original radial distance and the configuration is stable. Reversing the inequality in Eq. (34), the case at $r = r_2$, leads to the opposite results and the conclusion that this case is unstable.

In Fig. 7 we present the results of the path integration for jets initially tangential to the seamount at three different radial locations. The inner and outermost jets are located at the two solutions $(r_1$ and $r_2$, respectively) in Fig. 6. While the integration of the inner solution $(r = r_1)$, which is linearly stable, maintains its radius after many circuits, the small errors introduced in the integration (by using a finite stepsize along the path) are sufficient to cause the outer, linearly unstable, solution $(r = r_2)$ to escape the seamount before completing one orbit. The jet at the intermediate radius begins in a region where the dynamic curvature exceeds that of the isobaths. This jet turns more rapidly than the topography and traces a series of tight loops which progress around the seamount. Obviously, the thin-jet model breaks down at every path intersection, and the looping solution is not a physically acceptable solution. It does, however, prove the instability of the unperturbed path.

### 3.5. Elliptical seamount: closed paths

We can generalize the results for the circular seamount somewhat by allowing for elliptical seamounts, $\eta = \eta[(x^2/a^2) + (y^2/b^2)]$. In this case, we look for elliptical solutions for the closed path of the form:

$$\begin{align*}
x &= c \cos \phi \\
y &= d \sin \phi
\end{align*}$$

(35)

At any given point, the curvature of the jet path is given by

$$-\frac{1}{R} = \frac{x'y'' - y'x''}{\left[ (x')^2 + (y')^2 \right]^{3/2}} = \frac{cd}{\left( c^2 \sin^2 \phi + d^2 \cos^2 \phi \right)^{3/2}},$$ (36)
Fig. 7. Top right. Stable and unstable path solutions atop the circular Gaussian seamount of Fig. 6. All three paths begin tangentially with the innermost and outermost paths starting at $r = r_1$ and $r = r_2$ (see Fig. 6), respectively. Left and bottom. Depth profiles along the dashed lines through the center of the seamount.

where $'$ denotes differentiation with respect to the parametric variable $\phi$. This jet curvature, however, must match the dynamic curvature due to the local depth anomaly, $\eta$, so:

$$
\frac{d\theta}{ds} = \frac{f}{h_0U} \eta \left( \frac{c^2 \cos^2 \phi + d^2 \sin^2 \phi}{a^2 \cos^2 \phi + b^2 \sin^2 \phi} \right) = \frac{cd}{\left[ c^2 \sin^2 \phi + d^2 \cos^2 \phi \right]^{3/2}}. \tag{37}
$$

To match dependencies, we take $c^2 = \lambda^2 a^2 d^2$ and $d^2 = \lambda^2 b^2 c^2$ and, for convenience, define $\gamma^2 = d^2 \cos^2 \phi + e^2 \sin^2 \phi$ to get:

$$
\eta(\lambda^2 \gamma^2) = \frac{cdh_0U}{f\gamma^3}. \tag{38}
$$

It appears that for elliptical seamounts of arbitrary cross-sectional profiles, there will not be perfectly elliptical closed path solutions. However, if we flip the question around by asking what type of seamount structures yield elliptical solutions for the jet path, we see the obvious form $\eta(\mu) = \eta_0 \mu^{-3/2}$. While a seamount of this form may not be realistic (i.e. height $\to \infty$ as $\mu \to 0$), we only need the seamount to have this shape over the range of $\mu$ in which the jet travels. So, if there exists a band of sufficient width, $\mu_1 < \mu < \mu_2$, around the flanks of a seamount where the depth has this dependence, then closed
elliptical paths are possible. With this form for the topography, we obtain a system of equations for the unknown constants of the path, $c$, $d$, and $\lambda$, whose solution, for positive $a$, $b$, $c$, and $d$, is:

$$\lambda^2 = \frac{1}{ab},$$

$$c = \left[ \frac{f\eta_0}{h_o U} b \right]^{1/2} a,$$

$$d = \left[ \frac{f\eta_0}{h_o U} a \right]^{1/2} b.$$ 

Align the $x$-axis along the major axis of the seamount so that $a > b$ and $c > d$. Then, comparing the ratios of the squares of the major and minor axes between the topography and the elliptical path we find:

$$\frac{c^2}{d^2} / \frac{a^2}{b^2} = \frac{b}{a} < 1,$$

which states that eccentricity of the jet path is less than that of the isobaths. An example of this solution is shown in Fig. 8.

The existence of jet paths encircling topographic bumps, of exact circular shape or not, brings to mind a classic result of two-dimensional turbulence studies. In the presence of arbitrary topography, it has been found (Bretherton and Haidvogel, 1976; Holloway, 1978) that an initially random flow field with vorticity spontaneously organizes itself to generate jets around topographic features, all being anticyclonic around bumps. That those jets emerge from a disordered flow implies that they are stable, thus, on one side, we have found that jets along rectilinear topography with the deep fluid on their left (in the Northern Hemisphere) are unstable (Section 3.3) and, on the other side, we note the existence of stable jets along curved topography with the deep fluid on their left. There must therefore exist a critical topographic curvature separating the unstable jets (over straight or weakly curved topography) from the stable jets (over significantly curved topography). Perhaps, a local application of criterion Eq. (34), which was derived in the case of a perfectly circular bump, may prove to be helpful. A thorough investigation of this point remains to be undertaken.

### 3.6. Coastal example

Independently of this study, some of these authors have recently been simulating currents in the vicinity of Georges Bank in the Gulf of Maine, using a finite-element model developed at Dartmouth College (Lynch et al., 1996). In a particular simulation, with both wind and tidal forcing suppressed and the current along Georges Bank entirely driven by a barotropic inflow from the Scotian Shelf, the current distribution shown in Fig. 9 (bottom) was obtained. The situation is characterized by clockwise flow around the Bank beginning on the northern flank (where the Scotian inflow completes a gyre
around the Gulf of Maine), bending around the Bank, across the Great South Channel and exiting to the southwest parallel to the coastline.

Attention is drawn to the large retroflection of the current as it exits the southwestern edge of the Bank and finds its way across the Great South Channel to the Nantucket Shoals. It is evident from the velocity field that the current is strongly affected by the topography: the current turns to the right on the shallow tip of the bank, then veers left in the deeper waters of the Great South Channel enabling it to cross, and finally turns to the right again over the shallower waters of the Shoals and exits to the southwest. The locations of greatest rightward and leftward jet curvature do not coincide exactly with the shallowest and deepest spots along the path, as the preceding theory would require. However, the simulated jet is not only affected by topographic stretching, but also by bottom friction. Furthermore, the simulated jet is not thin with respect to its radius of curvature, and the topographic variations are not small, $\Delta h \sim O (40\text{–}60 \text{ m})$.

Nonetheless, despite the obvious limitations (and outright violations of some of the assumptions), the preceding thin-jet theory was applied to this flow. A thin-jet of stiffness $U = 1.08 \text{ m s}^{-1}$ with a no-curvature depth of $h_o = 65.1 \text{ m}$ was traced from the midpoint of the southeastern flank of Georges Bank using the true topography. The result is shown in Fig. 9 (top). While the thin jet path is not coincident with the

Fig. 8. Top right. Elliptical path solutions for three different jet stiffnesses ($U = 0.02$, $0.04$ and $0.06 \text{ m s}^{-1}$) around an elliptical seamount. The radial structure of seamount is defined in the text. Left and bottom. Depth profiles along the dashed lines through the center of the seamount.
modelled flow, it does show a similar pattern of retroflection in the Great South Channel. This demonstrates the role that relative vorticity can play in adjusting the local curvature of the jet and allow it to cross $f/h$ contours.
4. Conclusions

Using conservation laws on a particle-by-particle basis between an uncurved reference jet and any curved state, we derived a relation giving the curvature of the jet as an explicit function of the local fluid depth. We then showed how downstream integration of this relation permits the construction of the jet path. The underlying assumptions are: inviscid and barotropic fluid, $f$-plane dynamics, steadiness, absence of diabatic processes, and chiefly a jet width much shorter than the length scale over which the jet curves. This last assumption limits the applicability of the theory to weak topographic variations in both height and length scales. There is no restriction, however, on the jet speed, and its shear vorticity may be comparable with the planetary vorticity.

The path equation is similar to that derived by Robinson and Niiler (1967) when the latter is restricted by the same assumptions. It is also analogous to the equation derived via the same method by Cushman-Roisin (1993) in the case of the beta effect.

Actually, the path equation can be generalized to include both variable topography and beta effect. In dimensionless form, to highlight the analogy, the result is:

\[
\frac{U}{f_o R} = \frac{\Delta h}{h_o} - \frac{\Delta f}{f_o},
\]

where the jet curvature $1/R$ (positive in the trigonometric sense), the stiffness $U$ and the depth variation $\Delta h$ are all as defined earlier; $h_o$ and $f_o$ are respectively the depth and Coriolis parameter values at which the jet has (or would have) no curvature; finally, $\Delta f$ is the change in Coriolis parameter from $f_o$ caused by the meridional excursion of the jet. In principle, the equation applies to both atmospheric and oceanic jets.

The jet has no curvature ($1/R = 0$) not only when the depth is $h_o$ and the Coriolis parameter is simultaneously $f_o$, but whenever the topographic stretching compensates for change in planetary vorticity: $\Delta h/h_o = \Delta f/f_o$.

The relative simplicity of the path equation has allowed us to analyze cases with prototypical topographies such as rectilinear ridges, circular and elliptical seamounts. In particular, an analytical criterion has been established to predict under which conditions a jet can traverse a ridge or must be retroflected by it, an expression has been derived for the wavelength of steady meanders of a jet flowing nearly along isobaths, a solution and a stability criterion have been determined for closed circular jet paths over circular seamounts and finally, an elliptical path solution has been constructed for elliptical seamounts. Noteworthy also is the instability of thin jets flowing along isobaths with the deep fluid on their left (right) in the Northern (Southern) Hemisphere.

Finally, the thin-jet theory was applied to a barotropic coastal flow in the vicinity of Georges Bank. Although there are limitations to the application, the general trend of the jet retroflection in the Great South Channel is captured by this theory. This suggests that the role of relative vorticity in adjusting the jet curvature may be important for many coastal flows.
Acknowledgements

Support for this research was provided by NOAA grant NA36GP0299-0201 (for BCR and DTM) and by NSF Grant OCE-9118426 (for JAP). We would like to thank Drs. Phil Bodgen and Glenn Flierl for pertinent remarks at an oral presentation of this work which have helped the authors to clarify certain points. This is contribution 84 of the U.S. GLOBEC program, funded jointly by the NSF and NOAA.

Appendix A

We derive here the stability criterion of circular paths on a circular seamount. Denoting by $\eta(r)$ the elevation profile of the seamount above the no-curvature depth of the jet ($\eta = h_0 - h > 0$), which is not necessarily the abyssal depth around the seamount, and passing from Cartesian to polar coordinates ($x = r \cos \alpha$, $y = r \sin \alpha$), we can rewrite the path Eq. (20a), Eq. (20b), Eq. (20c) as follows:

\[
\frac{dr}{ds} = \cos(\theta - \alpha), \tag{A1a}
\]
\[
\frac{d\alpha}{ds} = \sin(\theta - \alpha), \tag{A1b}
\]
\[
\frac{d\theta}{ds} = -\frac{f}{h_s U} \eta(r). \tag{A1c}
\]

A circular-path solution obeys $r = r_s = \text{const.}$, $\alpha = -(s/r_s)$, $\theta = \alpha - (\pi/2)$, where $r_s$ is a solution to Eq. (33):

\[
\frac{1}{r_s} = \frac{f \eta(r_s)}{h_s U}. \tag{A2}
\]

A small perturbation to this solution ($r' \to r$, $\alpha' \to \alpha$, and $\theta' \to \theta$) is governed by the linearized equations:

\[
\frac{dr'}{ds} = \theta' - \alpha', \tag{A3a}
\]
\[
\frac{d\alpha'}{ds} = \frac{r'}{r_s^2}, \tag{A3b}
\]
\[
\frac{d\theta'}{ds} = -\frac{f \eta'}{h_s U} r', \tag{A3c}
\]

where $\eta' = (d\eta/dr)$ at $r = r_s$. Elimination of the variables $\alpha'$ and $\theta'$ yields a single equation for the perturbed radius:

\[
\frac{d^2 r'}{ds^2} + \left( \frac{f \eta'}{h_s U} + \frac{1}{r_s^2} \right) r' = 0. \tag{A4}
\]
The circular path is stable to infinitesimal perturbations if the perturbed radius exhibits oscillatory behavior, which occurs if
\[ \frac{f r'}{h_o U} + \frac{1}{r_s^2} > 0, \] (A5)
according to Eq. (A4). Since \( r' \) is the negative of the bottom slope on the flank of the seamount at the location of the circular path, this criterion can be expressed in the following terms: A circular path of radius \( r_s \) on a circular seamount is stable if the bottom slope is locally less than \( h_o U / fr_s^2 \). This is equivalent to criterion Eq. (34) deduced earlier from simple dynamical considerations.

Typically, for a convex seamount profile (such as a bump rounded on top), the innermost solution is the stable one.

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