Environmental Transport and Fate

# Chapter 3
# –
# Part 3 - Turbulent Jets

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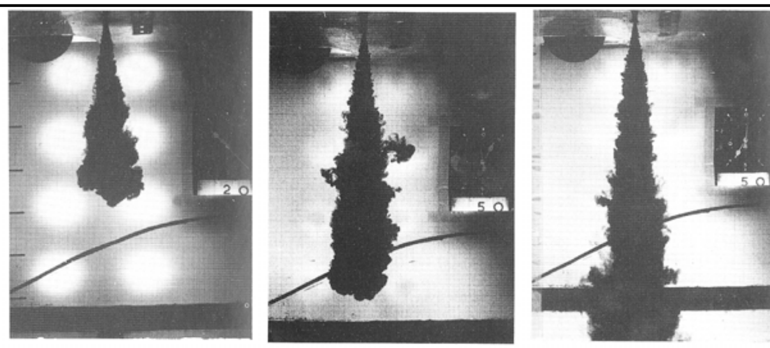
## General remarks about turbulent mixing

Mixing in fluids, such as the atmosphere or natural bodies of water, is generated by turbulence that is induced either mechanically or thermally, or both.
Relegating the problem of mixing in stratified fluids to a later section, we first consider here mixing in the absence of buoyancy forces, *i.e*. in homogeneous fluids.

In such fluids, turbulence arises from shear in the flow, that is the variation of the flow speed in a direction transverse to it. Examples of environmental significance are
- jets at the exit of underwater sewage release pipes
- factory smokestacks
- wind blowing over the water surface
- any encounter of flows with different velocities.

Since the properties of a turbulent flow greatly depend on the geometry of the flow domain and on the type of forces acting on the fluid, almost every situation is a separate problem requiring specific investigation. We therefore limit ourselves here to the most common case, that of a jet penetrating in an otherwise quiescent fluid.



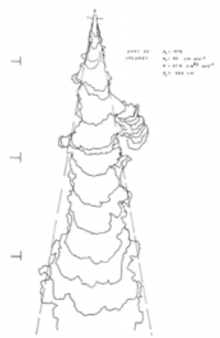
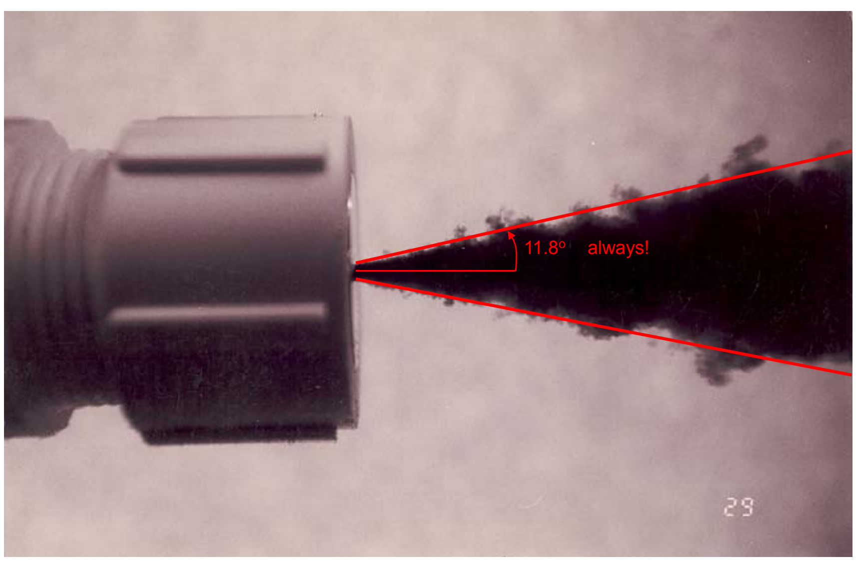



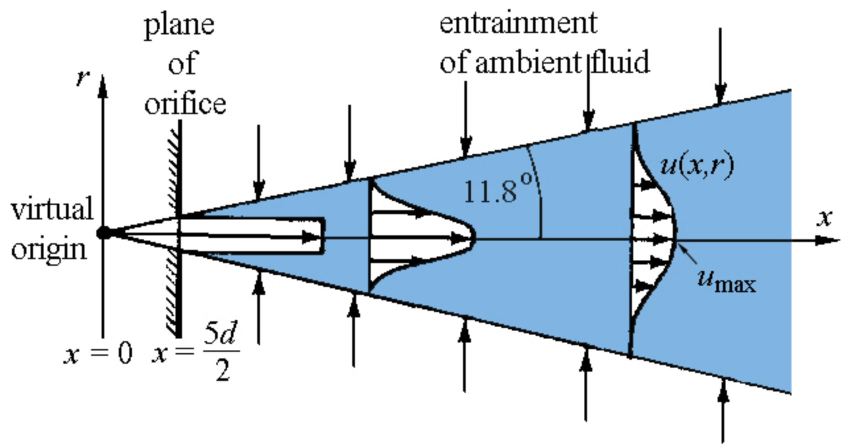
Fig. 10.9.iii The line diagram shows successive positions of the outline of a new jet (from Spackman [42]). A middle and a late position of the jet are shown. Occasional protuberances extend in the early stages beyond the cone indicated by the straight lines, but by the time the cap is far away they will have been entrained inside the cone.

A growing turbulent round jet

(Scorer, 1997, page 338)




A fully developed turbulent round jet (Photo taken in Thayer School Fluid's Lab)




Notation for the turbulent round jet.

Laboratory observations reveal that all turbulent round jets possess the same opening angle, regardless of fluid (air, water, other), orifice diameter ($d$) and injection speed ($U$).
The universal value is 11.8°, yielding a ratio radius-to-distance of 1-to-5. ⟵ $\tan 11.8^0 \approx \frac{1}{5}$

Therefore, for all turbulent jets
(when the jetting fluid has no buoyancy with respect to the ambient fluid):

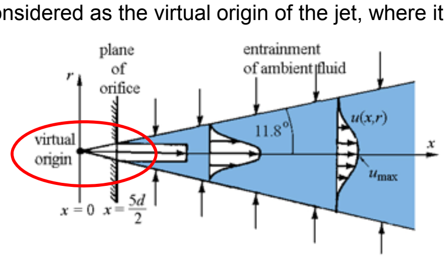
$$R = \frac{1}{5}x$$ in which $R$ = radius of jet at distance $x$

Note on the origin of $x$:

If the nozzle diameter is $d$, then the initial radius of the jet is not zero but $d/2$.

To enjoy the simple form above for the ($R$, $x$) relationship, we then need to place the origin of the $x$-axis at a distance of $5d/2 = 2.5d$ into the discharge pipe.

That origin of the $x$-axis may be considered as the virtual origin of the jet, where it begins from a precise point.



Observed velocity profiles in a turbulent jet

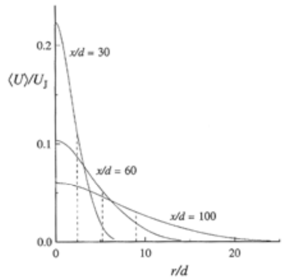



Fig. 5.2. Radial profiles of mean axial velocity in a turbulent round jet, Re = 95,500. The dashed lines indicate the half-width, $r_{1/2}(x)$, of the profiles. (Adapted from the data of Hussein *et al.* (1994).)

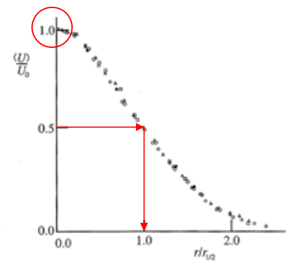



Fig. 5.3. Mean axial velocity against radial distance in a turbulent round jet, Re ≈ $10^5$; measurements of Wygnanski and Fiedler (1969). Symbols: ○, $x/d = 40$; △, $x/d = 50$; □, $x/d = 60$; ◇, $x/d = 75$; •, $x/d = 97.5$.

Velocity structure across a turbulent round jet.

$d$ = diameter of orifice (= inner diameter of discharge pipe)
$x$ = distance along jet's centerline
$r$ = radial distance from centerline
$r_{1/2}$ = radial distance where velocity falls to half of centerline speed
$\langle U \rangle$ = statistically-averaged velocity as function of distance $x$ along jet and radial distance from centerline $r$
$U_J$ = jet speed at orifice
$U_0$ = jet velocity along centerline, function of $x$

*Source*: Pope, 2000

Thus, we can express the velocity profile inside the jet as

$$u(r) = u_{\max} \exp\left( -\frac{r^2}{2\sigma^2} \right)$$

bell-curve profile with standard deviation $\sigma$ and maximum $u_{\max}$ .

Now, we know what the width of the jet is: $R = x/5$ → half-width = $2\sigma = R = x/5$

$$\sigma = \frac{x}{10} \quad \rightarrow \quad \frac{1}{2\sigma^2} = \frac{50}{x^2} \quad \rightarrow \quad u(x,r) = u_{\max} \exp\left( -\frac{50\, r^2}{x^2} \right)$$

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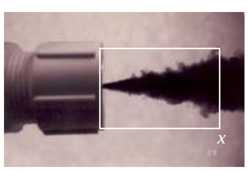
Next we need to determine the centerline velocity $u_{\max}$ .

For this, we perform a momentum budget on the jet.



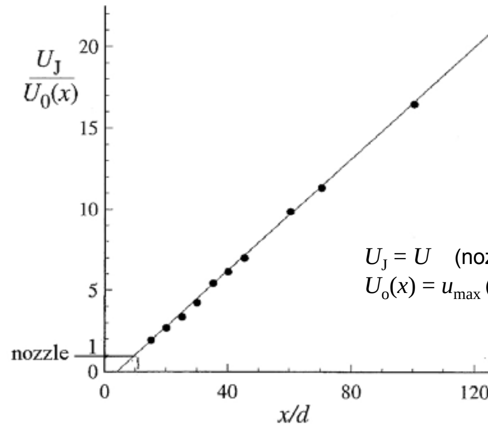

Momentum at position $x$ = Momentum exiting nozzle

$$\int_0^\infty \rho u \times 2\pi\, u\, r dr = \rho U \times U \frac{\pi d^2}{4}$$

$$\rightarrow \quad u_{\max} = \frac{5d}{x} U$$

Verification of

$$u_{\max} = \frac{5d}{x}U$$




$U_J = U$ (nozzle velocity)
$U_o(x) = u_{\max}$ (centerline velocity)

Fig. 5.4. The variation with axial distance of the mean velocity along the centerline in a turbulent round jet, Re = 95,500: symbols, experimental data of Hussein *et al.* (1994); and line, Eq. (5.6) with $x_0/d = 4$ and $B = 5.8$.

This plot shows that the centerline speed of a jet varies inversely with distance along the jet

*Source*: Pope, 2000

Average velocity

$$\bar{u} = \frac{1}{\pi R^2}\int_0^\infty u \times 2\pi r dr = \frac{5d}{2x}U = \frac{u_{\max}}{2}$$

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What about the mass budget?

$$\dot{m} = \rho Q = \int_0^\infty \rho u \times 2\pi\, r dr = \frac{\pi}{50}\rho u_{\max} x^2 = \frac{\pi}{10}\rho U d\, x$$

It seems strange to note that the mass carried by the jet increases with distance.
How can that be?

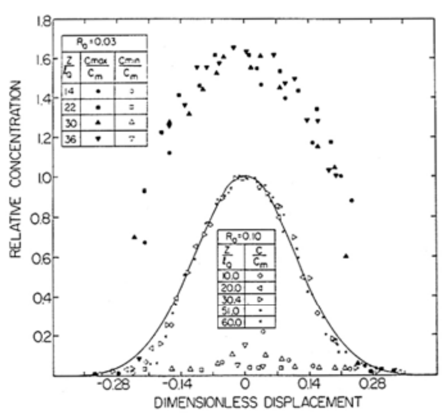
Answer: The jet entrains ambient fluid, and that it is why it grows in size.

We can defined the entrainment rate: $E = \dfrac{\text{change in volumetric flux}}{\text{distance}} = \dfrac{dQ}{dx} = \dfrac{\pi\, d\, U}{10}$

(Note that the rate of entrainment is constant down the jet.)

From Caitlin Johnson, February 2008

What about the concentration profile of a pollutant carried by the jet?

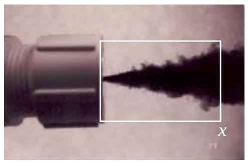



Looks like another classical bell curve!

$$c(x,r) = c_{\max}(x)\exp\left(-\frac{r^2}{2\sigma^2}\right)$$
$$= c_{\max}(x)\exp\left(-\frac{50\,r^2}{x^2}\right)$$

*Source*: Nikolas E. Kotsovinos, U. Thrace, Greece – see also *J. Fluid Mech.*, **87**, 55-63.

To determine the centerline concentration $c_{\max}$ ,
we perform a mass budget for the contaminant over a length of the jet:




Flux at position $x$ = Flux coming out of the nozzle

$$\int_0^\infty c \times 2\pi\, u\, r dr = c_0 \times U \frac{\pi d^2}{4}$$

$$\rightarrow \quad c_{\max} = \frac{5d}{x} c_0$$

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Putting the shape with the magnitude at the center, we have the concentration distribution along as well as across the jet:

$$c(x, r) = \frac{5d}{x} c_0 \exp\left( - \frac{50\, r^2}{x^2} \right)$$

Downstream distance along the jet starting at $x$ = 2.5 $d$ at the nozzle

Cross-jet distance measured from centerline outward

## Zone of excess concentration

Suppose that there is a concentration value $c_*$ not to be exceeded, and suppose it is exceeded at the discharge point (nozzle).

We can ask the question:

How extensive is the region down the jet where the concentration exceeds $c_*$ ?

To answer this question, we set the concentration $c$ equal to $c_*$ and consider the resulting relation between downstream distance $x$ and radial distance $r$.

$$c_* = \frac{5d}{x} c_0 \exp\left( - \frac{50\, r^2}{x^2} \right)$$

Solving this relationship for $r$ as a function of $x$, we obtain:

$$r^2 = \frac{x^2}{50} \ln\left( \frac{c_0}{c_*} \frac{5d}{x} \right)$$

Furthest location of the extent of excessive pollution is given by maximum $x$ value that yields $r^2 > 0$:

$$\frac{c_0}{c_*} \frac{5d}{x} = 1 \quad \rightarrow \quad x_{\max} = 5d \frac{c_0}{c_*}$$

## Effective diffusivity in a turbulent jet

Now that we know the rate of concentration dilution downstream of a turbulent jet, one may ask: What is the equivalent diffusivity?

To answer this question, we write the budget equation for the concentration:

$$\bar{u}\frac{\partial c}{\partial x} = D\left(\frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right)$$

consider advection by average velocity

assume highly advective situation no diffusion in $x$

diffusion in transverse direction

Highly advective situation demands: $Pe = \frac{Ux}{D} >> 1$ , to be verified *a posteriori*

Solution for steady source at $x = y = z = 0$ ,
with downstream distance playing the role of time:

$$t \to \frac{x}{\bar{u}} \qquad c(x,y,z) = \frac{S/\bar{u}}{\left(\sqrt{4\pi\, Dx/\bar{u}}\right)^2}\exp\left(-\frac{y^2+z^2}{4Dx/\bar{u}}\right) \qquad (y^2+z^2=r^2)$$

$$= \frac{S}{4\pi\, Dx}\exp\left(-\frac{\bar{u}\, r^2}{4Dx}\right)$$

With $\bar{u} = \frac{5d\,U}{2x}$ , the expression for $c$ becomes

$$c = \frac{S}{4\pi\, Dx}\exp\left(-\frac{5d\,U\,r^2}{8D\,x^2}\right)$$

Map this solution on the known distribution deduced from laboratory experiments:

$$c = c_{\max}\exp\left(-\frac{50\,r^2}{x^2}\right)$$

and equivalence demands

$$-\frac{5d\,U\,r^2}{8D\,x^2} = -\frac{50\,r^2}{x^2} \quad\to\quad \frac{d\,U}{8D} = 10 \quad\to\quad \boxed{D = \frac{d\,U}{80} = 0.0125\,d\,U}$$

Check on the Peclet

$$\left.\begin{array}{l} Pe = \dfrac{U\,x}{D} = \dfrac{U\,x}{d\,U/80} = 80\dfrac{x}{d} \\ x \geq 2.5d \quad \text{at a minimum} \end{array}\right\} \quad\to\quad Pe \geq 200 >> 1$$

Criterion is always met. A turbulent jet is always and everywhere highly advective.