Environmental Transport and Fate

Chapter 3

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Dispersion & Mixing

Part 2 – Turbulent dispersion

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Ubiquitous turbulence

Almost all environmental fluid flows (of air and water) are turbulent.

Therefore, the existing agitation that effectively mixes the fluid and diffuses pollutants is expected to be much greater than the underlying molecular agitation. This leads to an effective value of the diffusion coefficient D much larger than its molecular value.

The situation is not unlike that of shear flows, because erratic motions in the turbulent fluid can be thought of as many shear flows superimposed on one another.

Unfortunately, the analysis of shear-flow dispersion cannot be extended to turbulent flows because, unlike a simple shear flow where the velocity profile is known, the highly intermittent and irregular character of turbulent motions defies description.

In fact, there does not exist yet a unifying theory of turbulence, not even one for the turbulent statistical properties.

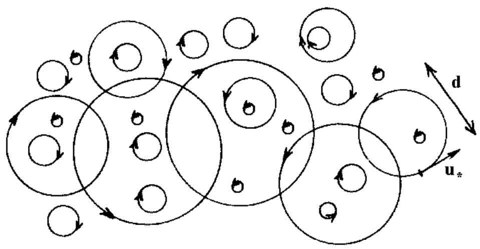
So, the approach will necessarily be much more empirical and heuristic.

Basic phenomenology of fluid turbulence

In a statistical description of turbulence, two variables play a fundamental role.
These are:

u_* = characteristic orbital velocity of fluid parcels in the turbulent eddies,

d = characteristic diameter of these eddies.



Actually, the turbulent fluid is populated by many eddies, of varying sizes and speeds, and as a result u_* and d do not assume each of a single value but vary within a certain realizable range.

http://geology.com/news/labels/Volcanoes.ht

Mayon Volcano, The Phillipines

Not a new concept!
(Sketch by Leonardo da Vinci circa 1502)

On the larger geophysical scale, too.
(Photo credit: NASA)

Turbulent smokestack plume
(photo by BCR)

Time scale and diffusivity

From the pair of quantities (d, u*) emerges a third one, an intrinsic time scale:

$$\tau = \frac{\pi\, d}{u_*} \sim \frac{d}{u_*}$$

τ is called the eddy turn-around time, for it is the time taken by a particle to cover the circumference πd at the speed u_* (nominal orbital velocity of eddy with diameter d).

From the quantities u_* , d and τ, we can estimate a diffusion coefficient D that characterizes the diffusion by eddies of size d.
We can arrive at the same result in three different ways.

1. <u>Dimensional analysis</u>: Since the dimensions of D must be length squared per time, its expression in terms of d and u_* can only be proportional to their product, *i.e*.

$$D \sim d\, u_*$$

where the symbol ~ means 'must be proportional to' or 'is on the order of'.
By virtue of the definition of the time scale τ, using the pair of variables (d , τ) or (u_* , τ) would have led to the same result.

2. Diffusive spreading: Under diffusion governed by diffusivity D, a patch grows in time according to

$$\text{patch size} = 4\sigma = 4\sqrt{2Dt}$$

In half an eddy turn-around time τ, a pollutant particle moves one diameter d away, and thus the patch size must be d by time $\tau/2$:

$$d \sim 4\sqrt{2D\frac{\tau}{2}} \sim 4\sqrt{D\frac{\pi d}{u_*}}$$

Solving this equation for D, we obtain:

$$D \sim \frac{1}{16\pi} d\, u_* \sim 0.020\, d\, u_*$$

3. Random-walk analogy: If we match eddies of size d with bins of width Δx, we ought to take the eddy turn-around time as the corresponding time step:

$$d = \Delta x, \quad \tau = \Delta t$$

Then, the random-walk model yields the diffusion coefficient:

$$D = \frac{\Delta x^2}{4\Delta t} = \frac{d^2}{4\tau} \sim d\, u_*$$

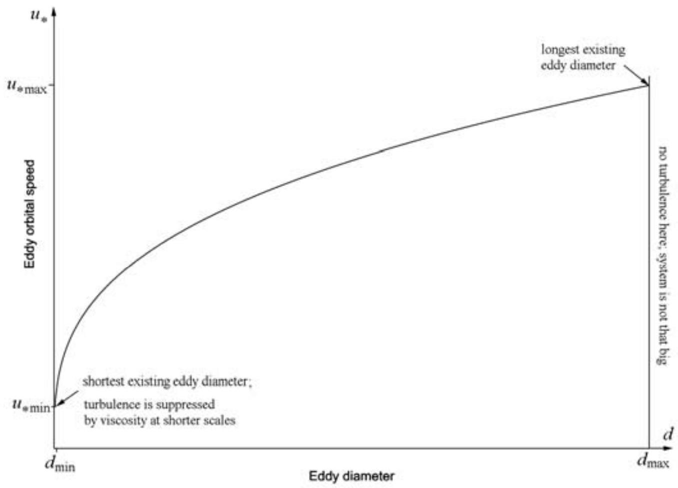
Therefore, all our considerations lead us to state that the turbulent diffusion coefficient caused by eddies of a given size is proportional to the product of the eddy diameter with the corresponding characteristic orbital velocity.

Turbulent motions naturally involve many eddies of various sizes and intensities, all embedded in one another. So, we must consider a continuous range of values for u_* and d. The question then arises as to which particular values should be used to construct the overall diffusion coefficient D. The answer is the pair of values (d , u_*) that maximizes D, because the operating diffusion (the one with the greatest influence) corresponds to the fastest spreading and thus to the largest value of D. Thus,

$$D = \max(d\, u_*)$$

Spectrum of length and velocity scales

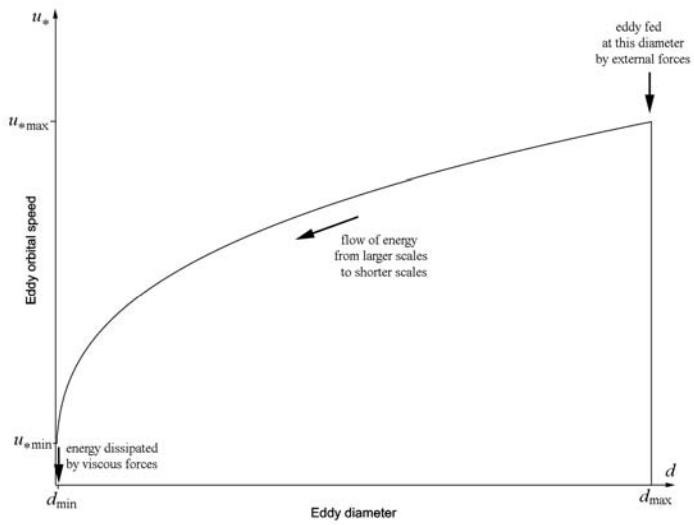
In stationary, homogeneous and isotropic turbulence (that is, a turbulent flow that statistically appears unchanging in time, uniform in space and without preferential direction), all eddies of a given size (same d) behave more or less in the same way and thus share the same characteristic velocity u_* . In other words, we expect u_* to be a function of d.



Let us try to figure out what that $u_*(d)$ function might be.

Concept of the turbulent eddy cascade

According to the theory established in the 1940s by A.N. Kolmogorov, turbulent motions span a wide range of scales ranging from a macroscale at which the energy is supplied, to a microscale at which energy is dissipated by viscosity.



The interaction among the eddies at the various scales passes energy from the larger eddies gradually to the smaller ones. This process is known as the turbulent energy cascade.

A type of cascade

Rainwater cascading down the steps at City Hall Plaza in Boston, Mass. on 1 April 2004 after 5 inches of precipitation
(Photo by David L. Ryan, Boston Globe)

As water goes down step by step in this cascade, so does energy go down the length scales in a turbulent flow.

Rate of energy cascade

If the state of turbulence is statistically steady (statistically unchanging turbulence intensity), then the rate of energy transfer from one scale to the next must be the same for all scales, so that no particular scale (= group of eddies at that scale) see its energy level increase or decrease over time.

It follows that the rate at which energy is supplied at the largest possible scale (d_{max}) is equal to that dissipated at the shortest scale (d_{min}). Let us denote by ε this rate of energy supply/dissipation, per unit mass of fluid:

ε = energy supplied to fluid per unit mass and time
= energy cascading from scale to scale, per unit mass and time
= energy dissipated by viscosity, per unit mass and time.

The dimensions of ε are:

$$[\varepsilon] = \frac{energy}{mass \times time} = \frac{ML^2T^{-2}}{M\ T} = L^2T^{-3}$$

(Units are: watts per kg or m^2 per s^3.)

With Kolmogorov, we further assume that the characteristics of the turbulent eddies of scale d depend solely on d itself (the eddies know how big they are) and on the energy cascade rate ε (the eddies know at which rate energy is supplied to them and at which rate they must supply it to the next smaller eddies in the cascade).

Thus the eddy orbital velocity u_* must depend only on d and ε.

Since [u_*] = LT^{-1} , [d] = L, and [ε] = L^2T^{-3}, the only possibility is:

$$u_* \sim (\varepsilon\, d)^{1/3}$$

Laboratory experiments suggest that the coefficient of proportionality is close to one:

$$u_* \approx 0.95(\varepsilon\, d)^{1/3}$$

The value for this coefficient, however, is poorly known, for it may be affected by a number of factors, including the manner by which the turbulent energy is fed in the fluid, the shape of the domain, *etc*.

$$u_* \approx 0.95(\varepsilon\, d)^{1/3}$$

The larger ε, the larger u_* . This makes sense, for the more energy that is supplied to the system, the more vigorous the eddies are.

The smaller d, the weaker u_* . This could not have been anticipated and must be considered as a result.

The implication is that the largest eddies have the largest speeds, and the smallest ones have the smallest speeds. Thus, turbulent intensity decreases with decreasing length scale.

Now that we know how u_* depends on d, we can also determine how the characteristic time scale, $\tau(d)$, and the diffusion coefficient, $D(d)$, vary with the eddy length scale d.

$$\tau = \frac{\pi d}{u_*} \sim \frac{d}{(\varepsilon\, d)^{1/3}} \quad \rightarrow \quad \tau \sim \varepsilon^{-1/3} d^{2/3}$$

$$D \sim d\, u_* \sim d(\varepsilon\, d)^{1/3} \quad \rightarrow \quad D \sim \varepsilon^{1/3} d^{4/3}$$

As stated earlier, the effective diffusivity of the turbulent flow is the maximum value of $D(d)$ over all possible values of d.

Since $D(d)$ is an increasing function of d, that value is obtained for the largest possible length scale, d_{max}. Thus,

$$D \sim \varepsilon^{1/3} d_{max}^{4/3}$$

In other words, the largest eddies regulate the rate of dispersion or, put the other way, turbulent dispersion is primarily effected by the largest eddies. By contrast, the molecular processes (viscosity), which affect only the shortest scales, are the least effective. Since in most turbulent flows d_{max} is much larger than d_{min} , the diffusivity caused by turbulence far exceeds the molecular diffusion coefficient.

Maximum length scale

Typically, the largest possible eddies are those that extend over the entire system and therefore

$$d_{max} = L$$

where L is the length scale of the system (such as the width of the domain). In natural flows, there is usually a noticeable scale disparity between a relatively short vertical extent (depth, height) and a relatively long horizontal extent (distance, length) of the system. Examples are:

Rivers: depth << width << length
Atmosphere: height << physically relevant horizontal distance.

In such situations, we must clearly distinguish between eddies that rotate in the vertical plane (about a horizontal axis) and those that rotate horizontally (about a vertical axis). In rivers, we must furthermore distinguish the transverse eddies from the longitudinal eddies.

Vertical diffusion (bottom-up and top-down mixing) is accomplished by eddies that bring fluid up from below and down from above, *i.e.* those that rotate in the vertical plane. The corresponding d_{max} is then H, the height of the system. The corresponding $u_*(H)$ is the scale for the velocity shear.
Horizontal diffusion is accomplished by horizontal eddies, and the corresponding d_{max} is the width or length of the system (depending on the direction of interest).

Minimum length scale

The shortest eddy scale is set by viscosity, because the shorter the eddy scale, the more important is the fluid's viscosity. Consequently, the shortest eddy scale can be defined as the length scale at which viscosity is dominant. Viscosity is, by definition, the molecular diffusion of linear momentum; it is traditionally denoted by ν, and its dimensions are identical to those of a diffusivity, L^2T^{-1} .

At the lowest scale, the turbulent diffusion of momentum reduces to viscosity and thus

$$\nu \sim \min[D(d)] \sim \varepsilon^{1/3} d_{\min}^{4/3}$$

Solving for $d_{\min}$, we obtain

$$d_{\min} \sim \nu^{3/4} \varepsilon^{-1/4}$$

Therefore, $d_{\min}$ depends on the energy level of the turbulence.
The greater the turbulence (the bigger ε), the shorter is the finest eddy scale.

Typically, $d_{\min}$ is on the order of a few millimeters or shorter.