

## Environmental Transport and Fate

### Chapter 3

–

### Dispersion & Mixing

#### Part 1 – Shear Dispersion

Benoit Cushman-Roisin  
Thayer School of Engineering  
Dartmouth College

#### Enhanced diffusion by non-uniform flows

In the preceding sections, we restricted our attention to advection by uniform flows.

While a uniform flow causes a mere translation of the pollutant (transport without distortion), a non-uniform flow can produce new and important effects. The cause is differential advection, and it goes as follows:

Different parts of the fluid flow at different rates, bringing into close proximity fluid parcels from different places and thus relatively independent concentrations. This intensifies concentration gradients and promotes diffusion.

At the extreme, a turbulent flow, by its very irregular structure, is highly favorable to mixing.

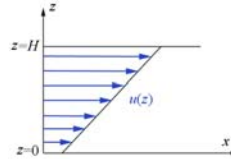
## Shear flow

A simpler case than turbulence, but one where diffusion is nonetheless greatly enhanced, is that of a shear flow.

In a shear flow, the velocity varies in the transverse direction, so that parcels on different flow lines travel at different speeds, the faster ones overtaking the slower ones.

The prototypical example is the uniform shear, where the velocity varies linearly in the transverse direction:

$$u(z) = a + bz$$



In this chapter, we denote the transverse direction by  $z$  (instead of  $y$ , the alphabetical choice) because environmental flows are typically sheared in the vertical.

Examples are rivers (where the current is weaker at depth because of bottom friction), lakes and oceans (where the flow is stronger near the surface under wind action), and the lower atmosphere (where winds are weaker near the ground and increase with height – Have you ever flown a kite?).

## Shear flow → Shear dispersion

Dispersion is simply another word for diffusion; but with a specific meaning.

We use the word dispersion to describe a process that appears as diffusion but does not exactly proceed according to Fick's law (*i.e.*, with a flux proportional to the concentration gradient).

The necessary ingredients for this process to occur are:

1. existence of a shear flow, say  $u(z)$ ,
2. diffusion in the direction *transverse* to the flow, and
3. presence of a substance (contaminant) that is carried and diffused.

In two dimensions (with  $x$  and  $z$  as the streamwise and transverse directions, respectively), the concentration distribution of a substance that is both carried differentially by the flow and diffused transversely is governed by the equation:

$$\frac{\partial c}{\partial t} + u(z) \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial z^2}$$

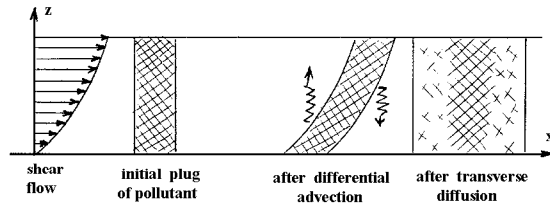
accumulation      sheared advection      transverse diffusion

We have neglected streamwise diffusion ( $D\partial^2c/\partial x^2$  term) by invoking a highly advective situation (high Peclet number:  $Pe = UL/D \gg 1$ ), something to be verified later.

To anticipate what happens, let us imagine that advection and diffusion act alternatively over short periods of time.

We then arrive at the scenario depicted below:

The plug of contaminant is differentially advected and distorted; transverse diffusion then smears cross-stream differences in concentration, and the new plug of contaminant has grown wider in the downstream direction.

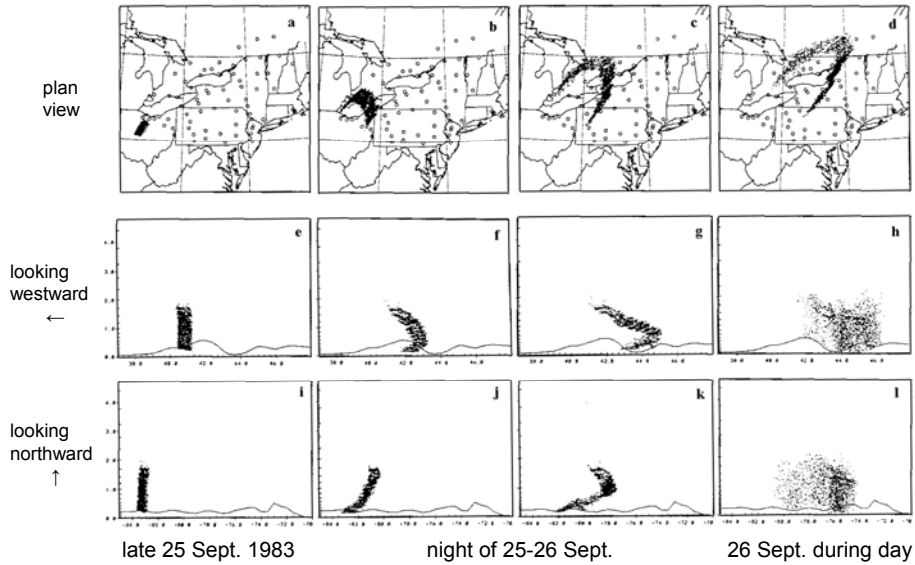


Consequently, the combined effect of differential advection and transverse diffusion is longitudinal spreading (that is, spreading along the flow).

Differential Advection + Transverse Diffusion = What appears as Longitudinal Diffusion

Shear dispersion noted in atmospheric dispersion model

- mostly differential advection at night (highly sheared wind & weak turbulence)
- uniform advection and vigorous vertical mixing during the day (strong turbulence)



The preceding example nicely illustrates the process of shear dispersion, by separating in time its two ingredients.  
 Almost always, however, differential advection and transverse diffusion act not alternatively but simultaneously.  
 Nonetheless, we can expect a very analogous outcome.

The question that arises naturally is whether the longitudinal spreading caused by the combined action of differential advection and transverse diffusion is similar to the spreading caused by a diffusive process.

In other words, can shear-induced longitudinal dispersion be represented by Fick's law of diffusion? The answer to this question is affirmative, in a limit: Longitudinal diffusion due to the shear effect obeys the law of Fickian diffusion when transverse diffusion is sufficiently rapid.

$$\frac{\partial c}{\partial t} + u(z) \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial z^2} \quad \Rightarrow \quad \frac{\partial \bar{c}}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} = K \frac{\partial^2 \bar{c}}{\partial x^2}$$

actual equation in  $(x, z, t)$ 
reduced equation in  $(x, t)$

advection by  
average speed  
of flow

effective diffusion  
in the downstream  
direction

If so, what should be the value of  $K$ ?

Shear dispersion appearing as equivalent to longitudinal diffusion

**Shear Dispersion reproduced by Random Walk**

Initial State: vertically homogenized	0	0	3	12	18	12	3	0	0
Action of differential advection	0	0	0	→ 3	→ 12	→ 18	→ 12	→ 3	0
Subsequent action of transverse diffusion	0	1	5	11	14	11	5	1	0
	0	1	5	11	14	11	5	1	0
	0	1	5	11	14	11	5	1	0

Note that the last state could have been obtained directly from the initial state by applying a  $1/3 - 1/3 - 1/3$  diffusion operation row by row. Thus,

$$\text{Sheared motion} + \text{Transverse mixing} = \text{Longitudinal diffusion}$$

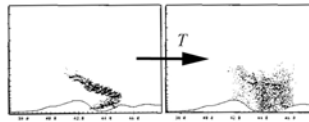
### Estimation of the effective diffusion

Here is a heuristic argument to establish an approximate value for the effective diffusivity  $K$ . (For a more precise determination, see later slides and course notes.)

The physical description given above relies in part on vertical homogenization under the action of transverse diffusion. Thus, we need to consider how long it takes for the pollutant to diffuse vertically across the system.

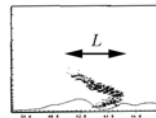
If  $H$  is the height of the domain, the time  $T$  over which vertical mixing occurs is given by:

$$T \approx 0.134 \frac{H^2}{D}$$



Now, the horizontal spread is related to how far apart two fluid parcels are sheared away during this time interval. If we introduce  $\Delta U$  as the scale for the velocity shear ( $\Delta U$  may be taken =  $u_{\max} - u_{\min}$ , and is generally not the average velocity,  $U$ ), two parcels with velocities differing by  $\Delta U$  become separated over the time interval  $T$  by the distance:

$$L = T \Delta U \approx 0.134 \frac{H^2 \Delta U}{D}$$



The length  $L$  is a measure of the horizontal spread of the pollutant patch. For an equivalent, effective diffusivity  $K$ , the same spread would be given by the  $4\sigma$ -rule:

$$\begin{aligned} \sigma &= \sqrt{2KT} \rightarrow L \approx 4\sigma = 4\sqrt{2KT} \\ &\rightarrow L \approx 4\sqrt{0.268 \frac{KH^2}{D}} \end{aligned}$$

Equating the two expressions for the same spread  $L$ , we obtain:

$$0.134 \frac{H^2 \Delta U}{D} \approx 4\sqrt{0.268 \frac{KH^2}{D}}$$

This, we solve for effective diffusivity  $K$  in terms of shear  $\Delta U$  and transverse diffusivity  $D$ :

$$K \approx 0.0042 \frac{H^2 \Delta U^2}{D}$$

Let us consider a few properties of the preceding result.

1) That  $K$  is inversely proportional to  $D$ , rather than proportional to it, is counter-intuitive.

Indeed, we could be inclined to think that the greater the transverse diffusion, the greater the longitudinal dispersion. Not so! It actually works this way:

A large  $D$  implies an efficient vertical mixing, which tends to erase (smear) the effect of differential advection; pollutant particles migrate up and down so fast that they essentially all move at the mean speed of the flow, and the shear is unimportant, causing only a weak longitudinal spreading.

At the other extreme, a small value of  $D$  implies a long time for vertical exchanges and thus ample time for differential advection to take effect; the pollutant patch is highly distorted while it diffuses moderately in the transverse direction, and longitudinal dispersion is large.

2) Check on the validity of the high-Peclet number assumption:

It was permitted to ignore the actual diffusion term if the new, effective-diffusion term is much larger, i.e. if

$$D \frac{\partial^2 c}{\partial x^2} \ll K \frac{\partial^2 \bar{c}}{\partial x^2} \Rightarrow D \ll K \Rightarrow D \ll 0.0042 \frac{H^2 \Delta U^2}{D} \Rightarrow \frac{H \Delta U}{D} \gg 15$$

### More refined theory (due to G. I. Taylor, 1921 & 1953)



1. Define transverse averages and departures from them:

$$\bar{u} = \frac{1}{H} \int_0^H u(z) dz \quad \bar{c} = \frac{1}{H} \int_0^H c dz \quad u(z) = \bar{u} + u'(z) \quad c = \bar{c} + c'$$

2. Substitute in budget equation and take its vertical average:

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial \bar{c}'}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} + \bar{u} \frac{\partial \bar{c}'}{\partial x} + u' \frac{\partial \bar{c}}{\partial x} + u' \frac{\partial \bar{c}'}{\partial x} = D \frac{\partial^2 \bar{c}}{\partial x^2} + D \frac{\partial^2 c'}{\partial z^2}$$

$$\frac{\partial \bar{c}}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} + \frac{1}{H} \int_0^H u' \frac{\partial \bar{c}'}{\partial x} dz = \left[ D \frac{\partial \bar{c}'}{\partial z^2} \right]_{z=0}^{z=H}$$

3. Rank terms according to size:

Assume near-uniform distribution in the transverse direction but finite shear:  $|c'| \ll \bar{c}$

Larger terms:  $u' \frac{\partial \bar{c}}{\partial x}$   $|u'| \sim \bar{u}$

Smaller terms:  $\frac{\partial \bar{c}}{\partial t}$ ,  $\bar{u} \frac{\partial \bar{c}}{\partial x}$ ,  $u' \frac{\partial c'}{\partial x}$

Even smaller terms:  $\frac{\partial c'}{\partial t}$ ,  $\bar{u} \frac{\partial c'}{\partial x}$

4. Budget equation minus its average, larger terms only:

$$u' \frac{\partial \bar{c}}{\partial x} = D \frac{\partial^2 c'}{\partial z^2}$$

Note how this reduced equation captures the two essential elements of shear dispersion:  
Differential advection ( $u'$ ) and transverse diffusion ( $D$ ).

5. Solve for  $c'$  by integrating twice over  $z$  :

$$\frac{\partial c'}{\partial z} = \frac{1}{D} \left[ \int_0^z u'(z') dz' \right] \frac{\partial \bar{c}}{\partial x} \quad (\text{No flux at impermeable bottom } z=0)$$

$$c' = \frac{1}{D} \left[ \int_0^z dz' \int_0^{z'} u'(z'') dz'' \right] \frac{\partial \bar{c}}{\partial x} + \text{constant}$$

6. Set the constant of integration so that  $c'$  has no transverse average:

$$c' = \frac{1}{D} \left[ \int_0^z dz' \int_0^{z'} u'(z'') dz'' - \frac{1}{H} \int_0^H dz \int_0^z dz' \int_0^{z'} u'(z'') dz'' \right] \frac{\partial \bar{c}}{\partial x}$$

7. Calculate the integral term for the average budget equation:

$$\frac{1}{H} \int_0^H u' \frac{\partial c'}{\partial x} dz = \frac{1}{DH} \left[ \int_0^H u'(z) dz \int_0^z dz' \int_0^{z'} u'(z'') dz'' \right] \frac{\partial^2 \bar{c}}{\partial x^2}$$

8. Average budget equation can now be expressed in the desired form:

$$\frac{\partial \bar{c}}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} = K \frac{\partial^2 \bar{c}}{\partial x^2}$$

with

$$K = \frac{-1}{DH} \int_0^H u'(z) dz \int_0^z dz' \int_0^{z'} u'(z'') dz''$$

Better, in terms, of intermediate variables:

$$\bar{u} = \frac{1}{H} \int_0^H u(z) dz \quad \tilde{u}(z) = \frac{1}{H} \int_0^z u'(z') dz'$$

$$K = \frac{H}{D} \int_0^H \tilde{u}^2(z) dz > 0$$