Environmental Transport and Fate

Chapter 3

Part 4 – Mixing in Stratified Fluids

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Thermal stratification in Wellington Reservoir, Western Australia
Heat affects fluid density. When temperature rises, air and water (>4°C) expand. Warmer fluid is less dense and seeks to float on top, while colder fluid is denser and seeks to sink to the bottom.

The gravitational force that a density difference produces is called the *buoyancy force*.

**Thermal expansion**

In the environment, temperature and density differences remain moderate, and a linear approximation is adequate:

$$\rho = \rho_0 [1 - \alpha (T - T_0)]$$

in which

- $\rho$ = density at temperature $T$
- $\rho_0$ = reference density at reference temperature $T_0$
- $\alpha$ = coefficient of thermal expansion

Freshwater: $\alpha = 2.57 \times 10^{-4} \, ^{\circ}C^{-1}$
Seawater: $\alpha = 1.7 \times 10^{-4} \, ^{\circ}C^{-1}$
Air (or any ideal gas): $\alpha = 1/T_0$ inverse of absolute temperature

**Typical diurnal cycle of thermal stratification in the lower atmosphere**

The thing to remember about the atmosphere is that it is not heated from the top by the sun but is ‘boiling’ from below by the ground (or sea) surface.

The radiating energy from the sun passes through the transparent atmosphere, is absorbed by the ground (or sea) and is re-radiated as heat to the atmosphere from below.

That the atmosphere is convecting from below is a physical complication but is very healthy from the environmental perspective because it provides ventilation.
Time evolution of the Kelvin-Helmholtz instability

Two zones of the same fluid at different densities and with different velocities tend to create a wave, which overturns and concludes with vertical mixing.

(Sketch from Graf & Mortimer, 1979)

Kelvin-Helmholtz occasionally manifested in the sky by judicious clouds
Laboratory simulation of the K-H instability in the laboratory. Here, downstream distance replaces time.

Kelvin-Helmholtz instability theory

Situation
- top fluid of density $\rho_1$ and moving at speed $u_1$
- bottom fluid of density $\rho_2$ ($\geq \rho_1$) and moving at speed $u_2$ ($\neq u_1$)
- system infinitely high, deep and wide

Result
Any existing perturbation turns into a wave component, of wavelength $\lambda$. The shorter waves, satisfying the condition below, grow in time (said to be unstable).

$$g\lambda (\rho_2^2 - \rho_1^2) < 2\pi \rho_1 \rho_2 (u_1 - u_2)^2$$

Here, $g = 9.81 \text{ m/s}^2$ is the gravitational acceleration on the earth.
There thus exists a critical wavelength that separates the shorter, unstable waves from the longer, stable waves:

\[
\lambda_{\text{crit}} = \frac{2\pi \rho \rho_i (u_i - u_2)^2}{g (\rho_2 - \rho_1)} = \frac{\pi \rho_i (u_i - u_2)^2}{g \Delta \rho} = \frac{\pi \Delta u^2}{\alpha g \Delta T}
\]

because \( \rho_1 \approx \rho_2 \approx \rho_0 \)

\[
\Delta \rho = \rho_0 - \rho_1 > 0
\]

\[
\Delta T = T_1 - T_2 > 0
\]

\[
\Delta u = |u_2 - u_1| > 0
\]

The amount of overturning is observed to occupy a vertical extent comparable to, but a little shorter than, the critical billow wavelength.

We write that vertical mixing proceeds over a height (depth) given by

\[
h = C \frac{\Delta u^2}{\alpha g \Delta T}
\]

with \( C \approx 0.3 \)

Computer simulation of the K-H instability
Energetics of vertical mixing

Question: Is the mixed state energetically possible from the initial state?

Answer: Yes under a certain condition.
Assume that thorough mixing occurs.

Then, the final state can be obtained by enforcing conservation of mass and momentum between initial and final states.

For simplicity, take \( H_1 = H_2 = H/2 \).

**Mass conservation:**

\[
p\rho H = \rho_1 H_1 + \rho_2 H_2 \quad \Rightarrow \quad \rho = \frac{\rho_1 + \rho_2}{2}
\]

**Momentum conservation:**

\[
p\rho UH = \rho_1 U_1 H_1 + \rho_2 U_2 H_2 \quad \Rightarrow \quad p_\rho UH = p_\rho U_1 H_1 + p_\rho U_2 H_2
\]

\[
\Rightarrow \quad U = \frac{U_1 H_1 + U_2 H_2}{H}
\]

\[
\Rightarrow \quad U = \frac{U_1 + U_2}{2}
\]

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<table>
<thead>
<tr>
<th>Kinetic Energy</th>
<th>Potential Energy</th>
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</thead>
<tbody>
<tr>
<td><strong>Before mixing</strong></td>
<td><strong>Before mixing</strong></td>
</tr>
<tr>
<td>( KE_{\text{initial}} = \frac{1}{2} \rho_1 U_1^2 H_1 + \frac{1}{2} \rho_2 U_2^2 H_2 )</td>
<td>( PE_{\text{initial}} = \int_{z_i}^{z_f} \rho_1 g z dz + \int_{z_i}^{z_f} \rho_2 g z dz )</td>
</tr>
<tr>
<td>( = \frac{1}{4} \rho_\rho (U_1^2 + U_2^2) H )</td>
<td>( = \frac{1}{8} \rho_\rho g H + \frac{3}{8} \rho_\rho g H )</td>
</tr>
<tr>
<td><strong>After mixing (if mixing occurs at all)</strong></td>
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</tr>
<tr>
<td>( KE_{\text{final}} = \frac{1}{2} \rho U^2 H = \frac{1}{2} \rho \left(\frac{U_1 + U_2}{2}\right)^2 H )</td>
<td>( PE_{\text{final}} = \int_{z_i}^{z_f} \rho g z dz + \int_{z_i}^{z_f} \rho \left(\frac{\rho_1 + \rho_2}{2}\right) g z dz )</td>
</tr>
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<td>( = \frac{1}{8} \rho_\rho (U_1 + U_2)^2 H )</td>
<td>( = \frac{1}{4} (\rho_1 + \rho_2) g H )</td>
</tr>
<tr>
<td><strong>Kinetic energy drop</strong></td>
<td><strong>Potential energy gain</strong></td>
</tr>
<tr>
<td>( KE_{\text{drop}} = KE_{\text{initial}} - KE_{\text{final}} )</td>
<td>( PE_{\text{gain}} = PE_{\text{final}} - PE_{\text{initial}} )</td>
</tr>
<tr>
<td>( = \frac{1}{8} \rho_\rho (U_1 - U_2)^2 H )</td>
<td>( = \frac{1}{8} (\rho_\rho - \rho_\rho) g H^2 )</td>
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<td><strong>Condition for spontaneous mixing:</strong></td>
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<td>( KE_{\text{drop}} &lt; PE_{\text{gain}} )</td>
<td>( \frac{1}{8} \rho_\rho (U_1 - U_2)^2 H &lt; \frac{1}{8} (\rho_\rho - \rho_\rho) g H^2 )</td>
</tr>
</tbody>
</table>
\[ KE_{\text{drop}} > PE_{\text{gain}} \]
\[ \rho_0 (U_1 - U_2)^2 > (\rho_2 - \rho_1) g H \]
\[ H < \frac{\rho_0 (U_1 - U_2)^2}{(\rho_2 - \rho_1) g} \]
\[ H < \frac{\rho_0 \Delta U^2}{g \Delta \rho} \quad \text{with} \quad \Delta U = |U_1 - U_2| \]
\[ \Delta \rho = \rho_2 - \rho_1 > 0 \]

In reality, not all the kinetic energy that is released goes into potential energy. Much of it (about 70%) goes into turbulent kinetic energy, which eventually dissipates. Only the remainder (about 30%) goes into potential energy. Thus, a practical criterion is:

\[ H < 0.3 \frac{\rho_0 \Delta U^2}{g \Delta \rho} \]