Environmental Transport and Fate

Chapter 3

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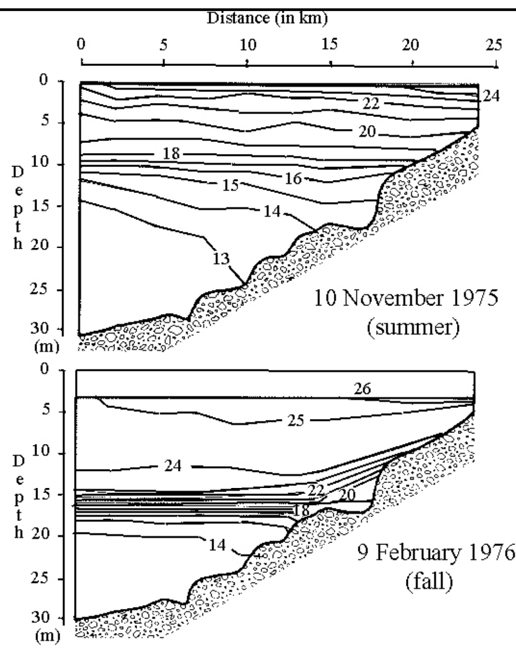
Part 4 – Mixing in Stratified Fluids

Benoit Cushman-Roisin
Thayer School of Engineering
Dartmouth College



Thermal stratification in Wellington Reservoir, Western Australia

Heat affects fluid density. When temperature rises, air and water (> 4°C) expand.
Warmer fluid is less dense and seeks to float on top,
while colder fluid is denser and seeks to sink to the bottom.

The gravitational force that a density difference produces is called the *buoyancy force*.

Thermal expansion

In the environment, temperature and density differences remain moderate,
and a linear approximation is adequate:

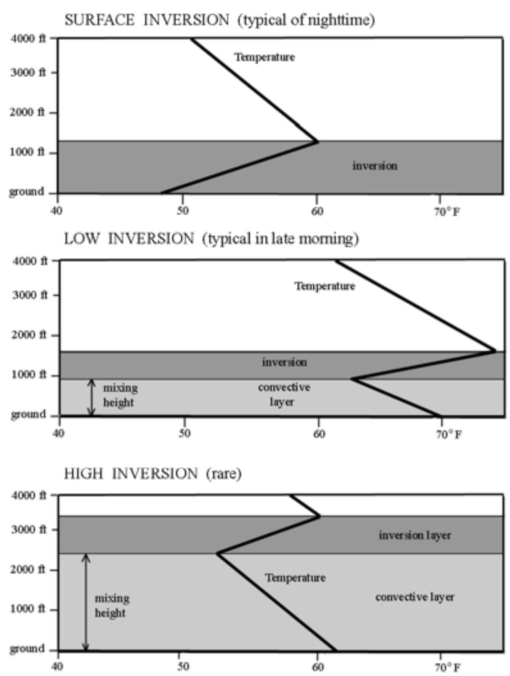
$$\rho = \rho_0 [1 - \alpha (T - T_0)]$$

in which

ρ = density at temperature T
ρ_0 = reference density at reference temperature T_0
α = coefficient of thermal expansion

Freshwater: α = 2.57 x 10^{-4} $^oC^{-1}$
Seawater: α = 1.7 x 10^{-4} $^oC^{-1}$

Air (or any ideal gas): $\alpha = 1/T_0$
inverse of absolute temperature



Typical diurnal cycle of thermal stratification in the lower atmosphere

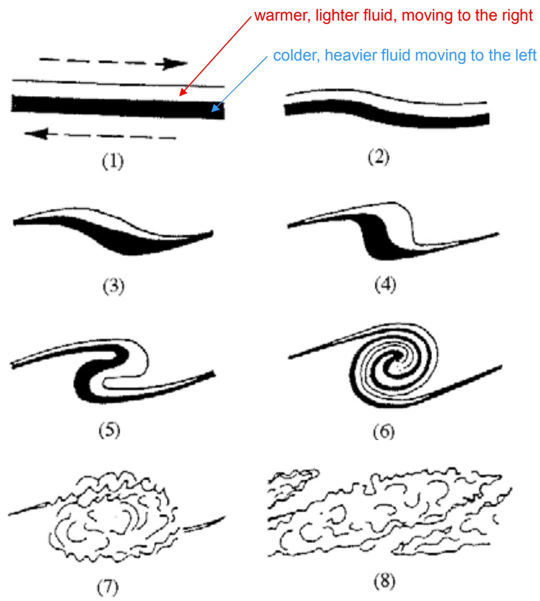
The thing to remember about the atmosphere is that it is not heated from the top by the sun but is 'boiling' from below by the ground (or sea) surface.

The radiating energy from the sun passes through the transparent atmosphere, is absorbed by the ground (or sea) and is re-radiated as heat to the atmosphere from below.

That the atmosphere is convecting from below is a physical complication but is very healthy from the environmental perspective because it provides ventilation.

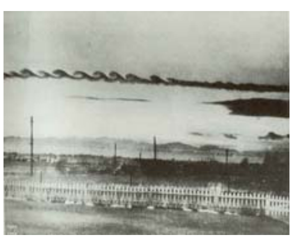
Time evolution of the Kelvin-Helmholtz instability

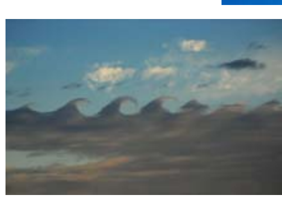
Two zones of the same fluid at different densities and with different velocities tend to create a wave, which overturns and concludes with vertical mixing.

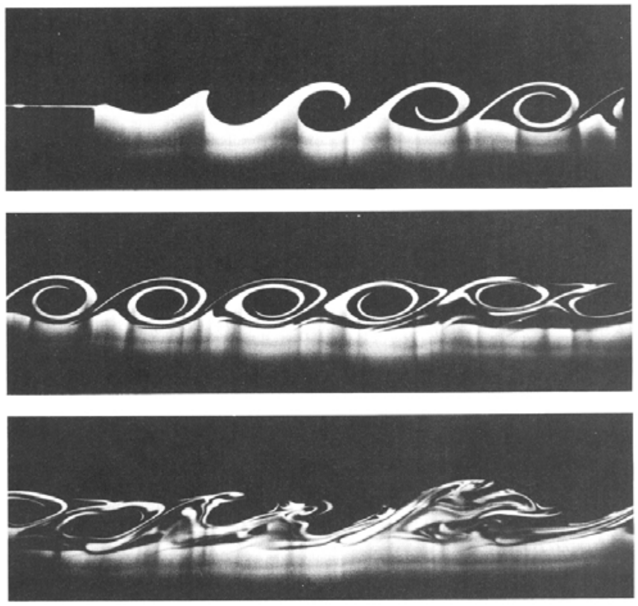


(Sketch from Graf & Mortimer, 1979)

Kelvin-Helmholtz occasionally manifested in the sky by judicious clouds



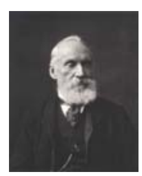




(Photo by Greg Lawrence, University of British Columbia)

Laboratory simulation of the K-H instability in the laboratory.
Here, downstream distance replaces time.

Kelvin-Helmholtz instability theory



William Thomson
Lord Kelvin
(1824-1907)

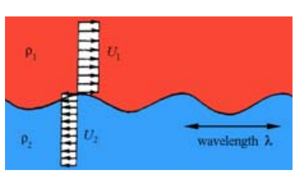
Hermann von Helmholtz
(1821-1894)

Situation

- top fluid of density ρ_1 and moving at speed u_1
- bottom fluid of density ρ_2 (> ρ_1) and moving at speed $u_2\,(\neq u_1)$
- system infinitely high, deep and wide

Result

Any existing perturbation turns into a wave component, of wavelength λ.
The shorter waves, satisfying the condition below, grow in time (said to be unstable).



$$g\lambda\,(\rho_2^2 - \rho_1^2) < 2\pi\,\rho_1\rho_2\,(u_1 - u_2)^2$$

Here, g = 9.81 m/s^2 is the gravitational acceleration on the earth.

There thus exists a critical wavelength that separates the shorter, unstable waves from the longer, stable waves:

$$\lambda_{\text{crit}} = \frac{2\pi\,\rho_1\rho_2\,(u_1 - u_2)^2}{g\,(\rho_2^2 - \rho_1^2)} \approx \frac{\pi\,\rho_0\,(u_1 - u_2)^2}{g\,\Delta\rho} = \frac{\pi\,\Delta u^2}{\alpha\,g\,\Delta T}$$

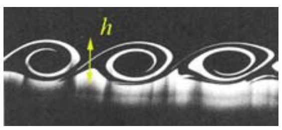
because $\rho_1 \approx \rho_2 \approx \rho_0$ $\quad \Delta\rho = \rho_2 - \rho_1 > 0$ $\quad \Delta\rho << \rho_0$

$\Delta T = T_1 - T_2 > 0$

$\Delta u = |u_2 - u_1| > 0$

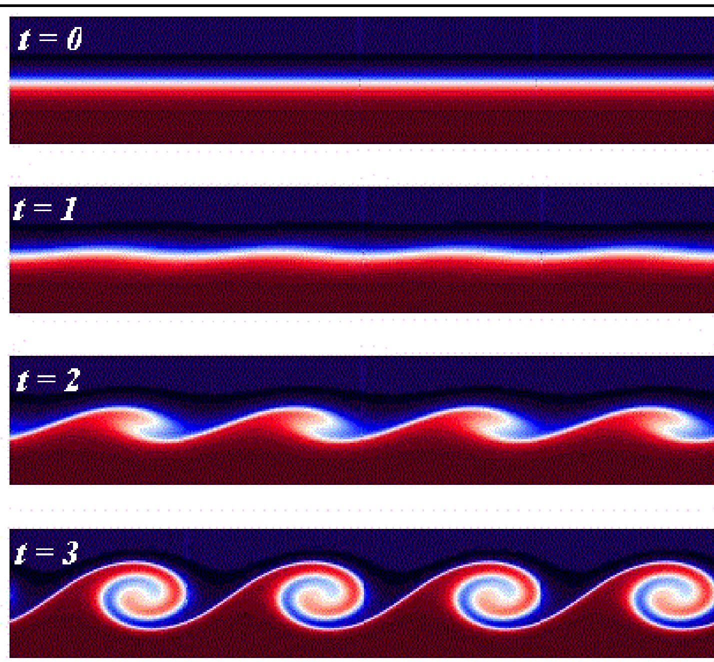
The amount of overturning is observed to occupy a vertical extent comparable to, but a little shorter than, the critical billow wavelength.

We write that vertical mixing proceeds over a height (depth) given by



$$h = C\,\frac{\Delta u^2}{\alpha\,g\,\Delta T}$$

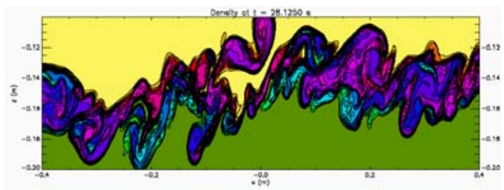
with $C \approx 0.3$



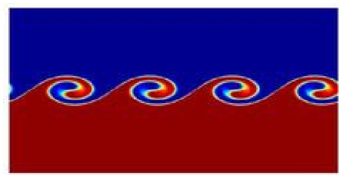
(http://www.riam.kyushu-u.ac.jp/ship/STAFF/hu/flow.html)

Computer simulation of the K-H instability

For nice animated computer simulations, see:

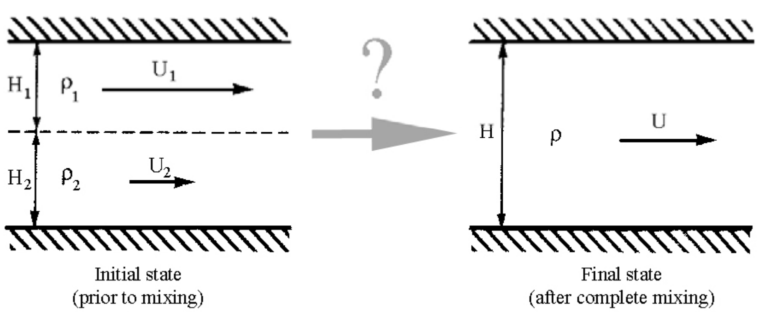


http://www.math.uwaterloo.ca/~kglamb/Hydrodynamic_Instability_animations.html



http://fluid.stanford.edu/~fringer/movies/shear_convect/shear.html

Energetics of vertical mixing



Question: Is the mixed state energetically possible from the initial state?

Answer: Yes under a certain condition.

Assume that thorough mixing occurs.

Then, the final state can be obtained by enforcing conservation of mass and momentum between initial and final states.

For simplicity, take $H_1 = H_2 = H/2$.

Mass conservation:

$$\rho H = \rho_1 H_1 + \rho_2 H_2 \quad \Rightarrow \quad \rho = \frac{\rho_1 + \rho_2}{2}$$

Momentum conservation:

$$\rho U H = \rho_1 U_1 H_1 + \rho_2 U_2 H_2 \quad \Rightarrow \quad \rho_0 U H \cong \rho_0 U_1 H_1 + \rho_0 U_2 H_2$$

$$\Rightarrow \quad U = \frac{U_1 H_1 + U_2 H_2}{H}$$

$$\Rightarrow \quad U = \frac{U_1 + U_2}{2}$$

Kinetic Energy

Before mixing

$$KE_{initial} = \frac{1}{2}\rho_0 U_1^2 H_1 + \frac{1}{2}\rho_0 U_2^2 H_2 = \frac{1}{4}\rho_0 (U_1^2 + U_2^2) H$$

After mixing (if mixing occurs at all)

$$KE_{final} = \frac{1}{2}\rho_0 U^2 H = \frac{1}{2}\rho_0 \left(\frac{U_1 + U_2}{2}\right)^2 H = \frac{1}{8}\rho_0 (U_1 + U_2)^2 H$$

Kinetic energy **drop**

$$KE_{drop} = KE_{initial} - KE_{final} = \frac{1}{8}\rho_0 (U_1 - U_2)^2 H$$

Potential Energy

Before mixing

$$PE_{initial} = \int_0^{H_2} \rho_2 g z \, dz + \int_{H_2}^{H_1 + H_2} \rho_1 g z \, dz = \frac{1}{8}\rho_2 g H^2 + \frac{3}{8}\rho_1 g H^2$$

After mixing (if mixing occurs at all)

$$PE_{final} = \int_0^H \rho g z \, dz = \int_0^H \frac{\rho_1 + \rho_2}{2} g z \, dz = \frac{1}{4}(\rho_1 + \rho_2) g H^2$$

Potential energy **gain**

$$PE_{gain} = PE_{final} - PE_{initial} = \frac{1}{8}(\rho_2 - \rho_1) g H^2$$

Condition for spontaneous mixing:

$$KE_{drop} > PE_{gain}$$

$$\frac{1}{8}\rho_0 (U_1 - U_2)^2 H > \frac{1}{8}(\rho_2 - \rho_1) g H^2$$

$$KE_{drop} > PE_{gain}$$

$$\Downarrow$$

$$\rho_0 (U_1 - U_2)^2 > (\rho_2 - \rho_1) g H$$

$$\Downarrow$$

$$H < \frac{\rho_0 (U_1 - U_2)^2}{(\rho_2 - \rho_1)\, g}$$

$$\Downarrow$$

$$H < \frac{\rho_0 \Delta U^2}{g \Delta \rho}$$ with $\Delta U = |U_1 - U_2|$, $\Delta\rho = \rho_2 - \rho_1 > 0$

In reality, not all the kinetic energy that is released goes into potential energy.
Much of it (about 70%) goes into turbulent kinetic energy, which eventually dissipates.
Only the remainder (about 30%) goes into potential energy.
Thus, a practical criterion is:

$$H < 0.3 \frac{\rho_0 \Delta U^2}{g \Delta \rho}$$